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A generalization of Higgs bundles to higher dimensional varieties

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Abstract. Let X be a smooth n-dimensional projective variety defined over $\mathbb{C}$ and let L be a line bundle on X. In this paper we shall construct a moduli space $\mathcal{P}(L)$ parametrizing $(n-1)$-cohomology L-twisted Higgs pairs, i.e., pairs $(E, \bar{\phi})$ where E is a vector bundle on X and $\bar{\phi} \in H^{n-1}(X, \mathcal{E}nd(E) \otimes L)$. If we take $L = \omega_X$, the canonical line bundle on X, the variety $\mathcal{P}(L)$ is canonically identified with the cotangent bundle of the smooth locus of the moduli space of stable vector bundles on X and, as such, it has a canonical symplectic structure. We prove that, in the general case, in correspondence to the choice of a non-zero section $s \in H^0(X, \omega_X^{-1} \otimes L)$, one can define, in a natural way, a Poisson structure θ_s on $\mathcal{P}(L)$. We also analyze the relations between this Poisson structure on $\mathcal{P}(L)$ and the canonical symplectic structure of the cotangent bundle to the smooth locus of the moduli space of parabolic bundles over X, with parabolic structure over the divisor D defined by the section s. These results generalize to the higher dimensional case similar results proved in [Bo1] in the case of curves.

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Introduction

Let C be a smooth projective curve of genus ≥ 2 defined over the complex field $\mathbb{C}$, and let us denote by $M(r, d)$ the moduli space of stable vector bundles of rank r and degree d over C.

It is well known that the cotangent bundle of $M(r, d)$ can be canonically identified with the set of isomorphism classes of pairs (E, ϕ), where $E \in$

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$M(r,d)$ and $\phi : E \to E \otimes \omega_C$ is a homomorphism of vector bundles. These pairs are known as *Higgs bundles*. Since the introduction of these objects by Hitchin in [Hi], moduli spaces of Higgs bundles have been studied by various authors, and several generalizations have been proposed.

In this paper we shall consider the general situation in which the curve C is replaced by a smooth n-dimensional projective variety X, defined over $\mathbb{C}$. Let us fix some "moduli data" and denote by M^s the moduli space of stable vector bundles on X. Since, in general, M^s is not a smooth variety, we shall restrict to its smooth locus M^s_{sm}. By standard infinitesimal deformation considerations, the cotangent bundle of M^s_{sm} can be canonically identified with the set of isomorphism classes of pairs $(E, \bar{\phi})$, where $E \in M^s_{sm}$ and $\bar{\phi} \in H^{n-1}(X, \mathcal{E}nd(E) \otimes \omega_X)$. We are thus naturally led to consider the more general situation of pairs $(E, \bar{\phi})$ with $\bar{\phi} \in H^{n-1}(X, \mathcal{E}nd(E) \otimes L)$, for some fixed line bundle L on X. We shall call these objects $(n-1)$-*cohomology L-twisted Higgs pairs*, or simply *pairs*, in the sequel.

The main result of this paper is the construction of a canonical family of (compatible) Poisson structures on the moduli space $\mathcal{P}(L)$ of $(n-1)$-cohomology L-twisted Higgs pairs, parametrized by the global sections of the line bundle $\omega_X^{-1} \otimes L$. If $L = \omega_X$ there is only one Poisson structure (up to scalars). This is actually non-degenerate, hence defines a symplectic structure on $\mathcal{P}(\omega_X)$, which coincides with the canonical symplectic structure of the cotangent bundle to M^s_{sm}. In the general case, if $L \cong \omega_X(D)$ for some effective divisor D, the Poisson structure of $\mathcal{P}(L)$ corresponding to the section s of $\omega_X^{-1} \otimes L$ defining the divisor D is related to the canonical symplectic structure of the cotangent bundle to the smooth locus of the moduli space of parabolic vector bundles with parabolic structure over the divisor D. These results generalize to the higher dimensional case the results obtained by Hitchin in [Hi] and by the present author in [Bo1].

The paper is organized as follows. In Sect. 1, we define the objects of our study, the $(n-1)$-cohomology L-twisted Higgs pairs, and construct the moduli space $\mathcal{P}(L)$; this variety has a natural structure of vector bundle over a suitable open subset of M^s.

In Sect. 2 we use infinitesimal deformation theory to study the tangent and cotangent bundles of $\mathcal{P}(L)$. In particular we prove that the tangent spaces to $\mathcal{P}(L)$ can be naturally identified with the first cohomology groups of certain complexes and, by means of duality theory, we also obtain an explicit description of the cotangent spaces. Then we use these results to define a homomorphism $B_s : T^*\mathcal{P}(L) \to T\mathcal{P}(L)$, depending on the choice of a non-zero global section s of $\omega_X^{-1} \otimes L$. This map defines an antisymmetric contravariant 2-tensor $\theta_s \in H^0(\mathcal{P}(L), \wedge^2 T\mathcal{P}(L))$, which will turn out to be a Poisson structure. The map B_s will be studied in Sect. 3.

In Sect. 4 we introduce the main technical tools used in the sequel. Precisely we show how tangent vector fields on $\mathcal{P}(L)$ can be expressed in terms of certain first order differential operators.

In the following section, Sect. 5, we recall some general results about symplectic and Poisson structures, and define the canonical Poisson structure on the dual of a vector bundle endowed with the structure of a locally free sheaf of Lie algebras. This construction is a generalization of the canonical symplectic structure of the cotangent bundle of a smooth variety.

In Sect. 6 we use the results obtained in the previous sections to prove that the antisymmetric contravariant 2-tensor θ_s actually defines a Poisson structure on $\mathcal{P}(L)$, that coincides with the canonical symplectic structure on $T^*M^s_{sm}$, when $L = \omega_X$ and s is the identity section.

In Sect. 7 we compare our definition of $(n-1)$-cohomology L-twisted Higgs pairs with the usual definition of Higgs bundles on higher dimensional varieties, as found, e.g., in [S1] or [S2].

Finally, in Sect. 8, we recall the construction of the moduli space $\mathcal{M}_{par}$ of parabolic vector bundles on X, with parabolic structure over an effective divisor D (defined by a section s), and describe the relations between the Poisson structure of $\mathcal{P}(L)$ (corresponding to the section s) and the canonical symplectic structure of the cotangent bundle to the smooth locus of $\mathcal{M}_{par}$.

1. $(n-1)$-cohomology Higgs pairs

Let X be a smooth n-dimensional projective variety defined over $\mathbb{C}$, with a very ample invertible sheaf $\mathcal{O}_X(1)$. For a coherent torsion-free $\mathcal{O}_X$-module E, we denote by $\mathrm{rk}(E)$ the rank of E at the generic point, by $\deg(E)$ the intersection number of $c_1(E)$ with $c_1(\mathcal{O}_X(1))^{n-1}$ and by $P_E = P_E(t)$ the Hilbert polynomial of E. Finally, the slope of E is defined by setting

$$\mu(E) = \frac{\deg(E)}{\mathrm{rk}(E)}.$$

Let $M = M^s_{sm}(P)$ denote the smooth locus of the moduli space of stable vector bundles on X with fixed Hilbert polynomial P. By abuse of notation we shall denote by E either a vector bundle on X or the point of M corresponding to the isomorphism class of E.

