Sheet 2, Exercise 8

Determine explicitly the flux of the following vector fields on $\mathbb{R}^2$.

(a) \( X(x, y) = y \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \)

Let \( P = (x_p, y_p) \in \mathbb{R}^2 \) and let us denote by \( \sigma^P(t) = (x(t), y(t)) \) the integral curve of the vector field \( X \) passing through \( P \) at \( t = 0 \).

The unknown functions \( x(t) \) and \( y(t) \) must satisfy the following system of differential equations:

\[
\begin{cases}
  x'(t) = y(t) \\
  y'(t) = 1 \\
  x(0) = x_p \quad ; \quad y(0) = y_p
\end{cases}
\]

It is very easy to find the solution:

\[
\begin{cases}
  x(t) = \frac{1}{2} t^2 + t y_p + x_p \\
  y(t) = t + y_p
\end{cases}
\]

By definition, the flux of \( X \) is the function

\( \Omega : \mathbb{R} \times \mathbb{R}^2 \longrightarrow \mathbb{R}^2 \)

\( (t, P=(x_p, y_p)) \mapsto \sigma^P(t) = \left( \frac{1}{2} t^2 + t y_p + x_p , t + y_p \right) \)

We can now safely remove the subscript \( P \), and write simply:

\( \Omega(t, x, y) = \left( \frac{1}{2} t^2 + t u + x , t + y \right) \)
(2) \( X(x, y) = x \frac{\partial}{\partial x} + 3y \frac{\partial}{\partial y} \)

By reasoning as before, we arrive at the following system of differential equations:

\[
\begin{align*}
\begin{cases}
x'(t) &= x(t) \\
y'(t) &= 3y(t) \\
x(0) &= x_p \\
y(0) &= y_p
\end{cases}
\end{align*}
\]

The solution is:

\[
\begin{align*}
x(t) &= x_p e^t \\
y(t) &= y_p e^{3t}
\end{align*}
\]

The flux is given by

\[\Phi(t, x, y) = (xe^t, ye^{3t})\]

(3) \( X(x, y) = x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y} \)

In this case we find the system of diff. equations

\[
\begin{align*}
\begin{cases}
x'(t) &= x(t) \\
y'(t) &= -y(t) \\
x(0) &= x_p \\
y(0) &= y_p
\end{cases}
\end{align*}
\]

The solution is:

\[
\begin{align*}
x(t) &= x_p e^t \\
y(t) &= y_p e^{-t}
\end{align*}
\]
The flux is given by

$$\mathcal{B}(t, x, y) = \left( x e^t, y e^{-t} \right)$$

(4) \[ x(t, y) = y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y} \]

In this case we find the system of diff. equations

\[
\begin{align*}
    x'(t) &= y(t) \\
    y'(t) &= x(t) \\
    x(0) &= x_p ; y(0) = y_p
\end{align*}
\]

Let us introduce vector notations:

\[
\mathbf{v}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} ; \quad \mathbf{v}'(t) = \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} \;
A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}
\]

Then the previous system of diff. equations can be written as:

\[
\begin{align*}
    \mathbf{v}'(t) &= A \mathbf{v}(t) \\
    \mathbf{v}(0) &= \begin{pmatrix} x_p \\ y_p \end{pmatrix}
\end{align*}
\]

The solution is now obvious:

\[
\mathbf{v}(t) = e^{At} \cdot \mathbf{v}(0)
\]

An easy computation shows that

\[
A t = e^{At} = \begin{pmatrix} \cosh(t) & \sinh(t) \\ \sinh(t) & \cosh(t) \end{pmatrix}
\]
hence we have:

\[
\begin{pmatrix}
  x(t) \\
  y(t)
\end{pmatrix} =
\begin{pmatrix}
  \cosh(t) & \sinh(t) \\
  \sinh(t) & \cosh(t)
\end{pmatrix}
\begin{pmatrix}
  \alpha_p \\
  \gamma_p
\end{pmatrix}
\]

we can rewrite this solution as follows:

\[
\begin{cases}
  x(t) = \alpha_p \cosh(t) + \gamma_p \sinh(t) \\
  y(t) = \alpha_p \sinh(t) + \gamma_p \cosh(t)
\end{cases}
\]

Finally, the flux is given by

\[
\Theta(t, x, y) = \begin{pmatrix}
  x \cosh(t) + y \sinh(t) \\
  x \sinh(t) + y \cosh(t)
\end{pmatrix}
\]