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Poisson Structures on Moduli Spaces of Framed Vector Bundles on Surfaces

By Francesco Bottacin of Padova

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Abstract. In this paper we prove that the moduli spaces of framed vector bundles over a surface X, satisfying certain conditions, admit a family of Poisson structures parametrized by the global sections of a certain line bundle on X. This generalizes to the case of framed vector bundles previous results obtained in [B2] for the moduli space of vector bundles over a Poisson surface. As a corollary of this result we prove that the moduli spaces of framed $\mathrm{SU}(r)$ – instantons on $S^4 = \mathbb{R}^4 \cup \{\infty\}$ admit a natural holomorphic symplectic structure.

1. Introduction

Let X be a non – singular projective surface defined over $\mathbb{C}$ and let $D \subset X$ be a reduced and irreducible curve. We shall fix a vector bundle F over D and consider vector bundles E over X such that $E\big|_D \cong F$.

A pair (E, ϕ) consisting of a vector bundle E over X and an isomorphism $\phi : E\big|_D \xrightarrow{\sim} F$ is called a framed vector bundle. More generally one can define a framed module to be a pair $(\mathcal{E}, \phi)$, where $\mathcal{E}$ is a coherent sheaf on a projective variety X and $\phi : \mathcal{E} \to \mathcal{F}$ is only a homomorphism of $\mathcal{O}_X$ – modules, for a fixed coherent sheaf $\mathcal{F}$.

Moduli spaces of framed vector bundles, or framed modules, were considered by various authors (see, e. g., [D], [HL1], [L]), from different points of view. In particular we would like to recall here their connection to the study of $\mathrm{SU}(r)$ – instantons on $S^4 = \mathbb{R}^4 \cup \{\infty\}$, discovered by S. Donaldson in [D]: there is a canonical identification between the moduli spaces of framed vector bundles on $\mathbf{P}^2$ (vector bundles whose restriction to a line is trivial), and the moduli spaces of framed $\mathrm{SU}(r)$ – instantons on S^4.

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In this paper we shall prove that, if the surface X and the divisor D are such that $H^0\big(X,\omega_X^{-1}(-2D)\big) \neq 0$, the choice of a non-zero section $\zeta \in H^0\big(X,\omega_X^{-1}(-2D)\big)$ determines, in a canonical way, a Poisson structure θ_ζ on the moduli space $\mathcal{FB}$ of framed vector bundles on X. If $X = \mathbf{P}^2$ and D is a line, there is a natural choice of the section $\zeta \in H^0\big(\mathbf{P}^2,\mathcal{O}(1)\big)$, namely we can choose ζ to be the section defining the divisor D (ζ is determined up to a scalar multiple). It turns out that, in this situation, the Poisson structure θ_ζ on $\mathcal{FB}$ is everywhere non-degenerate, hence it is equivalent to a symplectic structure. By recalling Donaldson's result, we have thus proved the existence of a canonical holomorphic symplectic structure on the moduli space of framed $\mathrm{SU}(r)$-instantons on S^4.

This paper is organized as follows. In Section 2 we recall some basic results on moduli spaces of framed modules on a smooth projective variety. Then we specialize to the case of framed vector bundles on a smooth complex projective surface X.

In Section 3 we recall some facts about symplectic and Poisson structures that we shall need later, then, in Section 4, we construct a family of Poisson structures θ_ζ on the moduli space $\mathcal{FB}$, parametrized by the global sections ζ of the line bundle $\omega_X^{-1}(-2D)$ on X.

Finally, in Section 5, we apply our results to the case $X = \mathbf{P}^2$, to prove that the moduli space of framed $\mathrm{SU}(r)$-instantons on S^4 admits a canonical holomorphic symplectic structure.

2. Moduli spaces of framed vector bundles

Moduli spaces of framed vector bundles were constructed and studied, using different techniques, by various authors (see, e. g., [HL1], [L]). The most general construction is probably the one given in [HL1], which we now briefly recall.

Let X be a non-singular complex projective algebraic variety of dimension d (more generally we can assume that X is defined over an algebraically closed field k of characteristic 0), and let H be an ample divisor on X. For a coherent $\mathcal{O}_X$-module $\mathcal{E}$ we will set $\mathcal{E}(n) = \mathcal{E} \otimes \mathcal{O}_X(nH)$, and denote by $P_{\mathcal{E}}(n) = \chi(\mathcal{E}(n))$ the Hilbert polynomial of $\mathcal{E}$.

Let $\mathcal{F}$ be a fixed coherent $\mathcal{O}_X$-module and $\delta(n) \in \mathbb{Q}[n]$ a polynomial with positive leading coefficient, of degree less than $\dim X$.

Definition 2.1. A *framed module* is a pair $(\mathcal{E},\phi)$, where $\mathcal{E}$ is a coherent $\mathcal{O}_X$-module and $\phi : \mathcal{E} \to \mathcal{F}$ is a homomorphism of $\mathcal{O}_X$-modules. The homomorphism ϕ is called the *framing*.

As usual, in order to construct moduli spaces of framed modules, we need a suitable notion of stability. This notion will depend on the polynomial $\delta(n)$.

