Sheet 1, Exercise 16

$F : \mathbb{R}^2 \to \mathbb{R}$ defined by $F(x, y) = x^3 + xy + y^3$

Which level sets of $F$ are embedded submanifolds of $\mathbb{R}^2$?

For each level set, prove that it is on that it is not an embedded submanifold.

Proof: The level sets of $F$ are the subsets of $\mathbb{R}^2$ of the form $F^{-1}(c) \subset \mathbb{R}^2$, for $c \in \mathbb{R}$.

So, $F^{-1}(c)$ is the plane curve defined by the equation

$$x^3 + xy + y^3 = c$$

The differential of $F$, $dF(x, y)$, is the linear map given by the Jacobian matrix

$$JF(x, y) = \begin{pmatrix} 3x^2 + y \\ 3y^2 + x \end{pmatrix}$$

hence $dF(x, y)$ is not surjective $\iff$ rank$(JF) = 0$

We have:

$$\begin{cases} 3x^2 + y = 0 \\ 3y^2 + x = 0 \end{cases} \implies \begin{cases} y = -3x^2 \\ x(27x^3 + 1) = 0 \end{cases}$$

There are two solutions, given by

$$\begin{cases} x = 0 \\ y = 0 \end{cases} \quad \begin{cases} x = -\frac{1}{3} \\ y = -\frac{1}{3} \end{cases}$$

This means that $F$ has two critical points: $(0, 0)$, $\left(-\frac{1}{3}, -\frac{1}{3}\right)$

The critical values of $F$ are: $F(0, 0) = 0$, $F\left(-\frac{1}{3}, -\frac{1}{3}\right) = \frac{1}{27}$

We can now apply Proposition 2.4.23 (page 95): the conclusion is that the level set $F^{-1}(c)$ is an embedded submanifold $\forall c \neq 0$, $\frac{1}{27}$.

For the level set $F^{-1}(0)$ something bad happens at the
critical point \((0,0)\), while for the level set \(F^{-1}(\frac{1}{27})\) something bad happens at the critical point \((-\frac{1}{3}, -\frac{1}{3})\).

Let's start by considering \(F^{-1}(0)\). This is the plane curve given by the equation \(x^3 + xy + y^3 = 0\).

We can immediately see that \((0,0)\) is a singular point for this curve (both partial derivatives vanish at \((0,0)\)).

The tangent cone at \((0,0)\) is the zero-set of the homogeneous term of lowest degree of the polynomial \(x^3 + xy + y^3\), hence the tangent cone is given by the equation \(xy = 0\). This means that at the point \((0,0)\) there are two tangent directions, corresponding to the lines \(x = 0\) and \(y = 0\). The point \((0,0)\) is then a node. Here is a sketch of the curve \(F^{-1}(0)\) near the point \((0,0)\):

![Sketch of the curve](image)

The presence of the singular point \((0,0)\) implies that \(F^{-1}(0)\) is not an embedded submanifold.

Now we consider the level set \(F^{-1}(\frac{1}{27})\).

This is the subset of \(\mathbb{R}^2\) described by the equation

\[x^3 + xy + y^3 = \frac{1}{27}\]

Since we are interested in studying what happens near the
critical point \((-\frac{1}{3}, -\frac{1}{3})\), we perform a change of variables (a translation) so that the critical point will be at the origin of a new coordinate system:

\[
\begin{align*}
\alpha &= X - \frac{4}{3} \\
y &= Y - \frac{4}{3}
\end{align*}
\]

With this change of variables the equation of the level set \(F^{-1}(\frac{4}{27})\) becomes:

\[X^3 + Y^3 - (X^2 + Y^2 - XY) = 0\]

As in the previous case, we recognize that the critical point \(\begin{cases} \alpha = -\frac{4}{3} \\ y = -\frac{4}{3} \end{cases}\) \(\Rightarrow\) \(X = 0\) is a singular point.

The tangent cone at the point \((X = 0, Y = 0)\) is given by the equation \(X^2 + Y^2 - XY = 0\).

This polynomial factors as follows:

\[X^2 + Y^2 - XY = \left[ X + \left(-\frac{1}{2} + \frac{i\sqrt{3}}{2}\right) Y \right] \left[ X + \left(-\frac{1}{2} - \frac{i\sqrt{3}}{2}\right) Y \right]\]

hence the tangent cone is the union of the two complex conjugate lines of equations

\[X + \left(-\frac{1}{2} + \frac{i\sqrt{3}}{2}\right) Y = 0 \quad ; \quad X + \left(-\frac{1}{2} - \frac{i\sqrt{3}}{2}\right) Y = 0\]

These lines meet at the unique real point \((X = 0, Y = 0)\).

Since in \(\mathbb{R}^2\) we only see points with real coordinates, the point \((X = 0, Y = 0)\) appears to be an isolated
(singular) point of the level set \( F^{-1}(\frac{1}{27}) \)

Here is a sketch of the level set \( F^{-1}(\frac{1}{27}) \) near the critical point:

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\[ \begin{array}{c}
\text{isolated critical point} \\
\text{complex tangent lines to } F^{-1}(\frac{1}{27}) \text{ at } \left( -\frac{1}{3}, -\frac{1}{3} \right) \\
\end{array} \]
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The presence of the singular point \( \left( -\frac{1}{3}, -\frac{1}{3} \right) \) implies that \( F^{-1}(\frac{1}{27}) \) is not an embedded submanifold.

Here is the graph of the function \( z = F(x, y) \)

The level sets \( F^{-1}(c) \) are obtained by intersecting the graph of \( z = F(x, y) \) with the plane \( z = c \).

Here are some level sets \( F^{-1}(c) \), for some different values of \( c \):