Sheet 2, Exercise 6

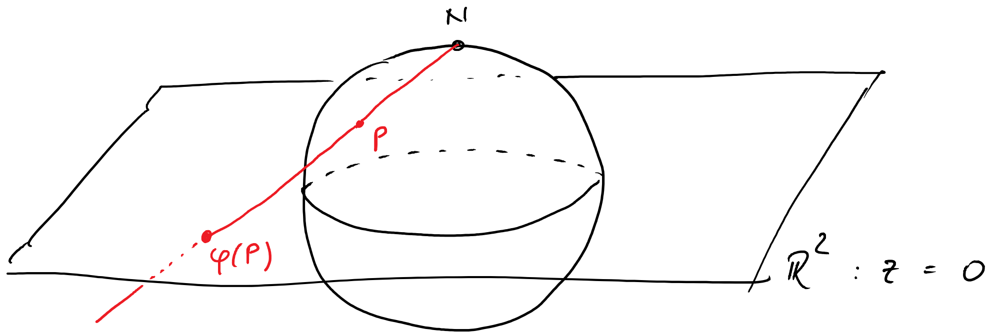
Show that there is a smooth vector field on S^2 that vanishes at exactly one point.

Proof Let $S^2 = \{ (x,y,z) \mid x^2+y^2+z^2 = 1 \} \subset \mathbb{R}^3$

Let $N = (0,0,1) \in S^2$ Let $U = S^2 - \{N\}$

Let $S = (0,0,-1) \in S^2$ Let $V = S^2 - \{S\}$

Let $\varphi : S^2 - \{N\} \xrightarrow{\sim} \mathbb{R}^2$ be the stereographic projection from N to the plane $\mathbb{R}^2$ given by $z = 0$



We shall use coordinates (x,y,z) on S^2 and (u,v) on $\mathbb{R}^2$. The local chart φ is given by:

$$\varphi : S^2 - \{N\} \xrightarrow{\sim} \mathbb{R}^2$$
$$(x,y,z) \longmapsto (u,v)$$

where: $\begin{cases} u = \dfrac{x}{1-z} \\ v = \dfrac{y}{1-z} \end{cases}$

The inverse map $\varphi^{-1} : \mathbb{R}^2 \xrightarrow{\sim} S^2 - \{N\}$ is given by

$$\begin{cases} x = \dfrac{2u}{1+u^2+v^2} \\ y = \dfrac{2v}{1+u^2+v^2} \end{cases}$$

$$\begin{cases} y = \dfrac{2v}{1+u^2+v^2} \\[2ex] z = \dfrac{u^2+v^2-1}{1+u^2+v^2} \end{cases}$$

Now let $\psi : S^2 - \{S\} \xrightarrow{\sim} \mathbb{R}^2$ be the stereographic projection from S to the same plane $\mathbb{R}^2 : z = 0$.
We shall call (α, β) the coordinates on this copy of $\mathbb{R}^2$.
The local chart $\psi : S^2 - \{S\} \xrightarrow{\sim} \mathbb{R}^2$ is given by:

$$(x, y, z) \longmapsto (\alpha, \beta)$$

where

$$\begin{cases} \alpha = \dfrac{x}{1+z} \\[2ex] \beta = \dfrac{y}{1+z} \end{cases}$$

The inverse map $\psi^{-1} : \mathbb{R}^2 \xrightarrow{\sim} S^2 - \{S\}$ is given by

$$\begin{cases} x = \dfrac{2\alpha}{1+\alpha^2+\beta^2} \\[2ex] y = \dfrac{2\beta}{1+\alpha^2+\beta^2} \\[2ex] z = \dfrac{1-\alpha^2-\beta^2}{1+\alpha^2+\beta^2} \end{cases}$$

On the intersection $U \cap V = S^2 - \{N, S\}$ we have the following commutative diagram:

$$\begin{tikzcd} & S^2 - \{N, S\} \arrow[dl, "\varphi"', "\sim"] \arrow[dr, "\psi", "\sim"'] & \\ {} & (x,y,z) & {} \end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{R}^2 \setminus \{(0,0)\} \arrow[r, "\sim", "\eta"'] & \mathbb{R}^2 \setminus \{(0,0)\} \\
(u,v) \arrow[r, mapsto] & (\alpha, \beta)
\end{tikzcd}$$

where $\eta = \psi \circ \varphi^{-1}$ is the "transition function".