Remark 1.1. In general, even if a universal family $\mathcal{E}$ does not exist on any Zariski open subset of $M^s(P)$, the sheaf $\mathcal{E}nd(\mathcal{E})$ is always defined. This follows from the construction of the moduli space $M^s(P)$ by Geometric Invariant Theory, by the same reasoning as in [Bo1, Remark 1.1.2]. As for the universal family $\mathcal{E}$, its local existence in the étale (or complex) topology, will be sufficient for our purposes.

It is well known that the tangent space $T_E M$ to M at a point E is canonically identified to $H^1(X, \mathcal{E}nd(E))$. By Serre duality, it follows that the cotangent space $T_E^* M$ is identified to $H^{n-1}(X, \mathcal{E}nd(E) \otimes \omega_X)$, where ω_X denotes the canonical line bundle on X. From this it follows that the cotangent bundle T^*M to M can be described set-theoretically as the set of isomorphism classes of pairs $(E, \bar{\phi})$, consisting of a vector bundle E on X and an element $\bar{\phi} \in H^{n-1}(X, \mathcal{E}nd(E) \otimes \omega_X)$. If X is a projective curve ($n = 1$), we obtain the classical notion of Higgs bundles, introduced by Hitchin in [Hi].

Let us fix a line bundle L on X (in the sequel we shall impose some conditions on L).

Definition 1.2. A $(n-1)$-cohomology L-twisted Higgs pair (simply called pair in the sequel) is a pair $(E, \bar{\phi})$, where E is a locally free sheaf on X and $\bar{\phi} \in H^{n-1}(X, \mathcal{E}nd(E) \otimes L)$.

If $\lambda : F \to E$ is a homomorphism of vector bundles, composition with λ on the right and with $\lambda \otimes \mathrm{id}_L$ on the left induce, respectively, the following homomorphisms

$$\cdot \circ \lambda : H^{n-1}(X, \mathcal{E}nd(E) \otimes L) \to H^{n-1}(X, \mathcal{H}om(F, E \otimes L)), \tag{1.1}$$

and

$$(\lambda \otimes \mathrm{id}_L) \circ \cdot : H^{n-1}(X, \mathcal{E}nd(F) \otimes L) \to H^{n-1}(X, \mathcal{H}om(F, E \otimes L)). \tag{1.2}$$

Then we have:

Definition 1.3. A homomorphism of $(n-1)$-cohomology L-twisted Higgs pairs $(F, \bar{\psi})$ and $(E, \bar{\phi})$ is a homomorphism of vector bundles $\lambda : F \to E$ such that the image of $\bar{\phi}$ by the map (1.1) is equal to the image of $\bar{\psi}$ by the map (1.2). If λ is an isomorphism of vector bundles, then we speak of an isomorphism of $(n-1)$-cohomology L-twisted Higgs pairs.

The notion of family of pairs is defined as follows:

Definition 1.4. A family of $(n-1)$-cohomology L-twisted Higgs pairs on X, parametrized by a noetherian scheme S, is the data of a vector bundle $\mathcal{E}$ on $S \times X$ and a global section $\bar{\Phi}$ of the sheaf $R^{n-1}q_*(\mathcal{H}om(\mathcal{E}, \mathcal{E} \otimes p^*L))$, where $p : X \times S \to X$ and $q : X \times S \to S$ are the canonical projections.

Let us come now to the definition of stability. We recall that, when X is a curve, a Higgs bundle (E, ϕ) is semistable (resp. stable) if $\mu(F) \leq \mu(E)$ (resp. $\mu(F) < \mu(E)$), for every ϕ-invariant proper subbundle F of E (see [N]).

There is an obvious generalization of this notion of stability to the case of $(n-1)$-cohomology L-twisted Higgs pairs on a higher dimensional variety;

we must only be careful to consider the right notion of subobjects of a pair $(E,\bar{\phi})$. In fact, when $\dim X > 1$, it is not sufficient to consider subpairs $(F,\bar{\psi})$ with F locally free, but we must allow F to be only a torsion-free sheaf. Correspondingly, $\bar{\psi}$ should be an element of $\mathrm{Ext}^{n-1}(F, F\otimes L)$. The definition of a homomorphism of these more general pairs is the obvious generalization of the one given in Definition 1.3. Hence we define subpairs as follows:

Definition 1.5. A subpair of a $(n-1)$-cohomology L-twisted Higgs pair $(E,\bar{\phi})$ is a pair $(F,\bar{\psi})$, where F is a coherent torsion-free subsheaf of E and $\bar{\psi} \in \mathrm{Ext}^{n-1}(F, F\otimes L)$, such that the inclusion $F \hookrightarrow E$ induces a homomorphism of pairs.

Now we can state the definition of (slope) stability:

Definition 1.6. A $(n-1)$-cohomology L-twisted Higgs pair $(E,\bar{\phi})$ is semistable (resp. stable) if $\mu(F) \leq \mu(E)$ (resp. $\mu(F) < \mu(E)$) for every proper subpair $(F,\bar{\psi})$.

Remark 1.7. In [BGP] the authors introduced a notion of stability (depending on parameters) for a more general class of objects called p-cohomology triples. It is easy to see that this definition of stability reduces to our definition in the case of $(n-1)$-cohomology L-twisted Higgs pairs (and the dependence on parameters disappears).

Remark 1.8. As usual, this definition of stability leads to the construction of moduli spaces of (semi)stable $(n-1)$-cohomology L-twisted Higgs pairs. In this paper, however, we shall restrict our attention to pairs $(E,\bar{\phi})$ such that E is a stable vector bundle. We note that, if E is stable, then $(E,\bar{\phi})$ is a stable pair for any $\bar{\phi} \in H^{n-1}(X, \mathcal{E}nd(E)\otimes L)$.

Remark 1.9. In the case of surfaces, a $(n-1)$-cohomology L-twisted Higgs pair $(E,\bar{\phi})$ is actually a 1-cohomology L-twisted Higgs pair, i.e. $\bar{\phi} \in H^1(X, \mathcal{E}nd(E)\otimes L) = \mathrm{Ext}^1(E, E\otimes L)$, hence it determines an extension of E by $E\otimes L$,

$$0 \to E\otimes L \to E_{\bar{\phi}} \to E \to 0.$$

Extensions of holomorphic vector bundles are studied in [BGP] and, particularly, in [DUW], where moduli spaces of extensions are constructed. There is a notion of stability for extensions, depending on a real parameter α, defined as follows: for an extension e given by an exact sequence of coherent torsion-free sheaves

$$e: \quad 0 \to \mathcal{E}_1 \to \mathcal{E} \to \mathcal{E}_2 \to 0,$$

we define the α-slope by setting

$$\mu_\alpha(e) = \mu(\mathcal{E}) + \alpha \frac{\mathrm{rk}(\mathcal{E}_2)}{\mathrm{rk}(\mathcal{E})}.$$

Then we say that an extension e is α-stable (resp. α-semistable) if, for every proper subextension e', we have

$$\mu_\alpha(e') < \mu_\alpha(e) \quad \text{(resp. } \leq\text{).}$$

It is easy to prove that this notion of α-stability, applied to the extension corresponding to a 1-cohomology L-twisted Higgs pair $(E, \bar{\phi})$ coincides with the notion of stability for pairs given in Definition 1.6. Again, we note that the dependence on the real parameter α has disappeared.