First of all, for a framed module $(\mathcal{E},\phi)$ we define $\epsilon(\phi)$ as follows:

$$\epsilon(\phi) = \begin{cases} 1 & \text{if} \quad \phi \neq 0\,, \\ 0 & \text{if} \quad \phi = 0\,. \end{cases}$$

Then we define the Hilbert polynomial of a framed module $(\mathcal{E},\phi)$ as follows:

$$P_{(\mathcal{E},\phi)}(n) \;=\; P_{\mathcal{E}}(n) - \epsilon(\phi)\delta(n)\,.$$

If $\mathrm{rk}(\mathcal{E}) \neq 0$ we also introduce, for convenience of notation, the normalized Hilbert polynomials of $\mathcal{E}$ and $(\mathcal{E},\phi)$:

$$p_{\mathcal{E}}(n) \;=\; \frac{P_{\mathcal{E}}(n)}{\mathrm{rk}(\mathcal{E})}\,, \quad p_{(\mathcal{E},\phi)}(n) \;=\; \frac{P_{(\mathcal{E},\phi)}(n)}{\mathrm{rk}(\mathcal{E})}\,.$$

If $(\mathcal{E},\phi)$ is a framed module and $\mathcal{E}'$ is a submodule of $\mathcal{E}$, then $\phi' = \phi\big|_{\mathcal{E}'}$ determines a natural framing on $\mathcal{E}'$.

We can now state the following definition:

Definition 2.2. A framed module $(\mathcal{E},\phi)$ is $\delta-$*stable* (resp. $\delta-$*semistable)* if, for every proper submodule $\mathcal{E}'$, with induced framing ϕ', we have $\mathrm{rk}(\mathcal{E})P_{(\mathcal{E}',\phi')}(n) < \mathrm{rk}(\mathcal{E}')P_{(\mathcal{E},\phi)}(n)$ (resp. $\leq$), for $n \gg 0$.

As for $\mathcal{O}_X-$modules, one can define the Jordan$-$Hölder filtration of a $\delta-$semistable framed module. This leads to the notion of $S-$equivalence of $\delta-$semistable framed modules.

The main result proved in [HL1] is the following:

Theorem 2.3. *There exists a projective scheme $\mathcal{FM}^{ss}_{\delta}(\mathcal{F},P)$ which is a coarse moduli space for isomorphism classes of $\delta-$semistable framed modules with Hilbert polynomial P. It has an open subscheme $\mathcal{FM}^{s}_{\delta}(\mathcal{F},P)$, which is a fine moduli space for isomorphism classes of $\delta-$stable framed modules. A closed point in $\mathcal{FM}^{ss}_{\delta}(\mathcal{F},P)$ represents an $S-$equivalence class of $\delta-$semistable framed modules.*

Remark 2.4. Note that the preceding theorem asserts, in particular, that over the moduli space of $\delta-$stable framed modules there always exists a universal family, i. e., a coherent sheaf $\mathcal{E}_{\mathcal{FM}}$ of $\mathcal{O}_{\mathcal{FM}^s_\delta(\mathcal{F},P)\times X}-$modules, flat over $\mathcal{FM}^s_\delta(\mathcal{F},P)$, together with a homomorphism $\Phi : \mathcal{E}_{\mathcal{FM}} \to \mathcal{F}\otimes\mathcal{O}_{\mathcal{FM}^s_\delta(\mathcal{F},P)}$, such that the restriction of $\mathcal{E}_{\mathcal{FM}}$ and Φ to $\{(\mathcal{E},\phi)\}\times X$ is isomorphic to the framed module $(\mathcal{E},\phi)$, for any $(\mathcal{E},\phi)\in\mathcal{FM}^s_\delta(\mathcal{F},P)$. This is a special property of moduli spaces of stable framed modules. In fact, in the usual case of moduli spaces of stable sheaves, a universal family does not exist in general, not even on any Zariski open subset of the moduli space. It only exists locally in the complex, or étale, topology.

In this paper we shall study moduli spaces of framed vector bundles on a surface, hence, from now on, X will denote a non$-$singular complex projective surface with an ample divisor H on it. We also fix an effective divisor D on X such that $D = \sum_{i=1}^n C_i$, where C_i are reduced and irreducible curves, and a locally free sheaf F on D. We shall simply write F instead of j_*F (where $j : D \hookrightarrow X$ is the inclusion map) whenever we consider F as a sheaf over X.

We recall that a locally free sheaf E is said to be stable (resp. semistable) if, for every proper coherent subsheaf E' of E, we have $p_{E'}(n) < p_E(n)$, (resp. $\leq$), for $n \gg 0$.

We state the following definition:

Definition 2.5. A *framed vector bundle* on X is a framed module (E,ϕ) such that E is a locally free sheaf and $\phi : E \to F$ is the composition of the natural restriction map $\rho : E \to E\big|_D$ with an isomorphism $\bar{\phi} : E\big|_D \xrightarrow{\sim} F$.

The following result, whose proof follows directly from the definitions by a standard computation using the theorem of Riemann–Roch, relates the stability of a framed vector bundle (E,ϕ) to the stability of E.