Explicitly, η is given by $(u,v) \longmapsto (\alpha, \beta)$,

where
$$\begin{cases} \alpha = \dfrac{u}{u^2+v^2} \\[2ex] \beta = \dfrac{v}{u^2+v^2} \end{cases}$$

The inverse map $\eta^{-1} : (\alpha,\beta) \longmapsto (u,v)$ is given by
$$\begin{cases} u = \dfrac{\alpha}{\alpha^2+\beta^2} \\[2ex] v = \dfrac{\beta}{\alpha^2+\beta^2} \end{cases}$$

If $P \in S^2 - \{N, S\}$, the commutative diagram above induces a commutative diagram at the level of tangent spaces:

$$\begin{tikzcd}
 & T_P S^2 \arrow[dl, "d\varphi_P"', "\sim"] \arrow[dr, "d\psi_P", "\sim"'] & \\
\mathbb{R}^2 \cong T_{\varphi(P)} \mathbb{R}^2 \arrow[rr, "\sim", "d\eta_{\varphi(P)}"'] & & T_{\psi(P)} \mathbb{R}^2 \cong \mathbb{R}^2
\end{tikzcd}$$

$d\varphi_P$ is the linear map given by the Jacobian matrix of φ at the point P:

$$J\varphi_P = \begin{pmatrix} \dfrac{1}{1-z_P} & 0 & \dfrac{x_P}{(1-z_P)^2} \\[2ex] 0 & \dfrac{1}{1-z_P} & \dfrac{y_P}{(1-z_P)^2} \end{pmatrix}$$

$$\begin{pmatrix} 0 & \frac{1}{1-z_p} & \frac{y_p}{(1-z_p)^2} \end{pmatrix}$$

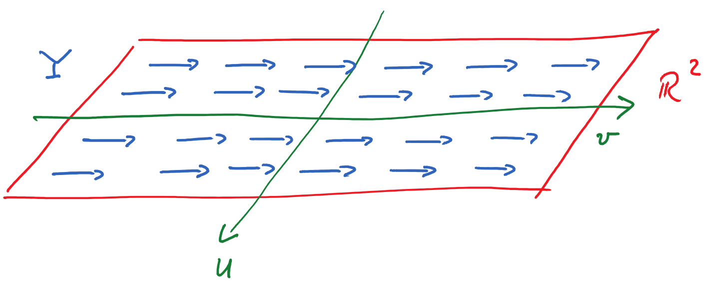
$d\psi_p$ is the linear map given by the Jacobian matrix of ψ at the point P:

$$J\psi_p = \begin{pmatrix} \frac{1}{1+z_p} & 0 & -\frac{x_p}{(1+z_p)^2} \\ 0 & \frac{1}{1+z_p} & -\frac{y_p}{(1+z_p)^2} \end{pmatrix}$$

$d\eta_{\varphi(P)}$ is the linear map given by the Jacobian matrix of η at the point $\varphi(P)$:

$$J\eta_{\varphi(P)} = \begin{pmatrix} \frac{v^2-u^2}{(u^2+v^2)^2} & -\frac{2uv}{(u^2+v^2)^2} \\ -\frac{2uv}{(u^2+v^2)^2} & \frac{u^2-v^2}{(u^2+v^2)^2} \end{pmatrix}$$

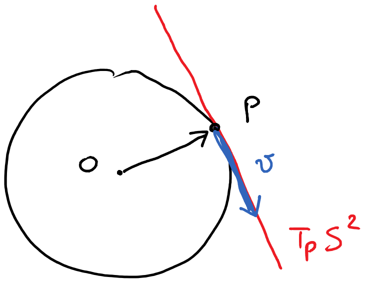
Now, the idea is the following: on $T_{\varphi(P)}\mathbb{R}^2 \cong \mathbb{R}^2$ $(\forall P \in S^2 - \{N\})$ take a constant tangent vector field. For example, we can take the constant vector field $Y(u,v) = (0,1)$, $\forall (u,v) \in \mathbb{R}^2$



Then we can use $(d\varphi_p)^{-1}$ to obtain, from Y, a tangent vector field to $S^2 - \{N\}$. This will be a smooth vector field on $S^2 - \{N\}$ that never vanishes. Finally we hope that this vector field on $S^2 - \{N\}$ can be extended to a smooth vector field on S^2, that vanishes exactly at the point N.

Let $P = (x, y, z) \in S^2$.

Since $S^2 \subset \mathbb{R}^3$ we can identify the tangent space $T_p S^2$ with a subspace of $T_p \mathbb{R}^3 \cong \mathbb{R}^3$.