Let us denote by $\mathcal{P}(L)$ the set of isomorphism classes of $(n-1)$-cohomology L-twisted Higgs pairs $(E, \bar{\phi})$ on X such that $E \in M$ (this means, in particular, that E is stable). There is a natural projection map $\pi : \mathcal{P}(L) \to M$, which sends a pair $(E, \bar{\phi})$ to E and whose fibers are $\pi^{-1}(E) = H^{n-1}(X, \mathcal{E}nd(E) \otimes L)$.

As we have previously seen, if we take $L = \omega_X$, $\mathcal{P}(\omega_X)$ is canonically identified with the cotangent bundle T^*M of the smooth locus M of the moduli space of stable vector bundles on X. This means, in particular, that the dimension of the fibers $\pi^{-1}(E) = H^{n-1}(X, \mathcal{E}nd(E) \otimes \omega_X)$ is constant as E varies in M.

Unfortunately, for a generic L, the dimension of $H^{n-1}(X, \mathcal{E}nd(E) \otimes L)$ will not be necessarily constant as E varies in M, hence it is not possible to regard $\mathcal{P}(L)$ as a vector bundle over M. However, by the semicontinuity theorem, there exists an open subset M' of M such that $\dim H^{n-1}(X, \mathcal{E}nd(E) \otimes L)$ is constant as E varies in M'. Now, if we consider pairs $(E, \bar{\phi})$ with $E \in M'$, and we use the symbol $\mathcal{P}(L)$ to denote the set of isomorphism classes of such pairs, it is easy to prove that $\mathcal{P}(L)$ can be given the structure of a vector bundle over M'.

In the language of algebraic geometry, this can be done as follows. Let us consider the universal sheaf $\mathcal{E}nd(\mathcal{E})$ on $M' \times X$ defined in Remark 1.1, and denote by $p : M' \times X \to X$ and $q : M' \times X \to M'$ the canonical projections. From the fact that the dimension of $H^{n-1}(X, \mathcal{E}nd(E) \otimes L)$ is constant as E varies in M', it follows that the sheaf

$$\mathcal{H} = R^{n-1} q_*(\mathcal{E}nd(\mathcal{E}) \otimes p^* L)$$

is a vector bundle on M', whose fibers are $\mathcal{H}_E \cong H^{n-1}(X, \mathcal{E}nd(E) \otimes L)$. We may then define the variety $\mathcal{P}(L)$ as follows:

$$\mathcal{P}(L) = \mathrm{Spec}(\mathrm{Sym}(\mathcal{H}^*)),$$

where $\mathrm{Sym}(\mathcal{H}^*)$ denotes the symmetric algebra of the dual sheaf of $\mathcal{H}$. Thus $\mathcal{P}(L)$ has a natural structure of vector bundle over M', with fibers isomorphic to $H^{n-1}(X, \mathcal{E}nd(E) \otimes L)$, for $E \in M'$.

On the variety $\mathcal{P}(L)$ there does not exist, in general, a "universal pair" $(\mathcal{E}, \bar{\Phi})$, since there does not even exist a universal vector bundle $\mathcal{E}$ on M'. We can however prove the following result:

Proposition 1.10. *If there exists a universal vector bundle $\mathcal{E}$ on M', then there exists a universal $(n-1)$-cohomology L-twisted Higgs pair on $\mathcal{P}(L)$.*

Proof. Let us consider the following commutative diagram:

$$\begin{tikzcd}
\mathcal{P}(L) \times X \arrow[r, "\pi'"] \arrow[d, "q'"'] & M' \times X \arrow[d, "q"] \\
\mathcal{P}(L) \arrow[r, "\pi"] & M',
\end{tikzcd}$$

where $\pi' = \pi \times \mathrm{id}_X$.

To construct a universal pair on $\mathcal{P}(L) \times X$, we first consider the vector bundle $\mathcal{E}' = \pi'^*(\mathcal{E})$, obtained by pulling-back the universal vector bundle $\mathcal{E}$ on $M' \times X$. Then we observe that the vector bundle $\pi^*(\mathcal{H})$ on $\mathcal{P}(L)$ has a canonical section $\bar{\Phi}$. By using the flatness of π and the fact that $\mathcal{E}$ is locally free of finite rank, we have:

$$\begin{aligned}
\pi^*(\mathcal{H}) &= \pi^*(R^{n-1} q_*(\mathcal{E}nd(\mathcal{E}) \otimes p^* L)) \\
&= R^{n-1} q'_* \pi'^*(\mathcal{E}nd(\mathcal{E}) \otimes p^* L) \\
&= R^{n-1} q'_*(\mathcal{E}nd(\mathcal{E}') \otimes p'^* L),
\end{aligned}$$

where $p' : \mathcal{P}(L) \times X \to X$ denotes the canonical projection.

It follows that $\bar{\Phi}$ can be considered as a section of $R^{n-1} q'_*(\mathcal{E}nd(\mathcal{E}') \otimes p'^* L)$. The pair $(\mathcal{E}', \bar{\Phi})$ has the property that, for every $(E, \bar{\phi}) \in \mathcal{P}(L)$, its restriction to $\{(E, \bar{\phi})\} \times X$ is isomorphic to the pair $(E, \bar{\phi})$. Now, by using the fact that $\mathcal{E}$ is a universal vector bundle on M', it is easy to prove that $(\mathcal{E}', \bar{\Phi})$ is a universal pair.

Remark 1.11. As we have already observed in Remark 1.1, even when there is no universal vector bundle $\mathcal{E}$ on M', the sheaf $\mathcal{E}nd(\mathcal{E})$ is always defined. It follows that, even if the universal sheaf $\mathcal{E}'$ on $\mathcal{P}(L) \times X$ may not exist, the sheaf $\mathcal{E}nd(\mathcal{E}')$ is always defined. By adapting the proof of the preceding proposition, it follows that the universal section $\bar{\Phi}$ of $R^{n-1} q'_*(\mathcal{E}nd(\mathcal{E}') \otimes p'^* L)$ is always defined.

2. Infinitesimal study of $\mathcal{P}(L)$

In this section we study infinitesimal deformations of Higgs pairs. Let $\mathbb{C}[\epsilon]/(\epsilon^2)$ be the ring of dual numbers over $\mathbb{C}$; in the sequel it will be denoted simply by $\mathbb{C}[\epsilon]$.

Definition 2.1. An infinitesimal deformation of a $(n-1)$-cohomology L-twisted Higgs pair $(E,\bar{\phi})$ is a family $(E_\epsilon,\bar{\phi}_\epsilon)$ of pairs parametrized by $\operatorname{Spec}(\mathbb{C}[\epsilon])$, together with an isomorphism of $(E,\bar{\phi})$ with the specialization of $(E_\epsilon,\bar{\phi}_\epsilon)$ (we shall say for short that $(E_\epsilon,\bar{\phi}_\epsilon)$ restricts to $(E,\bar{\phi})$).

Two infinitesimal deformations $(E'_\epsilon,\bar{\phi}'_\epsilon)$ and $(E''_\epsilon,\bar{\phi}''_\epsilon)$ of a pair $(E,\bar{\phi})$ are isomorphic if there exists an isomorphism of pairs $\lambda_\epsilon : (E'_\epsilon,\bar{\phi}'_\epsilon) \to (E''_\epsilon,\bar{\phi}''_\epsilon)$ which restricts to the identity over $(E,\bar{\phi})$.