Lemma 2.6. *Let us fix the rank r and the first Chern class $c_1 \in H^2(X,\mathbb{Z})$, and set $\bar{\delta}(n) = (D\cdot H)rn + c_1\cdot D - \frac{1}{2}D\cdot(D+K_X)$. Let (E,ϕ) be a framed vector bundle on X, of rank r and first Chern class c_1. Then, if E is semistable (resp. stable), (E,ϕ) is δ–semistable (resp. δ–stable), for all $\delta(n) \in \mathbb{Q}[n]$ (with positive leading coefficient) such that $\delta(n) \le \bar{\delta}(n)$ for $n \gg 0$.*

From now on we shall fix an integer $r \ge 2$ and Chern classes $c_1 \in H^2(X,\mathbb{Z})$, $c_2 \in H^4(X,\mathbb{Z}) \cong \mathbb{Z}$, and set

$$\delta(n) \;=\; (D\cdot H)rn + c_1\cdot D - \frac{1}{2}\,D\cdot(D+K_X)\,.$$

We shall denote by $\mathcal{FB} = \mathcal{FB}_\delta(r,c_1,c_2,D,F)$ the moduli space of δ–stable framed vector bundles and by $\mathcal{FB}^o$ the open subset of $\mathcal{FB}$ parametrizing framed vector bundles (E,ϕ) such that E is stable. All these varieties are quasi–projective.

By using standard techniques of infinitesimal deformation theory, or by specializing to this particular situation the general results of [HL1], it is not difficult to prove the following result:

Proposition 2.7. *For any $(E,\phi)\in\mathcal{FB}$, the tangent space $T_{(E,\phi)}\mathcal{FB}$ is canonically identified to $H^1(X,\mathcal{E}nd(E)\otimes\mathcal{O}_X(-D))$ and the obstruction to the smoothness of the moduli space $\mathcal{FB}$ at (E,ϕ) lies in $H^2(X,\mathcal{E}nd(E)\otimes\mathcal{O}_X(-D))$.*

Then, by Serre duality, we have:

Corollary 2.8. *For any $(E,\phi)\in\mathcal{FB}$, there is a canonical isomorphism $T^*_{(E,\phi)}\mathcal{FB} \cong H^1\big(X,\mathcal{E}nd(E)\otimes\omega_X(D)\big)$.*

In the sequel we shall also need the following result, whose proof follows from general properties of stable framed bundles (cf. [HL1]):

Lemma 2.9. *If (E,ϕ) is a δ–stable framed vector bundle, then*

$$H^0\big(X,\mathcal{E}nd(E)\otimes\mathcal{O}_X(-D)\big) \;=\; 0\,.$$

Remark 2.10. When the moduli space $\mathcal{FB}$ is non–singular the preceding results can be globalized as follows. First of all let us recall that there exists a universal framed vector bundle $(\mathcal{E}_{\mathcal{FB}}, \Phi)$ on $\mathcal{FB} \times X$, with the property that $(\mathcal{E}_{\mathcal{FB}}, \Phi)\big|_{\{(E,\phi)\}\times X} \cong (E, \phi)$, for any $(E, \phi) \in \mathcal{FB}$.

Let us denote by $p : \mathcal{FB} \times X \to \mathcal{FB}$ and $q : \mathcal{FB} \times X \to X$ the canonical projections. Then there are canonical isomorphisms

$$T\mathcal{FB} \;\cong\; R^1 p_*\big(\mathcal{E}nd(\mathcal{E}_{\mathcal{FB}}) \otimes q^*\mathcal{O}_X(-D)\big)\,, \tag{2.1}$$

and

$$T^*\mathcal{FB} \cong R^1 p_*\big(\mathcal{E}nd(\mathcal{E}_{\mathcal{FB}}) \otimes q^*\omega_X(D)\big)\,. \tag{2.2}$$

3. Poisson structures

For the convenience of the reader we recall here some basic definitions of symplectic geometry.

Definition 3.1. A *Poisson structure* on a non−singular complex variety X is a Lie algebra structure $\{\cdot,\cdot\}$ on the sheaf of regular functions $\mathcal{O}_X$ which is a derivation in each entry, i. e., satisfies $\{f, gh\} = \{f, g\}h + g\{f, h\}$.

It is easy to see that a Poisson structure on X is given equivalently by an antisymmetric contravariant 2−tensor $\theta \in H^0\big(X, \wedge^2 TX\big)$, by setting

$$\{f, g\} \;=\; \langle \theta, df \wedge dg\rangle\,. \tag{3.1}$$

The bracket defined in this way satisfies all the required properties, except for the Jacobi identity:

$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} \;=\; 0\,.$$

Note that giving $\theta \in H^0\big(X, \wedge^2 TX\big)$ is equivalent to giving a homomorphism of vector bundles $B_\theta : T^*X \to TX$, defined by $\langle \theta, \alpha \wedge \beta\rangle = \langle B_\theta(\alpha), \beta\rangle$, for 1−forms α and β.

Let us define now an operator $\tilde{d} : H^0\big(X, \wedge^2 TX\big) \to H^0\big(X, \wedge^3 TX\big)$ by setting

$$\begin{aligned} \tilde{d}\theta(\alpha, \beta, \gamma) \;=\; & B_\theta(\alpha)\theta(\beta, \gamma) - B_\theta(\beta)\theta(\alpha, \gamma) + B_\theta(\gamma)\theta(\alpha, \beta) \\ & - \langle [B_\theta(\alpha), B_\theta(\beta)], \gamma\rangle + \langle [B_\theta(\alpha), B_\theta(\gamma)], \beta\rangle - \langle [B_\theta(\beta), B_\theta(\gamma)], \alpha\rangle\,, \end{aligned} \tag{3.2}$$

for 1−forms α, β, γ, where $[\,\cdot,\cdot\,]$ denotes the usual commutator of vector fields. We have the following result, whose proof consists in a straightforward computation using local coordinates.