$T_p S^2$ is the subspace of all vectors $v = (a, b, c)$ that are orthogonal to the vector $\vec{OP} = (x, y, z)$, hence

$$T_p S^2 = \{ (a, b, c) \in \mathbb{R}^3 \mid ax + by + cz = 0 \}$$

Let $P = (x, y, z) \in S^2 - \{N\}$ and let us consider the map

$$d\varphi_p : T_p S^2 \xrightarrow{\sim} T_{\varphi(P)} \mathbb{R}^2 = \mathbb{R}^2$$

Let us denote by X the tangent vector field to $S^2 \setminus \{N\}$, $X : P \longmapsto X_p \in T_p S^2$, that corresponds to the vector field $Y(u, v) = (0, 1)$ on $\mathbb{R}^2$, via the isomorphism $d\varphi_p$.

If we write $X_p = (a, b, c) \in T_p S^2$, we must have $d\varphi_p (X_p) = (0, 1)$.

By using the Jacobian matrix $J\varphi_p$, this means

$$\begin{pmatrix} \frac{1}{1-z} & 0 & \frac{x}{(1-z)^2} \\ 0 & \frac{1}{1-z} & \frac{y}{(1-z)^2} \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

which gives the system

$$\begin{cases} \frac{a}{1-z} + \frac{cx}{(1-z)^2} = 0 \\ \frac{b}{1-z} + \frac{cy}{(1-z)^2} = 1 \\ ax + by + cz = 0 \quad \text{because } (a,b,c) \in T_p S^2 \end{cases}$$

The solution is: $\begin{cases} a = -xy \\ b = 1 - z - y^2 \\ c = y(1-z) \end{cases}$

The vector field $X \in \mathfrak{T}(S^2 \setminus \{N\})$ is then given by

$$X = \left(-xy,\ 1 - z - y^2,\ y(1-z) \right)$$

We can immediately see that this vector field is actually defined also at the point $N = (0, 0, 1)$:

$$X(0,0,1) = (0, 0, 0)$$

In fact, X is a smooth vector field on the whole sphere, $X \in \mathfrak{T}(S^2)$, and it vanishes exactly at the point N.

Let's see what happens in the chart φ.

Let us consider the map

$$d\psi_p : T_p S^2 \xrightarrow{\sim} T_{\psi(P)} \mathbb{R}^2 \cong \mathbb{R}^2$$

and let us denote by $Z = Z(\alpha, \beta)$ the vector field on $\mathbb{R}^2$ that corresponds to X via $d\psi$.

By the commutativity of the diagram

$$\begin{tikzcd}
& T_p S^2 \arrow[dl, "d\varphi_p"', "\sim"] \arrow[dr, "d\psi_p", "\sim"'] & \\
\mathbb{R}^2 \cong T_{\varphi(P)} \mathbb{R}^2 \arrow[rr, "\sim", "d\eta_{\varphi(P)}"'] & & T_{\psi(P)} \mathbb{R}^2 \cong \mathbb{R}^2
\end{tikzcd}$$

the vector field $Z(\alpha, \beta)$ on $\mathbb{R}^2$ is the one that corresponds to the vector field $Y(u,v) = (0,1)$ on the other copy of $\mathbb{R}^2$, via the map $d\eta_{\varphi(P)}$

The components z_1, z_2 of $Z = (z_1, z_2)$ are obtained by multiplying the Jacobian matrix $J\eta_{\varphi(P)}$ with the vector $Y(u,v) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$:

$$\begin{pmatrix} \dfrac{v^2 - u^2}{(u^2+v^2)^2} & -\dfrac{2uv}{(u^2+v^2)^2} \\[2ex] -\dfrac{2uv}{(u^2+v^2)^2} & \dfrac{u^2-v^2}{(u^2+v^2)^2} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

Hence we find $z_1 = -\dfrac{2uv}{(u^2+v^2)^2}$; $z_2 = \dfrac{u^2 - v^2}{(u^2+v^2)^2}$

Since the vector field $Z = (z_1, z_2)$ lives on the copy of $\mathbb{R}^2$ with coordinates (α, β), we must rewrite z_1 and z_2 in terms of α, β.

To do this, we use the relations

$$u = \frac{\alpha}{\alpha^2+\beta^2} \quad , \quad v = \frac{\beta}{\alpha^2+\beta^2} \qquad \text{(see above)}$$

We find:

$$Z_1 = -\frac{2uv}{(u^2+v^2)^2} = -2\alpha\beta$$

$$Z_2 = \frac{u^2-v^2}{(u^2+v^2)^2} = \alpha^2-\beta^2$$

hence $Z(\alpha,\beta) = (-2\alpha\beta \,,\, \alpha^2-\beta^2)$

From this expression we see that Z is a C^∞ vector field, and $Z(0,0) = (0,0)$.

Remember that the point $(\alpha,\beta)=(0,0)$ is the one that corresponds to the point $N=(0,0,1) \in S^2$ in the local chart ψ.