Sheet 1, Exercise 16

$F : \mathbb{R}^2 \longrightarrow \mathbb{R}$ defined by $F(x,y) = x^3 + xy + y^3$

Which level sets of F are embedded submanifolds of $\mathbb{R}^2$?

For each level set, prove that it is or that it is not an embedded submanifold.

Proof The level sets of F are the subsets of $\mathbb{R}^2$ of the form $F^{-1}(c) \subset \mathbb{R}^2$, for $c \in \mathbb{R}$.

So, $F^{-1}(c)$ is the plane curve defined by the equation

$$x^3 + xy + y^3 = c$$

The differential of F, $dF(x,y)$, is the linear map given by the Jacobian matrix

$$JF(x,y) = \left(3x^2 + y \ , \ 3y^2 + x \right)$$

hence $dF(x,y)$ is <u>not surjective</u> $\iff$ $\operatorname{rank}(JF) = 0$

We have: $\begin{cases} 3x^2 + y = 0 \\ 3y^2 + x = 0 \end{cases} \iff \begin{cases} y = -3x^2 \\ x(27x^3 + 1) = 0 \end{cases}$

There are two solutions, given by $\begin{cases} x = 0 \\ y = 0 \end{cases}$; $\begin{cases} x = -\frac{1}{3} \\ y = -\frac{1}{3} \end{cases}$

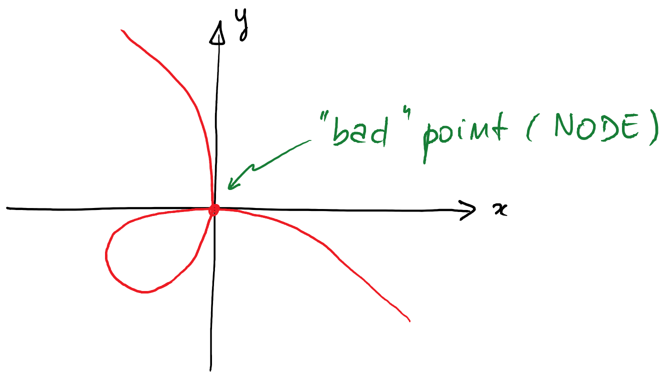
This means that F has two <u>critical points</u>: $(0,0)$, $\left(-\frac{1}{3}, -\frac{1}{3}\right)$

The <u>critical values</u> of F are: $F(0,0) = 0$, $F\left(-\frac{1}{3}, -\frac{1}{3}\right) = \frac{1}{27}$

We can now apply Proposition 2.4.23 (page 95): the conclusion is that the level set $F^{-1}(c)$ is an embedded submanifold $\forall\, c \neq 0, \frac{1}{27}$.

For the level set $F^{-1}(0)$ something bad happens at the critical point $(0,0)$, while for the level set $F^{-1}\left(\frac{1}{27}\right)$ something bad happens at the critical point $\left(-\frac{1}{3}, -\frac{1}{3}\right)$.

Let's start by considering $F^{-1}(0)$. This is the plane curve given by the equation $x^3 + xy + y^3 = 0$.
We can immediately see that $(0,0)$ is a singular point for this curve (both partial derivatives vanish at $(0,0)$).
The tangent cone at $(0,0)$ is the zero-set of the homogeneous term of lowest degree of the polynomial $x^3 + xy + y^3$, hence the tangent cone is given by the equation $xy = 0$. This means that at the point $(0,0)$ there are two tangent directions, corresponding to the lines $x = 0$ and $y = 0$. The point $(0,0)$ is then a node. Here is a sketch of the curve $F^{-1}(0)$ near the point $(0,0)$:

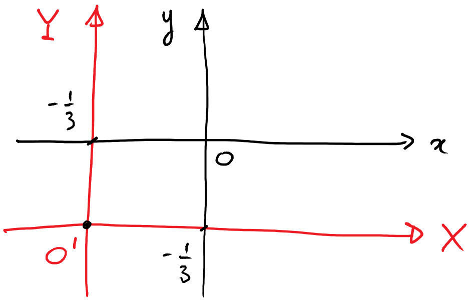


The presence of the singular point $(0,0)$ implies that $F^{-1}(0)$ is not an embedded submanifold.