Let $(E,\bar{\phi})$ be a $(n-1)$-cohomology L-twisted Higgs pair in $\mathcal{P}(L)$. We shall use the Čech complexes $C^\cdot(\mathcal{U},\mathcal{E}nd(E))$ and $C^\cdot(\mathcal{U},\mathcal{E}nd(E)\otimes L)$, with respect to a suitable affine open covering $\mathcal{U}=(U_i)_{i\in I}$ of X, to compute the cohomology of these sheaves (in the sequel, to simplify the notation, the indication of the open covering $\mathcal{U}$ will be omitted). The cohomology class $\bar{\phi}\in H^{n-1}(X,\mathcal{E}nd(E)\otimes L)$ can thus be represented by a Čech $(n-1)$-cocycle $\{\phi_{i_0,\dots,i_{n-1}}\}$ in $C^{n-1}(\mathcal{E}nd(E)\otimes L)$.

For $\{\alpha_{j_0,\dots,j_p}\}\in C^p(\mathcal{E}nd(E))$, we define

$$\{[\alpha,\phi]_{i_0,\dots,i_{p+n-1}}\}\in C^{p+n-1}(\mathcal{E}nd(E)\otimes L)$$

by setting

$$\begin{aligned}[\alpha,\phi]_{i_0,\dots,i_{p+n-1}} &= [\alpha_{i_0,\dots,i_p},\phi_{i_p,\dots,i_{p+n-1}}]\\ &= (\alpha_{i_0,\dots,i_p}\otimes \mathrm{id}_L)\circ\phi_{i_p,\dots,i_{p+n-1}} - \phi_{i_p,\dots,i_{p+n-1}}\\ &\quad\circ\alpha_{i_0,\dots,i_p}.\end{aligned}$$

It is easy to check that the maps

$$[\cdot,\phi] : C^i(\mathcal{E}nd(E))\to C^{i+n-1}(\mathcal{E}nd(E)\otimes L) \tag{2.1}$$

define a homomorphism (of degree $n-1$) of Čech complexes.

We now define a new complex $C^\cdot([\cdot,\phi])$ by setting

$$C^i([\cdot,\phi]) = C^i(\mathcal{E}nd(E))\oplus C^{i+n-2}(\mathcal{E}nd(E)\otimes L),$$

with coboundary $d^i : C^i([\cdot,\phi])\to C^{i+1}([\cdot,\phi])$ given by

$$d^i=\begin{pmatrix}\delta^i & 0\\ [\cdot,\phi] & -\delta^{i+n-2}\end{pmatrix}.$$

It is straightforward to verify that $C^\cdot([\cdot,\phi])$ is actually a complex and that its first cohomology group $H^1(C^\cdot([\cdot,\phi]))$ is the set of equivalence classes of pairs $(\alpha, \eta) = (\{\alpha_{i_0,\dots,i_{n-1}}\}, \{\eta_{j_0,j_1}\}) \in C^{n-1}(\mathcal{E}nd(E) \otimes L) \oplus C^1(\mathcal{E}nd(E))$ such that $\delta\eta = 0$ and $\delta\alpha = [\eta, \phi]$, modulo the equivalence relation defined by $(\alpha, \eta) \sim (\alpha', \eta')$ if there exist $\beta \in C^{n-2}(\mathcal{E}nd(E) \otimes L)$ and $\zeta \in C^0(\mathcal{E}nd(E))$ such that $\alpha' = \alpha - \delta\beta + [\zeta, \phi]$ and $\eta' = \eta + \delta\zeta$.

Let us fix our notation here. In terms of Čech cocycles, the equality $\delta\alpha = [\eta, \phi]$ means precisely

$$(\delta\alpha)_{i_0,\dots,i_n} = [\eta_{i_0,i_1}, \phi_{i_1,\dots,i_n}],$$

and $\alpha' = \alpha - \delta\beta + [\zeta, \phi]$ means

$$\alpha'_{i_0,\dots,i_{n-1}} = \alpha_{i_0,\dots,i_{n-1}} - (\delta\beta)_{i_0,\dots,i_{n-1}} + [\zeta_{i_0}, \phi_{i_0,\dots,i_{n-1}}].$$

Remark 2.2. The complex $C^{\cdot}([\cdot, \phi])$ is essentially the "mapping cone" of the homomorphism of Čech complexes defined by (2.1), with a shift of -1 in the degrees, if we take as definition of the mapping cone the one given in [KS].

The complex $C^{\cdot}([\cdot, \phi])$ actually depends on the representative ϕ of the cohomology class $\bar{\phi}$. The following lemma shows that, on the other hand, the cohomology of this complex depends only on the cohomology class $\bar{\phi}$.

Lemma 2.3. *Let ϕ and ϕ' be two $(n-1)$-cocycles which represent the same cohomology class $\bar{\phi} \in H^{n-1}(X, \mathcal{E}nd(E) \otimes L)$, and let $C^{\cdot}([\cdot, \phi])$ and $C^{\cdot}([\cdot, \phi'])$ denote the corresponding complexes. Then the cohomology groups $H^i(C^{\cdot}([\cdot, \phi]))$ and $H^i(C^{\cdot}([\cdot, \phi']))$ are canonically isomorphic.*

Proof. We shall prove the result only for $i = 1$, the proof in the general case being similar. Let us write $\phi' = \phi + \delta\psi$, for $\psi \in C^{n-2}(\mathcal{E}nd(E) \otimes L)$. Using the explicit description of the first cohomology group $H^1(C^{\cdot}([\cdot, \phi]))$ given before, it is straightforward to prove that the map $H^1(C^{\cdot}([\cdot, \phi'])) \to H^1(C^{\cdot}([\cdot, \phi]))$, which sends the element (α, η) to $(\alpha + [\eta, \psi], \eta)$ is well-defined and is an isomorphism of cohomology groups.

Now, by an infinitesimal deformation computation similar to the one used in [Bo1] in the proof of Proposition 3.1.2, we can prove the following result:

Theorem 2.4. *The set of isomorphism classes of infinitesimal deformations of a pair $(E, \bar{\phi})$ is canonically identified with the first cohomology group $H^1(C^{\cdot}([\cdot, \phi]))$.*

From the existence of a local universal family of $(n-1)$-cohomology L-twisted Higgs pairs, we obtain:

Corollary 2.5. *The tangent space $T_{(E,\bar{\phi})}\mathcal{P}(L)$ to $\mathcal{P}(L)$ at the point $(E, \bar{\phi})$ is canonically isomorphic to $H^1(C^{\cdot}([\cdot, \phi]))$.*

Remark 2.6. We remark, without giving any details, that the methods used to study infinitesimal deformations of $(n-1)$-cohomology L-twisted Higgs pairs can be applied, with only minor modifications, to the study of the infinitesimal deformations of p-cohomology triples, defined in [BGP].

