# Sheet 2, Exercise 7

$\forall z \in S^{2n-1} \subset \mathbb{C}^n$, let $\sigma_z : \mathbb{R} \to S^{2n-1}$ be the curve $\sigma_z(t) = e^{it} z$. Prove that by setting $X(z) = \sigma_z'(0)$ we get a nowhere vanishing vector field $X \in \mathfrak{X}(S^{2n-1})$.

Proof. We shall use the identification $\mathbb{C}^n \cong \mathbb{R}^{2n}$ given by

$$\mathbb{C}^n \ni (z_1, \dots, z_n) \longleftrightarrow (x_1, y_1, x_2, y_2, \dots, x_n, y_n) \in \mathbb{R}^{2n}$$

where $\quad z_j = x_j + \sqrt{-1}\, y_j \qquad ( i = \sqrt{-1} )$

Note that, if $z = (z_1, \dots, z_n) \in \mathbb{C}^n$, we have

$$|z|^2 = |z_1|^2 + \dots + |z_n|^2 = |x_1|^2 + |y_1|^2 + \dots + |x_n|^2 + |y_n|^2$$

hence $S^{2n-1} \subset \mathbb{R}^{2n}$ can be described as

$$S^{2n-1} = \{ z \in \mathbb{C}^n \mid |z| = 1 \}.$$

Since $|e^{it}| = 1$, $\forall t \in \mathbb{R}$, if $z \in S^{2n-1}$ we have

$$|\sigma_z(t)| = |e^{it}| \cdot |z| = 1, \quad \text{hence } \sigma_z(t) \in S^{2n-1}$$

$\forall t \in \mathbb{R}$ and $\forall z \in S^{2n-1}$ (this means that $t \mapsto \sigma_z(t)$ is actually a smooth curve in $S^{2n-1}$).

We have: $X(z) = \sigma_z'(0) = i z e^{it}\Big|_{t=0} = iz = (iz_1, \dots, iz_n)$

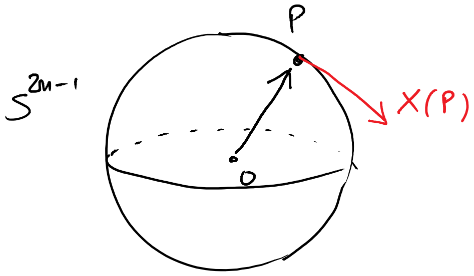
By writing everything in real coordinates $x_j, y_j$, we have: $\quad iz_j = i(x_j + iy_j) = -y_j + i x_j$

hence the vector field $X$ is given explicitly by:

$$X(x_1, y_1, \dots, x_n, y_n) = (-y_1, x_1, -y_2, x_2, \dots, -y_n, x_n)$$

$$\forall (x_1, y_1, \dots, x_m, y_m) \in S^{2m-1}$$

If we set $P = (x_1, y_1, \dots, x_m, y_m)$ the picture is as follows:




The scalar product of the vectors $\vec{OP}$ and $X(P)$ is:

$$\vec{OP} \cdot X(P) = (x_1, y_1, \dots, x_m, y_m) \cdot (-y_1, x_1, \dots, -y_m, x_m) = 0$$

hence the vector $X(P)$ is orthogonal to $\vec{OP}$, $\forall P \in S^{2m-1}$.

This means that the vector $X(P)$ is tangent to the sphere!

$$X(P) \in T_P S^{2m-1}, \quad \forall P \in S^{2m-1}$$

hence $X$ is a smooth tangent vector field to $S^{2m-1}$.

$X(P) = 0$ if and only if $P = (0, 0, \dots, 0)$, which is <u>not</u> a point on the sphere.

Hence $X$ is nowhere vanishing on $S^{2m-1}$.

<u>Remark</u> Note that in the previous construction it is essential that the dimension of the sphere is odd!

If the dimension of the sphere is <u>even</u>, say $\dim = 2m$ for some integer $m$, then it is possible to prove that there does not exist a smooth nowhere vanishing tangent vector field on $S^{2m}$.

This is known as the "hairy ball theorem".