Sheet 2, Exercise 8

Determine explicitly the flux of the following vector fields on $\mathbb{R}^2$.

(1) $X(x,y) = y \dfrac{\partial}{\partial x} + \dfrac{\partial}{\partial y}$

Let $P = (x_P, y_P) \in \mathbb{R}^2$ and let us denote by

$\sigma_P(t) = (x(t), y(t))$ the integral curve of the vector field X passing through P at $t = 0$.

The unknown functions $x(t)$ and $y(t)$ must satisfy the following system of differential equations:

$$\begin{cases} x'(t) = y(t) \\ y'(t) = 1 \\ x(0) = x_P \quad ; \quad y(0) = y_P \end{cases}$$

It is very easy to find the solution:

$$\begin{cases} x(t) = \dfrac{1}{2} t^2 + t y_P + x_P \\[2ex] y(t) = t + y_P \end{cases}$$

By definition, the <u>flux</u> of X is the function

$$\Theta : \mathbb{R} \times \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

$$(t, P = (x_P, y_P)) \longmapsto \sigma_P(t) = \left(\tfrac{1}{2} t^2 + t y_P + x_P , t + y_P \right)$$

We can now safely remove the subscript P, and write simply:

$$\Theta(t, x, y) = \left(\tfrac{1}{2} t^2 + t y + x , t + y \right)$$

(2) $X(x,y) = x\frac{\partial}{\partial x} + 3y\frac{\partial}{\partial y}$

By reasoning as before, we arrive at the following system of differential equations:

$$\begin{cases} x'(t) = x(t) \\ y'(t) = 3y(t) \\ x(0) = x_p \; ; \; y(0) = y_p \end{cases}$$

The solution is: $\begin{cases} x(t) = x_p e^{t} \\ y(t) = y_p e^{3t} \end{cases}$

The flux is given by

$$\Theta(t,x,y) = \left(x e^{t} \, , \, y e^{3t} \right)$$

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(3) $X(x,y) = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}$

In this case we find the system of diff. equations

$$\begin{cases} x'(t) = x(t) \\ y'(t) = -y(t) \\ x(0) = x_p \; ; \; y(0) = y_p \end{cases}$$

The solution is: $\begin{cases} x(t) = x_p e^{t} \\ y(t) = y_p e^{-t} \end{cases}$

The flux is given by

$$\Theta(t,x,y) = \left(x e^{t} \, , \, y e^{-t} \right)$$

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(4) $\quad X(x,y) = y \dfrac{\partial}{\partial x} + x \dfrac{\partial}{\partial y}$

In this case we find the system of diff. equations

$$\begin{cases} x'(t) = y(t) \\ y'(t) = x(t) \\ x(0) = x_p \; ; \; y(0) = y_p \end{cases}$$

Let us introduce vector notations :

$$v(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \; ; \; v'(t) = \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} \; ; \; A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Then the previous system of diff. equations can be written as :

$$\begin{cases} v'(t) = A\, v(t) \\ v(0) = \begin{pmatrix} x_p \\ y_p \end{pmatrix} \end{cases}$$

The solution is now obvious :

$$\underline{v(t) = e^{At} \cdot v(0)}$$

An easy computation shows that

$$e^{At} = \begin{pmatrix} \cosh(t) & \sinh(t) \\ \sinh(t) & \cosh(t) \end{pmatrix}$$

hence we have :

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} \cosh(t) & \sinh(t) \\ \sinh(t) & \cosh(t) \end{pmatrix} \begin{pmatrix} x_p \\ y_p \end{pmatrix}$$

We can rewrite this solution as follows :

$$\begin{cases} x(t) = x_p \cosh(t) + y_p \sinh(t) \\ y(t) = x_p \sinh(t) + y_p \cosh(t) \end{cases}$$

Finally, the flux is given by

$$\Theta(t, x, y) = \left(x \cosh(t) + y \sinh(t) \; , \; x \sinh(t) + y \cosh(t) \right)$$