Now we turn to the study of the cotangent space $T^*_{(E,\bar{\phi})}\mathcal{P}(L)$. For this we have to "dualize" all the constructions we have previously done. We begin by constructing the "dual complex" $\check{C}^\cdot([\phi,\cdot])$ of $C^\cdot([\cdot,\phi])$. This is defined as follows:

$$\check{C}^i([\phi,\cdot]) = C^i(\mathcal{E}nd(E)\otimes L^{-1}\otimes\omega_X)\oplus C^{i+n-2}(\mathcal{E}nd(E)\otimes\omega_X),$$

with coboundary given by

$$d^i = \begin{pmatrix} \delta^i & 0 \\ [\phi,\cdot] & -\delta^{i+n-2} \end{pmatrix}.$$

Remark 2.7. The reason for considering the map $[\phi,\cdot]$ instead of $[\cdot,\phi]$ in the dual complex is the following: when we dualize, we obtain sheaves $(\mathcal{E}nd(E))^*\otimes L^{-1}\otimes\omega_X$ and $(\mathcal{E}nd(E))^*\otimes\omega_X$. Now, there is a canonical identification between $(\mathcal{E}nd(E))^*$ and $\mathcal{E}nd(E)$ given by the pairing trace. Under this identification, the transpose of the map $[\cdot,\phi]$ coincides precisely with the map $[\phi,\cdot]$.

The following result follows now from the general theory of duality (but can also be proved directly):

Proposition 2.8. *The dual of the i-th cohomology group $H^i(C^\cdot([\cdot,\phi]))$ is canonically identified with $H^{2-i}(\check{C}^\cdot([\phi,\cdot]))$. In particular, we have a canonical isomorphism*

$$H^1(C^\cdot([\cdot,\phi]))^*\cong H^1(\check{C}^\cdot([\phi,\cdot])).$$

We can now state the following result:

Corollary 2.9. *The cotangent space $T^*_{(E,\bar{\phi})}\mathcal{P}(L)$ is canonically isomorphic to the first cohomology group $H^1(\check{C}^\cdot([\phi,\cdot]))$.*

Remark 2.10. An analogue of Lemma 2.3 holds for the complex $\check{C}^\cdot([\phi,\cdot])$, i.e., its cohomology groups actually depend only on the cohomology class $\bar{\phi}$.

Remark 2.11. In terms of Čech cocycles, the group $H^1(\check{C}^\cdot([\phi,\cdot]))$ may be described in a way perfectly similar to the group $H^1(C^\cdot([\cdot,\phi]))$: it is the set of equivalence classes (modulo an obvious equivalence relation, that we do not write explicitly) of pairs $(\beta,\zeta) = (\{\beta_{i_0,\dots,i_{n-1}}\},\{\zeta_{j_0,j_1}\}) \in C^{n-1}(\mathcal{E}nd(E)\otimes\omega_X)\oplus C^1(\mathcal{E}nd(E)\otimes L^{-1}\otimes\omega_X)$ such that $\delta\zeta = 0$ and $\delta\beta = [\phi,\zeta]$, where this equality means precisely

$$(\delta\beta)_{i_0,\dots,i_n} = [\phi_{i_0,\dots,i_{n-1}},\zeta_{i_{n-1},i_n}].$$

Remark 2.12. By using the description of the cohomology groups in terms of Čech cocycles, it is possible to give an explicit description of the duality pairing

$$H^1(C^{\cdot}([\cdot,\phi])) \times H^1(\check{C}^{\cdot}([\phi,\cdot])) \to H^n(X,\omega_X) \cong \mathbb{C}. \tag{2.2}$$

If $(\alpha,\eta) \in H^1(C^{\cdot}([\cdot,\phi]))$ and $(\beta,\zeta) \in H^1(\check{C}^{\cdot}([\phi,\cdot]))$ are represented by cocycles $(\{\alpha_{i_0,\dots,i_{n-1}}\},\{\eta_{j_0,j_1}\})$ and $(\{\beta_{i_0,\dots,i_{n-1}}\},\{\zeta_{j_0,j_1}\})$, respectively, it is easy to verify that

$$\{\mathrm{tr}(\alpha_{i_0,\dots,i_{n-1}}\zeta_{i_{n-1},i_n} + \eta_{i_0,i_1}\beta_{i_1,\dots,i_n})\}_{i_0,\dots,i_n}$$

defines a n-cocycle with values in ω_X, hence determines an element of $H^n(X,\omega_X) \cong \mathbb{C}$. It follows that the duality pairing (2.2) may be written explicitly as follows:

$$\begin{aligned}\langle(\alpha,\eta),(\beta,\zeta)\rangle &= \{\mathrm{tr}(\alpha_{i_0,\dots,i_{n-1}}\zeta_{i_{n-1},i_n} + \eta_{i_0,i_1}\beta_{i_1,\dots,i_n})\}\\ &= \mathrm{tr}(\alpha \cup \zeta + \eta \cup \beta),\end{aligned}$$

where we denote by $\cup$ the “cup product”.

Remark 2.13. We can also globalize the preceding constructions to the whole tangent and cotangent bundles to $\mathcal{P}(L)$.

For simplicity, let us denote by $\mathcal{E}nd(\mathcal{E})$ the sheaf on $\mathcal{P}(L) \times X$ which was previously denoted by $\mathcal{E}nd(\mathcal{E}')$, in Remark 1.11. Let $\bar{\Phi}$ be the canonical section of $R^{n-1}q_*(\mathcal{E}nd(\mathcal{E}) \otimes p^*L)$, where we now denote by $p : \mathcal{P}(L) \times X \to X$ and $q : \mathcal{P}(L) \times X \to \mathcal{P}(L)$ the canonical projections. The section $\bar{\Phi}$ can be represented by a Čech $(n-1)$-cocycle with values in $\mathcal{E}nd(\mathcal{E}) \otimes p^*L$, with respect to a suitable affine open covering of $\mathcal{P}(L) \times X$. Let us consider the resolutions of the sheaves $\mathcal{E}nd(\mathcal{E})$ and $\mathcal{E}nd(\mathcal{E}) \otimes p^*L$ given by the Čech complexes of sheaves $\mathcal{C}^{\cdot}(\mathcal{E}nd(\mathcal{E}))$ and $\mathcal{C}^{\cdot}(\mathcal{E}nd(\mathcal{E}) \otimes p^*L)$.

We are now in a position to define a sheafified version of the complex $C^{\cdot}([\cdot,\phi])$. We set

$$\mathcal{C}^i([\cdot,\Phi]) = \mathcal{C}^i(\mathcal{E}nd(\mathcal{E})) \oplus \mathcal{C}^{i+n-2}(\mathcal{E}nd(\mathcal{E}) \otimes p^*L),$$

with coboundary d^i given by

$$d^i = \begin{pmatrix} \delta^i & 0 \\ [\cdot,\Phi] & -\delta^{i+n-2} \end{pmatrix}.$$

Note that this is a complex of sheaves on $\mathcal{P}(L) \times X$; its restriction to $\{(E,\bar{\phi})\} \times X$ gives a sheafified version of the complex $C^{\cdot}([\cdot,\phi])$ on X.

From what we have previously seen, it follows that we have a canonical identification

$$T\mathcal{P}(L) \cong R^1q_*(\mathcal{C}^{\cdot}([\cdot,\Phi])). \tag{2.3}$$

In a similar way, we can define the global dual complex $\check{\mathcal{C}}^{\cdot}([\Phi,\cdot])$, and obtain a global isomorphism

$$T^*\mathcal{P}(L) \cong R^1 q_*(\check{\mathcal{C}}^{\cdot}([\Phi,\cdot])). \tag{2.4}$$

There is also an obvious global analogue of the explicit expression of the duality pairing given in Remark 2.12.