Proposition 3.2. *The bracket* $\{\cdot,\cdot\}$ *defined by an element* $\theta \in H^0\big(X, \wedge^2 TX\big)$ *as in* (3.1) *is a Poisson structure, i. e., satisfies the Jacobi identity, if and only if* $\tilde{d}\theta = 0$*.*

Remark 3.3. When θ has maximal rank everywhere, i. e., when $B_\theta : T^*X \to TX$ is an isomorphism, to give θ is equivalent to giving its inverse 2−form $\omega \in \Omega^2_X$, which corresponds to the inverse isomorphism $B_\theta^{-1} : TX \to T^*X$. It is easy to check that, in this situation, the condition $\tilde{d}\theta = 0$ is equivalent to $d\omega = 0$, i. e., to the closure of the 2 − form ω. In this case we say that ω defines a symplectic structure on X, or simply, that the Poisson structure is symplectic. Note that a necessary condition for the existence of a symplectic structure on X is that the dimension of X be even.

If X is a smooth surface, a Poisson structure on X is given by a section of $\wedge^2 TX \cong \omega_X^{-1}$, i. e., by an element $\theta \in H^0\big(X, \omega_X^{-1}\big)$. In this case the "closure" condition $\tilde{d}\theta = 0$ is trivially satisfied, since $\wedge^3 TX = 0$.

4. Poisson structures on moduli spaces

We have proved in [B2] that the choice of a Poisson structure on a surface X naturally determines a Poisson structure on the moduli space $\mathcal{M}$ of stable vector bundles on X. We shall see now how this result generalizes to the moduli space of framed vector bundles.

Let X be a non − singular complex connected projective surface and D an effective divisor on X such that $D = \sum_{i=1}^{n} C_i$, where C_i are reduced and irreducible curves. We shall assume that $H^0\big(X, \omega_X^{-1}(-2D)\big) \neq 0$, and choose a non − zero section $\zeta \in H^0\big(X, \omega_X^{-1}(-2D)\big)$. Let us denote by D_ζ the divisor defined by ζ.

Note that, since D is effective, the sheaf $\omega_X^{-1}(-2D)$ injects in ω_X^{-1}, hence the section ζ defines a Poisson structure on X (which is never symplectic); X is then naturally a Poisson surface.

Example 4.1. Let us give some examples of such surfaces X and divisors D.

(i) Let $X = \mathbf{P}^2$ and D be a line. Then $\omega_X^{-1}(-2D) \cong \mathcal{O}_X(1)$. If we fix homogeneous coordinates (x_0, x_1, x_2) on $\mathbf{P}^2$ such that the divisor D is defined by $x_0 = 0$, the natural injection map $\omega_X^{-1}(-2D) \hookrightarrow \omega_X^{-1}$ is identified to the map $\mathcal{O}_X(1) \to \mathcal{O}_X(3)$ determined by the multiplication by x_0^2. The choice of a section $\zeta \in H^0\big(X, \mathcal{O}_X(1)\big)$ determines a Poisson structure on X corresponding to the section $x_0^2\zeta$. Note however that in this situation there is an obvious natural choice for the section ζ; namely we can choose $\zeta = x_0$. The resulting Poisson structure on $\mathbf{P}^2$ is then given by x_0^3. This Poisson structure induces on $\mathbb{C}^2 = \mathbf{P}^2 \setminus D$ the usual symplectic structure given by the 2 − form $dX \wedge dY$, where $X = x_1/x_0$ and $Y = x_2/x_0$.

(ii) Let $X = \mathbf{P}^1 \times \mathbf{P}^1$. We have $\omega_X = p_1^*(\mathcal{O}_{\mathbf{P}^1}(-2)) \otimes p_2^*(\mathcal{O}_{\mathbf{P}^1}(-2))$, where p_1 and p_2 are the canonical projections from the product $\mathbf{P}^1 \times \mathbf{P}^1$ to its factors. The canonical divisor K_X has type $(-2, -2)$, hence $-K_X$ is linearly equivalent to $2D$, where D is a divisor of type $(1, 1)$. In this case there is only one non − zero section $\zeta \in H^0\big(X, \omega_X^{-1}(-2D)\big)$, up to scalars.

(iii) Let $X = \mathbf{P}^1 \times C$, where C is an elliptic curve. Then $\omega_X \cong p_1^*(\mathcal{O}_{\mathbf{P}^1}(-2))$, hence $-K_X$ is linearly equivalent to twice a divisor D corresponding to the line bundle $p_1^*(\mathcal{O}_{\mathbf{P}^1}(1))$. Again there is only one non − zero section $\zeta \in H^0\big(X, \omega_X^{-1}(-2D)\big)$, up to scalars.

In [B2] we have proved that the moduli space $\mathcal{M}$ of stable vector bundles on a Poisson surface is always non − singular. For the special class of surfaces we are considering here, the same result holds also for the moduli space of framed vector bundles.