Now we consider the level set $F^{-1}\left(\frac{1}{27}\right)$.
This is the subset of $\mathbb{R}^2$ described by the equation

$$x^3 + xy + y^3 = \frac{1}{27}$$

Since we are interested in studying what happens near the critical point $\left(-\frac{1}{3}, -\frac{1}{3}\right)$, we perform a change of variables (a translation) so that the critical point will be at the origin of a new coordinate system:



$$\begin{cases} x = X - \frac{1}{3} \\ y = Y - \frac{1}{3} \end{cases}$$

With this change of variables the equation of the level set $F^{-1}\left(\frac{1}{27}\right)$ becomes:

$$X^3 + Y^3 - (X^2 + Y^2 - XY) = 0$$

As in the previous case, we recognise that the critical point $\begin{cases} x = -\frac{1}{3} \\ y = -\frac{1}{3} \end{cases} \iff \begin{cases} X = 0 \\ Y = 0 \end{cases}$ is a <u>singular point</u>

The <u>tangent cone</u> at the point $(X=0, Y=0)$ is given by the equation $X^2 + Y^2 - XY = 0$.

This polynomial factors as follows:

$$X^2 + Y^2 - XY = \left[X + \left(-\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)Y\right] \cdot \left[X + \left(-\frac{1}{2} - \frac{i\sqrt{3}}{2}\right)Y\right]$$

hence the tangent cone is the union of the two complex conjugate lines of equations

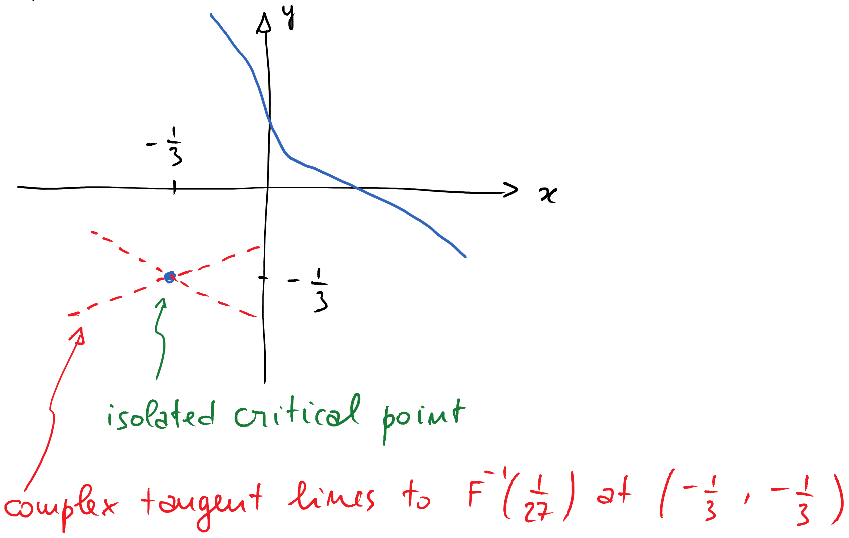
$$X + \left(-\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)Y = 0 \quad ; \quad X + \left(-\frac{1}{2} - \frac{i\sqrt{3}}{2}\right)Y = 0$$

These lines meet at <u>the unique real</u> point $(X=0, Y=0)$

Since in $\mathbb{R}^2$ we only see points with <u>real</u> coordinates, the point $(X=0, Y=0)$ appears to be an <u>isolated</u>

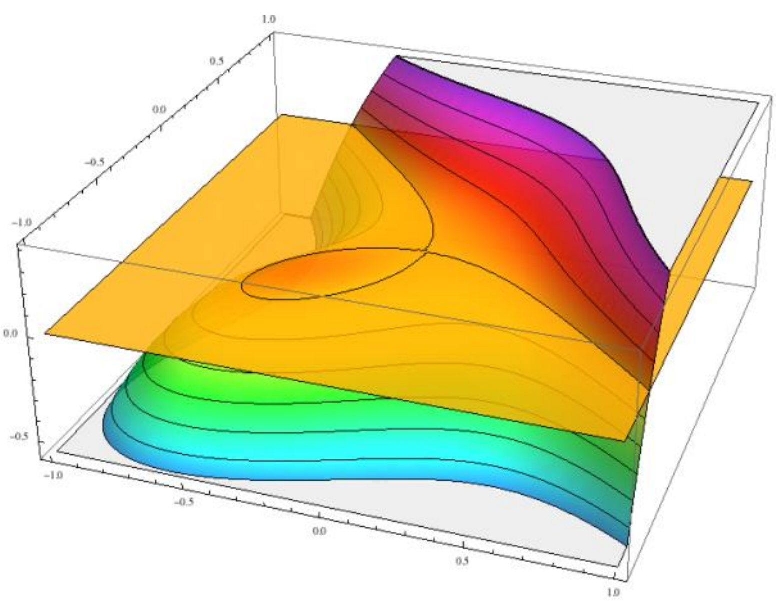
(singular) point of the level set $F^{-1}\left(\frac{1}{27}\right)$

Here is a sketch of the level set $F^{-1}\left(\frac{1}{27}\right)$ near the critical point:



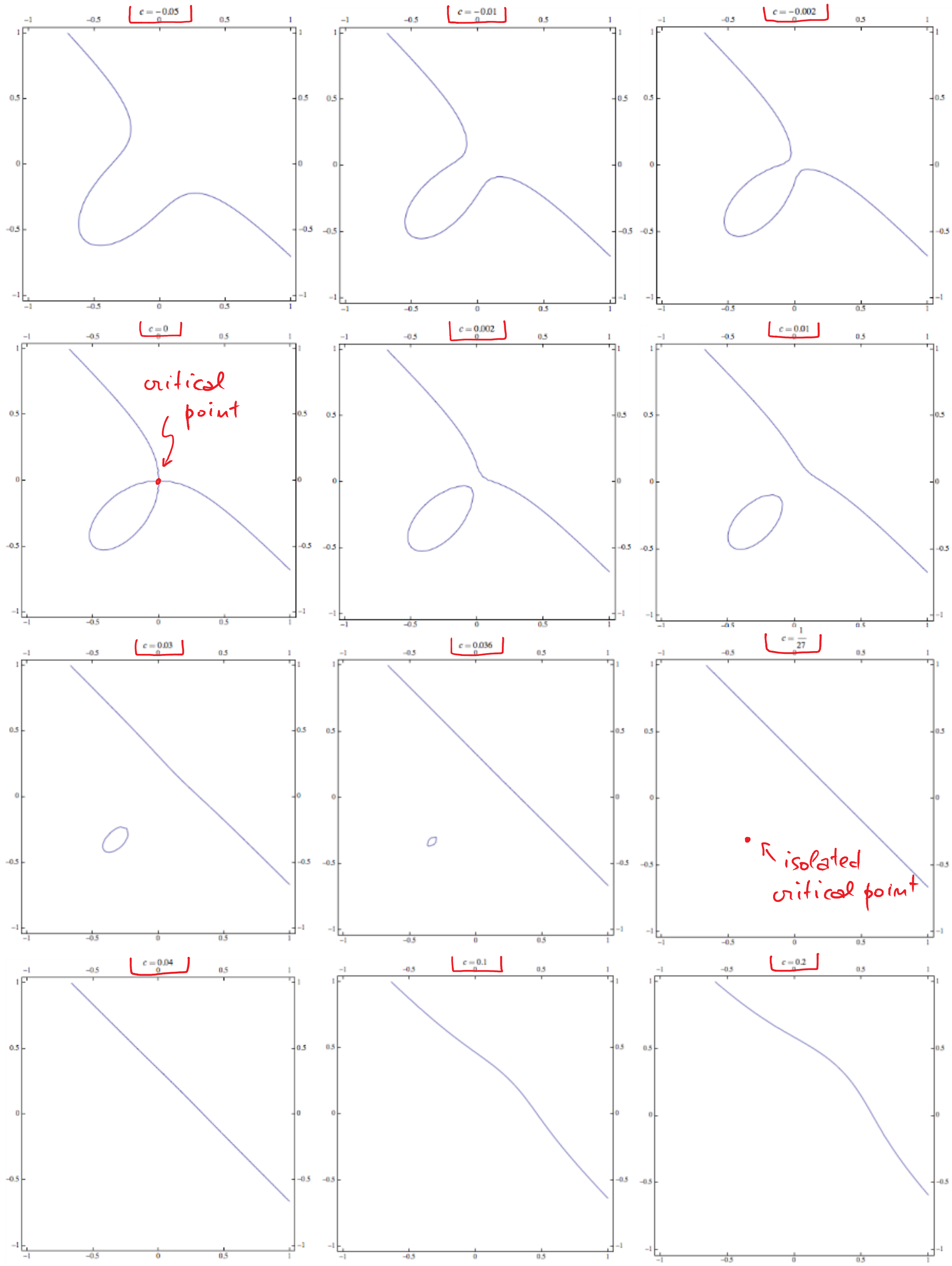
The presence of the singular point $\left(-\frac{1}{3}, -\frac{1}{3}\right)$ implies that $F^{-1}\left(\frac{1}{27}\right)$ is <u>not</u> an embedded submanifold.

Here is the graph of the function $z = F(x, y)$



The level sets $F^{-1}(c)$ are obtained by intersecting the graph of $z = F(x,y)$ with the plane $z = c$.

Here are some level sets $F^{-1}(c)$, for some different values of c:


c = −0.05
c = −0.01
c = −0.002
c = 0
critical point
c = 0.002
c = 0.01
c = 0.03
c = 0.036
c = 1/27
isolated critical point
c = 0.04
c = 0.1
c = 0.2