Let us now choose a non-zero section $s \in H^0(X, \omega_X^{-1} \otimes L)$. Multiplication by s and $-s$ respectively, induces homomorphisms of complexes

$$\mathcal{C}^{\cdot}(\mathcal{E}nd(\mathcal{E}) \otimes p^*\omega_X) \xrightarrow{s} \mathcal{C}^{\cdot}(\mathcal{E}nd(\mathcal{E}) \otimes p^*L)$$

and

$$\mathcal{C}^{\cdot}(\mathcal{E}nd(\mathcal{E}) \otimes p^*(L^{-1} \otimes \omega_X)) \xrightarrow{-s} \mathcal{C}^{\cdot}(\mathcal{E}nd(\mathcal{E})).$$

From these maps we obtain a homomorphism of complexes

$$B_s : \check{\mathcal{C}}^{\cdot}([\Phi,\cdot]) \to \mathcal{C}^{\cdot}([\cdot,\Phi]), \tag{2.5}$$

which, in turn, induces a homomorphism

$$B_s : R^1 q_*(\check{\mathcal{C}}^{\cdot}([\Phi,\cdot])) \to R^1 q_*(\mathcal{C}^{\cdot}([\cdot,\Phi])).$$

By recalling the natural identifications (2.3) and (2.4), we can define a contravariant 2-tensor $\theta_s \in H^0(\mathcal{P}(L), \otimes^2 T\mathcal{P}(L))$ by setting

$$\theta_s(w_1, w_2) = \langle w_1, B_s(w_2)\rangle,$$

for 1-forms w_1 and w_2 considered as sections of $R^1 q_*(\check{\mathcal{C}}^{\cdot}([\Phi,\cdot]))$, where $\langle\cdot,\cdot\rangle$ denotes the duality pairing between $T\mathcal{P}(L)$ and $T^*\mathcal{P}(L)$.

For any $(E, \bar{\phi}) \in \mathcal{P}(L)$, we obtain from (2.5) a homomorphism of complexes

$$B_s : \check{C}^{\cdot}([\phi,\cdot]) \to C^{\cdot}([\cdot,\phi]),$$

which determines a homomorphism

$$B_s : H^1(\check{C}^{\cdot}([\phi,\cdot])) \to H^1(C^{\cdot}([\cdot,\phi])). \tag{2.6}$$

By recalling the description of the cohomology groups in terms of Čech cocycles, the map (2.6) can be written as follows: for $(\alpha,\eta) = (\{\alpha_{i_0,\dots,i_{n-1}}\}, \{\eta_{j_0,j_1}\}) \in H^1(\check{C}^{\cdot}([\phi,\cdot]))$, we have

$$B_s(\alpha,\eta) = (\{s\alpha_{i_0,\dots,i_{n-1}}\}, \{-s\eta_{j_0,j_1}\}) \in H^1(C^{\cdot}([\cdot,\phi])).$$

It is now immediate to prove that B_s is skew-symmetric, hence θ_s is actually an antisymmetric contravariant 2-tensor, i.e., $\theta_s \in H^0(\mathcal{P}(L), \wedge^2 T\mathcal{P}(L))$.

Remark 2.14. If we suppose that $\deg(L) < \deg(\omega_X)$ and that there exists a non-zero global section s of $\omega_X \otimes L^{-1}$, we are in a situation similar to the one just described, but with all the arrows reversed.

Precisely, by multiplying by s and $-s$ respectively, we obtain two homomorphisms of complexes

$$\mathcal{C}^{\cdot}(\mathcal{E}nd(\mathcal{E}) \otimes p^*L) \xrightarrow{s} \mathcal{C}^{\cdot}(\mathcal{E}nd(\mathcal{E}) \otimes p^*\omega_X)$$

and

$$\mathcal{C}^{\cdot}(\mathcal{E}nd(\mathcal{E})) \xrightarrow{-s} \mathcal{C}^{\cdot}(\mathcal{E}nd(\mathcal{E}) \otimes p^*(L^{-1} \otimes \omega_X)),$$

hence a homomorphism

$$B_s : \mathcal{C}^{\cdot}([\cdot, \Phi]) \to \check{\mathcal{C}}^{\cdot}([\Phi, \cdot]).$$

This, in turn, determines a homomorphism

$$B_s : R^1q_*(\mathcal{C}^{\cdot}([\cdot, \Phi])) \to R^1q_*(\check{\mathcal{C}}^{\cdot}([\Phi, \cdot])),$$

which is equivalent to giving a 2-form $\omega_s \in H^0(\mathcal{P}(L), \wedge^2 T^*\mathcal{P}(L))$, defined by setting

$$\omega_s(\xi_1, \xi_2) = \langle \xi_1, B_s(\xi_2) \rangle,$$

for two vector fields ξ_1 and ξ_2, considered as sections of $R^1q_*(\mathcal{C}^{\cdot}([\cdot, \Phi]))$.

Remark 2.15. As a final remark we point out that all the constructions carried out in this section could be done, in a more intrinsic way, using the language of derived categories.

3. The map B_s

Let us study more closely the morphism

$$B_s : H^1(\check{C}^{\cdot}([\phi, \cdot])) \to H^1(C^{\cdot}([\cdot, \phi])).$$

The section $s \in H^0(X, \omega_X^{-1} \otimes L)$ defines an effective divisor D_s on X, such that $\mathcal{O}_X(D_s) = \omega_X^{-1} \otimes L$. For any sheaf F on X let us denote by F_{D_s} the sheaf $F \otimes \mathcal{O}_{D_s}$.

From the exact sequences

$$0 \to \mathcal{E}nd(E) \otimes \omega_X \xrightarrow{s} \mathcal{E}nd(E) \otimes L \to \mathcal{E}nd(E) \otimes L_{D_s} \to 0 \tag{3.1}$$

and

$$0 \to \mathcal{E}nd(E) \otimes L^{-1} \otimes \omega_X \xrightarrow{-s} \mathcal{E}nd(E) \to \mathcal{E}nd(E)_{D_s} \to 0, \tag{3.2}$$

we obtain an exact sequence of complexes

$$0 \to \check{C}^{\cdot}([\phi, \cdot]) \xrightarrow{B_s} C^{\cdot}([\cdot, \phi]) \to C^{\cdot}([\cdot, \phi]_{D_s}) \to 0, \tag{3.3}$$

where $C^{\cdot}([\cdot,\phi]_{D_s})$ is defined similarly to $C^{\cdot}([\cdot,\phi])$, by replacing the sheaves $\mathcal{E}nd(E)$ and $\mathcal{E}nd(E)\otimes L$ with their restrictions to D_s, $\mathcal{E}nd(E)_{D_s}$ and $\mathcal{E}nd(E)\otimes L_{D_s}$, respectively.