Proposition 4.2. *Let X be a non – singular projective surface and D be an effective divisor as above. Then the moduli space $\mathcal{FB}$ of δ – stable framed vector bundles on X is a non – singular quasi – projective variety of dimension* $2rc_2 + (1-r)c_1^2 - \frac{1}{2} r^2 D \cdot (D + K_X) - r^2 \chi(\mathcal{O}_X)$.

The proof of this result is easy and is left to the reader: the smoothness of the moduli space follows from the vanishing of the obstruction group $H^2\big(X, \mathcal{E}nd(E) \otimes \mathcal{O}_X(-D)\big)$, and the dimension can be computed using the theorem of Riemann – Roch.

Now, for any non – zero section $\zeta \in H^0\big(X, \omega_X^{-1}(-2D)\big)$, we define the following bilinear map:

$$\begin{aligned} \theta_\zeta(E,\phi) : H^1\big(X, \mathcal{E}nd(E) \otimes \omega_X(D)\big) \times H^1\big(X, \mathcal{E}nd(E) \otimes \omega_X(D)\big) \\ \xrightarrow{\circ} H^2\big(X, \mathcal{E}nd(E) \otimes \omega_X^2(2D)\big) \xrightarrow{\zeta} H^2\big(X, \mathcal{E}nd(E) \otimes \omega_X\big) \xrightarrow{\mathrm{tr}} H^2\big(X, \omega_X\big) \cong \mathbb{C}\,, \end{aligned}$$

where the first map is the composition map followed by the cup – product of two cohomology classes, the second is induced by the multiplication by ζ and the third is induced by the trace map.

By recalling the identifications of the tangent and cotangent spaces to the moduli space $\mathcal{FB}$ in terms of cohomology groups, we see that we have thus defined a bilinear map

$$\theta_\zeta(E,\phi) \ : \ T^*_{(E,\phi)}\mathcal{FB} \times T^*_{(E,\phi)}\mathcal{FB} \ \longrightarrow \ \mathbb{C}\ .$$

Since $\mathcal{FB}$ is non – singular, the maps $\theta_\zeta(E,\phi)$ determine a global section

$$\theta_\zeta \in H^0\big(\mathcal{FB}, \otimes^2 T\mathcal{FB}\big)\,.$$

We recall that giving $\theta_\zeta \in H^0\big(\mathcal{FB}, \otimes^2 T\mathcal{FB}\big)$ is equivalent to giving a homomorphism of vector bundles

$$B_\zeta \ : \ T^*\mathcal{FB} \ \longrightarrow \ T\mathcal{FB}\,,$$

defined by setting $\theta_\zeta(\alpha \otimes \beta) = \langle B_\zeta(\alpha), \beta\rangle$, for 1 – forms α and β.

It is easy to see that the homomorphism B_ζ corresponding to the section θ_ζ defined above is induced by the homomorphism $\omega_X(D) \to \mathcal{O}_X(-D)$ determined by the multiplication by ζ. Precisely, on the fibers over a point (E,ϕ), we have:

$$B_\zeta(E,\phi) \ : \ H^1\big(X, \mathcal{E}nd(E) \otimes \omega_X(D)\big) \ \xrightarrow{\ \zeta\ } \ H^1\big(X, \mathcal{E}nd(E) \otimes \mathcal{O}_X(-D)\big)\,.$$

Since $\theta_\zeta(E,\phi)$ is essentially given by the cup – product of two cohomology classes, followed by the multiplication by ζ, it follows from the graded commutativity property of the cup – product (see, e. g., [GH, p. 707] or [HL2, p. 217]) that θ_ζ is skew – symmetric, i. e., $\theta_\zeta \in H^0\big(\mathcal{FB}, \wedge^2 T\mathcal{FB}\big)$. This is our candidate to define a Poisson structure on the moduli space $\mathcal{FB}$. The only thing it remains to prove is that θ_ζ satisfies the "closure" condition $\tilde{d}\theta_\zeta = 0$. This can be proved by adapting to the present situation the proof given in [B2, Section 5] for the moduli space of stable vector bundles. We shall describe here the relevant modifications.

First of all let us recall some preliminary results. Let $\pi : X \to Y$ be a morphism (locally of finite presentation) of schemes, and F, G two locally free sheaves on X.

We denote by $\mathcal{D}\mathit{iff}^{\,1}_{X/Y}(F,G)$ the sheaf of relative differential operators from F to G of order less than or equal to 1. We have the following short exact sequence

$$0 \longrightarrow \mathcal{H}om_X(F,G) \longrightarrow \mathcal{D}\mathit{iff}^{\,1}_{X/Y}(F,G) \xrightarrow{\sigma} \mathcal{D}er_Y(\mathcal{O}_X)\otimes\mathcal{H}om_X(F,G) \longrightarrow 0\,,$$

where σ is the symbol morphism. From this, if $F=G$ and we restrict to differential operators with scalar symbol, written $\mathcal{D}^1_{X/Y}(F)$, we obtain the exact sequence

$$0 \longrightarrow \mathcal{E}nd_X(F) \longrightarrow \mathcal{D}^1_{X/Y}(F) \xrightarrow{\sigma} \mathcal{D}er_Y(\mathcal{O}_X) \longrightarrow 0\,.$$

Let $p:\mathcal{FB}\times X\to\mathcal{FB}$ and $q:\mathcal{FB}\times X\to X$ denote the canonical projections. By applying the preceding results to the map $q:\mathcal{FB}\times X\to X$, with F equal to the universal vector bundle $\mathcal{E}_{\mathcal{FB}}$, we obtain the following exact sequence

$$0 \longrightarrow \mathcal{E}nd(\mathcal{E}_{\mathcal{FB}}) \longrightarrow \mathcal{D}^1_X(\mathcal{E}_{\mathcal{FB}}) \longrightarrow p^*T\mathcal{FB} \longrightarrow 0\,,$$

where $\mathcal{D}^1_X(\mathcal{E}_{\mathcal{FB}})=\mathcal{D}^1_{\mathcal{FB}\times X/X}(\mathcal{E}_{\mathcal{FB}})$ denotes the sheaf of first – order differential operators on $\mathcal{E}_{\mathcal{FB}}$ with scalar symbol, that are $q^*\mathcal{O}_X$ – linear.

By applying p_*, and noting that $p_*p^*T\mathcal{FB}\cong T\mathcal{FB}$ since p is a proper morphism, we get a long exact sequence, a piece of which is

$$\cdots \longrightarrow T\mathcal{FB} \longrightarrow R^1p_*\big(\mathcal{E}nd\big(\mathcal{E}_{\mathcal{FB}}\big)\big) \longrightarrow R^1p_*\big(\mathcal{D}^1_X\big(\mathcal{E}_{\mathcal{FB}}\big)\big) \longrightarrow \cdots\,. \tag{4.1}$$

By recalling the isomorphism (2.1), it is easy to prove that the map

$$T\mathcal{FB} \longrightarrow R^1p_*\big(\mathcal{E}nd\big(\mathcal{E}_{\mathcal{FB}}\big)\big)$$

coincides with the map

$$R^1p_*\big(\mathcal{E}nd\big(\mathcal{E}_{\mathcal{FB}}\big)\otimes q^*\mathcal{O}_X(-D)\big) \longrightarrow R^1p_*\big(\mathcal{E}nd\big(\mathcal{E}_{\mathcal{FB}}\big)\big)$$

induced by the natural inclusion $\mathcal{O}_X(-D)\hookrightarrow\mathcal{O}_X$. From the preceding exact sequence it follows that the image of a global section $\{\eta_{ij}\}$ of $R^1p_*\big(\mathcal{E}nd\big(\mathcal{E}_{\mathcal{FB}}\big)\otimes q^*\mathcal{O}_X(-D)\big)$ in $R^1p_*\big(\mathcal{D}^1_X\big(\mathcal{E}_{\mathcal{FB}}\big)\big)$ is zero, hence there exist sections $\dot D_i$ of $\mathcal{D}^1_X\big(\mathcal{E}_{\mathcal{FB}}\big)$ over suitable open subsets U_i, such that, on $U_i\cap U_j$, we have

$$\eta_{ij} \;=\; \dot D_j-\dot D_i\,.$$

Note that this equation is formally the same as Equation (5.2) of [B2]. From now on all the subsequent discussion carried out in [B2, Section 5] can be repeated, almost literally, for the moduli space $\mathcal{FB}$. The proof of the closure condition $\tilde d\theta_\zeta=0$ is now practically identical to the proof of [B2, Theorem 5.1].

We have thus proved the following result:

Theorem 4.3. *For any non – zero section $\zeta\in H^0\big(X,\omega_X^{-1}(-2D)\big)$, the antisymmetric contravariant 2 – tensor $\theta_\zeta\in H^0\big(\mathcal{FB},\wedge^2T\mathcal{FB}\big)$ defines a Poisson structure on the moduli space $\mathcal{FB}$.*

Remark 4.4. Let X, D and $\zeta\in H^0\big(X,\omega_X^{-1}(-2D)\big)$ be as above. We have already observed, in the beginning of Section 4, that ζ defines a Poisson structure on X, hence, by the results of [B2], it also defines a Poisson structure on the moduli space $\mathcal{M}$ of stable vector bundles on X. If we restrict to the open subscheme $\mathcal{FB}^o$ of $\mathcal{FB}$, and consider the natural projection map $\pi : \mathcal{FB}^o \to \mathcal{M}$ sending a framed vector bundle (E, ϕ) to E, it is easy to prove that π is a Poisson morphism, i. e., it is compatible with the Poisson structures of $\mathcal{FB}^o$ and $\mathcal{M}$ determined by ζ.