From the exact sequence (3.3) we obtain a long exact sequence of cohomology groups

(3.4)
$$\begin{aligned} &0 \to H^0(\check{C}^{\cdot}([\phi,\cdot])) \to H^0(C^{\cdot}([\cdot,\phi])) \to H^0(C^{\cdot}([\cdot,\phi]_{D_s})) \\ &\quad \to H^1(\check{C}^{\cdot}([\phi,\cdot])) \xrightarrow{B_s} H^1(C^{\cdot}([\cdot,\phi])) \to H^1(C^{\cdot}([\cdot,\phi]_{D_s})) \to \cdots . \end{aligned}$$

The stability of E implies that

$$H^0(C^{\cdot}([\cdot,\phi])) \cong H^{n-2}(X,\mathcal{E}nd(E)\otimes L)\oplus\mathbb{C},$$

and

$$\begin{aligned} &H^0(\check{C}^{\cdot}([\phi,\cdot])) \\ &\cong \begin{cases} \mathbb{C}\oplus H^{n-2}(X,\mathcal{E}nd(E)\otimes\omega_X) & \text{if } D_s=0, \text{ i.e. } L\cong\omega_X, \\ H^{n-2}(X,\mathcal{E}nd(E)\otimes\omega_X) & \text{if } D_s\neq 0. \end{cases} \end{aligned}$$

Analogously, if we suppose that the restriction of E to D_s is again stable, it follows that

$$H^0(C^{\cdot}([\cdot,\phi]_{D_s})) \cong H^{n-2}(D_s,\mathcal{E}nd(E)\otimes L_{D_s})\oplus\mathbb{C}.$$

From the long exact sequence (3.4) it follows that

$$\begin{aligned} &\ker(B_s : H^1(\check{C}^{\cdot}([\phi,\cdot])) \to H^1(C^{\cdot}([\cdot,\phi]))) \\ &\cong \frac{H^{n-2}(D_s,\mathcal{E}nd(E)\otimes L_{D_s})}{H^{n-2}(X,\mathcal{E}nd(E)\otimes L)}, \end{aligned}$$

which, in turn, is isomorphic to

$$\ker(H^{n-1}(X,\mathcal{E}nd(E)\otimes\omega_X) \xrightarrow{s} H^{n-1}(X,\mathcal{E}nd(E)\otimes L)),$$

as follows at once from the exact sequence (3.1).

Note that, if $L\cong\omega_X$, we obtain $\ker(B_s)=0$, which corresponds to the fact (obvious, from the very definition of B_s) that B_s is an isomorphism.

Remark 3.1. The situation is more complicated if we suppose that $H^0(X, \omega_X\otimes L^{-1})\neq 0$. In this case, in correspondence to the choice of a non-zero section $s\in H^0(X,\omega_X\otimes L^{-1})$, we get two exact sequences

$$0\to\mathcal{E}nd(E)\otimes L \xrightarrow{s} \mathcal{E}nd(E)\otimes\omega_X \to \mathcal{E}nd(E)\otimes\omega_X\otimes\mathcal{O}_{D_s}\to 0$$

and

$$0\to\mathcal{E}nd(E) \xrightarrow{-s} \mathcal{E}nd(E)\otimes L^{-1}\otimes\omega_X \to \mathcal{E}nd(E)\otimes L^{-1}\otimes\omega_X\otimes\mathcal{O}_{D_s}\to 0,$$

where D_s denotes the effective divisor defined by s.

These sequences determine an exact sequence of complexes

$$0 \to C^{\cdot}([\cdot,\phi]) \xrightarrow{B_s} \check{C}^{\cdot}([\phi,\cdot]) \to \check{C}^{\cdot}([\phi,\cdot]_{D_s}) \to 0,$$

where $\check{C}^{\cdot}([\phi,\cdot]_{D_s})$ is the complex obtained from $\check{C}^{\cdot}([\phi,\cdot])$ by replacing the sheaves $\mathcal{E}nd(E)\otimes\omega_X$ and $\mathcal{E}nd(E)\otimes L^{-1}\otimes\omega_X$ with their restrictions to D_s.

The corresponding long exact cohomology sequence

$$\begin{aligned} 0 &\to H^0(C^{\cdot}([\cdot,\phi])) \to H^0(\check{C}^{\cdot}([\phi,\cdot])) \to H^0(\check{C}^{\cdot}([\phi,\cdot]_{D_s})) \\ &\to H^1(C^{\cdot}([\cdot,\phi])) \xrightarrow{B_s} H^1(\check{C}^{\cdot}([\phi,\cdot])) \to H^1(\check{C}^{\cdot}([\phi,\cdot]_{D_s})) \to \cdots \end{aligned}$$

may be used to study the kernel of the map B_s, i.e., the degeneracy locus of the corresponding 2-form ω_s.

4. Vector fields on $\mathcal{P}(L)$

In this section we extend to the n-dimensional case the results proved in [Bo1, Sect. 3.3]. The following, somewhat technical, results are needed for our subsequent computations. Since most of what follows is a rather straightforward generalization of what we proved in *loc. cit.*, the proofs of the following results are only sketched.

Let us start by recalling some general facts. If Y is a k-scheme, a tangent vector field on Y may be thought of as an automorphism over $\mathrm{Spec}(k[\epsilon])$

$$\begin{tikzcd} Y\times\mathrm{Spec}(k[\epsilon]) \arrow[rr, "D"] \arrow[dr] & & Y\times\mathrm{Spec}(k[\epsilon]) \arrow[dl] \\ & \mathrm{Spec}(k[\epsilon]), & \end{tikzcd}$$

that restricts to the identity morphism of Y when one looks at the fibers over $\mathrm{Spec}(k)$.

Let now D be a tangent vector field on $\mathcal{P}(L)$. If we denote by $(\mathcal{E},\bar{\Phi})$ the local universal family on $\mathcal{P}(L)\times X$, and by $(\mathcal{E}[\epsilon],\bar{\Phi}[\epsilon])$ its pull-back to $\mathcal{P}(L)\times\mathrm{Spec}(k[\epsilon])\times X$, the vector field D may be described, locally, by giving the infinitesimal deformation $(\mathcal{E}_\epsilon,\bar{\Phi}_\epsilon)=(D\times\mathrm{id}_X)^*(\mathcal{E}[\epsilon],\bar{\Phi}[\epsilon])$ of the local universal family $(\mathcal{E},\bar{\Phi})$. At a point $(E,\bar{\phi})\in\mathcal{P}(L)$, the corresponding tangent vector is given by $(E_\epsilon,\bar{\phi}_\epsilon)=(\mathcal{E}_\epsilon,\bar{\Phi}_\epsilon)|_{\{(E,\bar{\phi})\}\times X}$, which is an infinitesimal deformation of the pair $(E,\bar{\phi})$.

From what we have seen in Sect. 2, the tangent field given by $(\mathcal{E}_\epsilon,\bar{\Phi}_\epsilon)$ corresponds to a global section $(\alpha,\eta)=(\{\alpha_{i_0,\dots,i_{n-1}}\},\{\eta_{j_0,j_1}\})$ of $R^1q_*(\mathcal{C}^{\cdot}([\cdot,$

$\Phi]))$. We shall see in the sequel how this section can be expressed in terms of first order differential operators.