Remark 4.5. Let D_ζ be the divisor defined by the section $\zeta \in H^0\big(X, \omega_X^{-1}(-2D)\big)$, and let us consider the following exact sequence:

$$0 \longrightarrow \mathcal{E}nd(E) \otimes \omega_X(D) \xrightarrow{\zeta} \mathcal{E}nd(E) \otimes \mathcal{O}_X(-D) \longrightarrow \mathcal{E}nd(E) \otimes \mathcal{O}_X(-D)\big|_{D_\zeta} \longrightarrow 0\,.$$

By recalling that $H^0\big(X, \mathcal{E}nd(E) \otimes \mathcal{O}_X(-D)\big) = 0$ (cf. Lemma 2.9), from the long exact sequence of cohomology groups we obtain

$$\begin{aligned} 0 \longrightarrow H^0\Big(D_\zeta, \mathcal{E}nd(E) \otimes \mathcal{O}_X(-D)\big|_{D_\zeta}\Big) \longrightarrow \\ H^1\big(X, \mathcal{E}nd(E) \otimes \omega_X(D)\big) \xrightarrow{B_\zeta} H^1\big(X, \mathcal{E}nd(E) \otimes \mathcal{O}_X(-D)\big)\,, \end{aligned} \tag{4.2}$$

which shows that $\ker\big(B_\zeta(E, \phi)\big) = H^0\Big(D_\zeta, \mathcal{E}nd(E) \otimes \mathcal{O}_X(-D)\big|_{D_\zeta}\Big)$. This gives information about the rank of the Poisson structure of the moduli space $\mathcal{FB}$ at the point (E, ϕ), i. e., the dimension of the symplectic leaf passing through (E, ϕ).

5. Moduli spaces of instantons

In this section we specialize our preceding constructions to the case of framed vector bundles on the complex projective plane. We shall see that our results imply that the moduli space of framed $\mathrm{SU}(r)$ − instantons on $S^4 = \mathbb{R}^4 \cup \{\infty\}$ has a canonical holomorphic symplectic structure.

Let $X = \mathbf{P}^2$ and D be a line. We take as F a trivial vector bundle on D, $F = \mathcal{O}_D^{\oplus r}$. We also denote by $\sigma \in H^0(X, \mathcal{O}(1))$ a section defining the divisor D.

Note that if E is a vector bundle on $\mathbf{P}^2$ whose restriction to a line is trivial, then $c_1(E) = 0$. Hence, in this section, we shall consider the moduli space $\mathcal{FB} = \mathcal{FB}\big(r, 0, c_2, D, \mathcal{O}_D^{\oplus r}\big)$ of stable rank − r framed vector bundles with $c_1 = 0$.

We know from Proposition 4.2 that $\mathcal{FB}$ is a non − singular quasi − projective variety of dimension $2rc_2$, and we have seen that for any non − zero global section ζ of $\omega_X^{-1}(-2D) \cong \mathcal{O}(1)$, we get a Poisson structure θ_ζ on $\mathcal{FB}$. In this situation, however, there is an obvious natural choice (up to a scalar multiple) of the section ζ, namely we can choose $\zeta = \sigma$, the section defining the divisor D. In this way we obtain a canonical Poisson structure θ_σ on $\mathcal{FB}$, depending (up to a scalar multiple) only on the initial choice of the divisor D. It turns out that this Poisson structure is actually symplectic, as we shall now prove.

Proposition 5.1. *The Poisson structure θ_σ on $\mathcal{FB}$ is non − degenerate, hence defines a symplectic structure on $\mathcal{FB}$.*

Proof. We have seen in Remark 4.5 that the kernel of the map $B_\sigma : T^*\mathcal{FB} \to T\mathcal{FB}$ at a point (E,ϕ) is isomorphic to $H^0\big(D, \mathcal{E}nd(E)\otimes\mathcal{O}_X(-D)\big|_D\big)$. By recalling that $D \cong \mathbf{P}^1$ and that $E|_D \cong \mathcal{O}_D^{\oplus r}$, we have

$$\ker\big(B_\sigma(E,\phi)\big) \;\cong\; H^0\left(\mathbf{P}^1, \mathcal{O}(-1)^{\oplus r^2}\right) \;=\; 0\,.$$

It follows that B_σ is an isomorphism, hence θ_σ is everywhere non-degenerate. □

Remark 5.2. We recall here that DONALDSON has proved in [D] the following result. Let P be a principal $\mathrm{SU}(r)$-bundle over $S^4 = \mathbb{R}^4 \cup \{\infty\}$, with Pontryagin index $-c_2$, and let $\widetilde{M}$ denote the moduli space parametrizing isomorphism classes of pairs consisting of an anti self-dual $\mathrm{SU}(r)$-connection on P together with an isomorphism between the fiber P_∞ and the group $\mathrm{SU}(r)$. This is also called the moduli space of framed instantons of charge c_2. Then, if we fix a complex structure on $\mathbb{R}^4$ compatible with the metric, and consider $\mathbb{R}^4 \cong \mathbb{C}^2 = \mathbf{P}^2 \setminus \ell$, there is a natural one-to-one correspondence between the moduli space $\widetilde{M}$ and the moduli space $\mathcal{FB}$ of rank-r framed vector bundles with second Chern class c_2 whose restriction to the line ℓ is trivial. In this way we get a natural complex structure on the moduli space $\widetilde{M}$ of framed $\mathrm{SU}(r)$-instantons. As a corollary of our preceding results, we now find that on $\widetilde{M}$ there is also a natural holomorphic symplectic structure.

To end this section, we shall investigate what happens for a different choice of the section $\zeta \in H^0(X, \mathcal{O}(1))$.

The kernel of the map $B_\zeta : T^*\mathcal{FB} \to T\mathcal{FB}$ at a point (E,ϕ) is now isomorphic to $H^0\left(D_\zeta, \mathcal{E}nd(E)\otimes\mathcal{O}_X(-D)\big|_{D_\zeta}\right)$, where D_ζ is the divisor defined by ζ, hence the rank of the Poisson structure θ_ζ at a point (E,ϕ) depends on the restriction of E to the line D_ζ, which is different from the line D on which the restriction of E is trivial. It is nevertheless true that, for any choice of ζ and for a generic $(E,\phi) \in \mathcal{FB}$, the restriction of E to D_ζ is trivial. In fact, for a vector bundle E, the property that its restriction to a line is trivial is an open property. It follows that, for a generic line ℓ, the restriction to ℓ of a generic vector bundle E with $c_1(E) = 0$ is trivial. In our case, however, we are considering vector bundles whose restriction to a fixed line D is trivial, and then we picked another line D_ζ. The following result shows that the generic such bundle is trivial also on D_ζ.

Lemma 5.3. *Let ℓ_1 and ℓ_2 be two distinct lines in $\mathbf{P}^2$. If E is a generic stable vector bundle whose restriction to ℓ_1 is trivial, then also its restriction to ℓ_2 is trivial.*

Proof. Take a generic stable vector bundle F such that $F\big|_{\ell_1}$ is trivial. We know that its restriction to a generic line is also trivial, hence there exists a line ℓ' such that $F\big|_{\ell'}$ is trivial. Let $f : \mathbf{P}^2 \to \mathbf{P}^2$ be a projective isomorphism fixing ℓ_1 and sending ℓ_2 to ℓ' and let $E = f^*(F)$. It follows that $E\big|_{\ell_2} = F\big|_{\ell'}$ is trivial. Knowing that there exists a stable vector bundle E whose restrictions to ℓ_1 and ℓ_2 are trivial, it follows, by considering deformations of E that keep the restrictions of E to these two lines fixed, that this property is satisfied also by the generic such vector bundle. □

Coming back to our situation, it follows from the preceding discussion that, for any choice of the (non−zero) section $\zeta \in H^0(X, \mathcal{O}(1))$, the kernel of the map $B_\zeta(E, \phi)$ is zero, for a generic $(E, \phi) \in \mathcal{FB}$. This means that the Poisson structure θ_ζ on $\mathcal{FB}$ is generically symplectic. On the other hand, the dimension of the kernel of B_ζ at a point (E, ϕ) such that $E\big|_{D_\zeta} = \mathcal{O}(a_1) \oplus \mathcal{O}(a_2) \oplus \cdots \oplus \mathcal{O}(a_r)$, with $a_1 \geq a_2 \geq \cdots \geq a_r$ and $a_1 + a_2 + \cdots + a_r = 0$, is given by:

$$\begin{aligned}
\dim \ker(B_\zeta(E, \phi)) &= h^0\Big(D_\zeta, \mathcal{E}nd(E) \otimes \mathcal{O}_X(-D)\big|_{D_\zeta}\Big) \\
&= h^0\left(\mathbf{P}^1, \left(\bigoplus_{i=1}^{r} \mathcal{O}(a_i)\right) \otimes \left(\bigoplus_{j=1}^{r} \mathcal{O}(-a_j)\right) \otimes \mathcal{O}(-1)\right) \\
&= h^0\left(\mathbf{P}^1, \bigoplus_{i,j} \mathcal{O}\big(a_i - a_j - 1\big)\right) \\
&= \sum_{i<j} \big(a_i - a_j\big)\,.
\end{aligned}$$

For example, if $r = 2$ and $(E, \phi) \in \mathcal{FB}$ is such that $E\big|_{D_\zeta} = \mathcal{O}(k) \oplus \mathcal{O}(-k)$, with $k > 0$, we have

$$\dim \ker(B_\zeta(E, \phi)) \;=\; 2k\,.$$

This shows that the rank of the Poisson structure θ_ζ on $\mathcal{FB}$ drops by $2k$ at the points (E, ϕ) such that the restriction of E to the line D_ζ splits as $\mathcal{O}(k) \oplus \mathcal{O}(-k)$.

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References

[B1] Bottacin, F.: Symplectic Geometry on Moduli Spaces of Stable Pairs, Ann. Scient. Éc. Norm. Sup. **28** (1995), 391 – 433

[B2] Bottacin, F.: Poisson Structures on Moduli Spaces of Sheaves over Poisson Surfaces, Invent. Math. **121** (1995), 421 – 436

[D] Donaldson, S. K.: Instantons and Geometric Invariant Theory, Comm. Math. Phys. **93** (1984), 453 – 460

[GH] Griffiths, P., and Harris, J.: Principles of Algebraic Geometry, Wiley, New York, 1978

[HL1] Huybrechts, D., and Lehn, M.: Framed Modules and Their Moduli, Internat. J. Math. **6** (1995), 297 – 324

[HL2] Huybrechts, D., and Lehn, M.: The Geometry of Moduli Spaces of Sheaves, Aspects of Mathematics **E 31**, Vieweg, 1997

[L] Lübke, M.: The Analytic Moduli Space of Framed Vector Bundles, J. Reine Angew. Math. **441** (1993), 45 – 59

Dip. di Mat. Pura e Appl.
Via Belzoni, 7
I – 35131, Padova
Italy
E – mail:
bottacin@math.unipd.it