First we need another general fact. Let $\pi : X \to Y$ be a morphism (locally of finite presentation) of schemes, and F, G two locally free sheaves on X. We denote by $\mathcal{D}\mathit{iff}^{\,1}_{X/Y}(F,G)$ the sheaf of relative differential operators from F to G of order ≤ 1. There is an exact sequence (cf. [EGA, Ch. IV, §16.8])

$$0 \to \mathcal{H}\mathit{om}(F,G) \to \mathcal{D}\mathit{iff}^{\,1}_{X/Y}(F,G) \xrightarrow{\sigma} \mathcal{D}\mathit{er}_Y(\mathcal{O}_X) \otimes \mathcal{H}\mathit{om}(F,G) \to 0,$$

where σ is the "symbol" morphism. Then, if $F = G$ and we restrict to differential operators with scalar symbol, which we denote by $\mathcal{D}^1_{X/Y}(F)$, we get the exact sequence

$$0 \to \mathcal{E}\mathit{nd}(F) \to \mathcal{D}^1_{X/Y}(F) \xrightarrow{\sigma} \mathcal{D}\mathit{er}_Y(\mathcal{O}_X) \to 0. \tag{4.1}$$

Now we shall apply these results to the morphism $p : \mathcal{P}(L) \times X \to X$. The idea is to take as F the universal family $\mathcal{E}$ on $\mathcal{P}(L) \times X$; actually it may not exist but, as already remarked, the sheaf $\mathcal{E}\mathit{nd}(\mathcal{E})$ is always defined. By a similar argument, it follows that also the sheaf $\mathcal{D}^1_X(\mathcal{E}) = \mathcal{D}^1_{\mathcal{P}(L)\times X/X}(\mathcal{E})$, of first order differential operators with scalar symbol which are $p^*(\mathcal{O}_X)$-linear, is always defined. From (4.1) we thus obtain the exact sequence

$$0 \to \mathcal{E}\mathit{nd}(\mathcal{E}) \to \mathcal{D}^1_X(\mathcal{E}) \to q^*T\mathcal{P}(L) \to 0, \tag{4.2}$$

where $q : \mathcal{P}(L) \times X \to \mathcal{P}(L)$ is the canonical projection.

Exactly as in Sect. 2, we can consider the resolutions of the sheaves $\mathcal{E}\mathit{nd}(\mathcal{E})$ and $\mathcal{D}^1_X(\mathcal{E})$ given by the Čech complexes $\mathcal{C}^\cdot(\mathcal{E}\mathit{nd}(\mathcal{E}))$ and $\mathcal{C}^\cdot(\mathcal{D}^1_X(\mathcal{E}))$. In this case too there is a well-defined map

$$[\cdot,\Phi] : \mathcal{C}^i(\mathcal{D}^1_X(\mathcal{E})) \to \mathcal{C}^{i+n-1}(\mathcal{E}\mathit{nd}(\mathcal{E}) \otimes p^*L).$$

(To this respect, note that if $\dot{D}$ is a first order differential operator, then $[\dot{D},\Phi]$ is a differential operator of order 0, hence a homomorphism of sheaves.)

It follows that we can define a new complex $\mathcal{D}^\cdot([\cdot,\Phi])$ by setting

$$\mathcal{D}^i([\cdot,\Phi]) = \mathcal{C}^i(\mathcal{D}^1_X(\mathcal{E})) \oplus \mathcal{C}^{i+n-2}(\mathcal{E}\mathit{nd}(\mathcal{E}) \otimes p^*L),$$

with coboundary given by

$$d^i = \begin{pmatrix} \delta^i & 0 \\ [\cdot,\Phi] & -\delta^{i+n-2} \end{pmatrix}.$$

Now, from the exact sequence (4.2), we deduce that there is an exact sequence of complexes

$$0 \to \mathcal{C}^\cdot([\cdot,\Phi]) \to \mathcal{D}^\cdot([\cdot,\Phi]) \to \mathcal{C}^\cdot(q^*T\mathcal{P}(L)) \to 0,$$

where $\mathcal{C}^{\cdot}(q^*T\mathcal{P}(L))$ is a suitable Čech resolution of the sheaf $q^*T\mathcal{P}(L)$ (this result could be better stated in the derived category $D^b(\mathcal{P}(L)\times X)$; in this case we actually have an exact sequence of complexes

$$0 \to \mathcal{C}^{\cdot}([\cdot,\Phi]) \to \mathcal{D}^{\cdot}([\cdot,\Phi]) \to q^*T\mathcal{P}(L) \to 0,$$

where $q^*T\mathcal{P}(L)$ is regarded as a complex concentrated in degree 0).

By applying the functor q_*, and noting that $q_*q^*T\mathcal{P}(L) \cong T\mathcal{P}(L)$, since q is a proper morphism, we get a long exact sequence, a piece of which is

$$(4.3) \qquad \cdots \to T\mathcal{P}(L) \to R^1q_*(\mathcal{C}^{\cdot}([\cdot,\Phi])) \to R^1q_*(\mathcal{D}^{\cdot}([\cdot,\Phi])) \to \cdots.$$

It is evident that the map $T\mathcal{P}(L) \to R^1q_*(\mathcal{C}^{\cdot}([\cdot,\Phi]))$ coincides with the isomorphism (2.3), hence the image of $R^1q_*(\mathcal{C}^{\cdot}([\cdot,\Phi]))$ in $R^1q_*(\mathcal{D}^{\cdot}([\cdot,\Phi]))$ is zero. This means that for each section $(\{\alpha_{i_0,\dots,i_{n-1}}\},\{\eta_{j_0,j_1}\})$ of $R^1q_*(\mathcal{C}^{\cdot}([\cdot,\Phi]))$, there exist differential operators $\dot{D}_i$ and homomorphisms $\beta_{i_0,\dots,i_{n-2}}$, determining sections of $\mathcal{C}^0(\mathcal{D}^1_X(\mathcal{E}))$ and $\mathcal{C}^{n-2}(\mathcal{E}nd(\mathcal{E})\otimes p^*L)$ respectively, such that

$$\eta_{j_0,j_1} = \dot{D}_{j_1} - \dot{D}_{j_0}$$

and

$$\alpha_{i_0,\dots,i_{n-1}} = [\dot{D}_{i_0},\Phi_{i_0,\dots,i_{n-1}}] - (\delta\beta)_{i_0,\dots,i_{n-1}}.$$

Finally, by observing that the cocycles (α,η) and $(\alpha-\delta\beta,\eta)$ represent the same section of $R^1q_*(\mathcal{C}^{\cdot}([\cdot,\Phi]))$, we obtain a proof of the following result:

Proposition 4.1. *For any tangent vector field D on $\mathcal{P}(L)$, corresponding to an infinitesimal deformation $(\mathcal{E}_\epsilon,\bar{\Phi}_\epsilon)$ of $(\mathcal{E},\bar{\Phi})$, described by a global section (α,η) of $R^1q_*(\mathcal{C}^{\cdot}([\cdot,\Phi]))$, there exist a suitable open affine covering $\mathcal{U} = (U_i)_{i\in I}$ of $\mathcal{P}(L)\times X$ and first order differential operators $\dot{D}_i \in \Gamma(U_i,\mathcal{D}^1_X(\mathcal{E}))$ such that the section (α,η) is represented by a Čech cocycle $(\{\alpha_{i_0,\dots,i_{n-1}}\},\{\eta_{j_0,j_1}\})$, with*

$$\alpha_{i_0,\dots,i_{n-1}} = [\dot{D}_{i_0},\Phi_{i_0,\dots,i_{n-1}}]$$

and

$$\eta_{j_0,j_1} = \dot{D}_{j_1} - \dot{D}_{j_0}.$$

To end this section, let us remark that all the considerations made in [Bo1, Remark 3.3.4], concerning a different proof of the analogue of Proposition 4.1, and also those expressed in Remark 3.3.5 of *loc. cit.*, can be generalized to the present situation. We leave the details to the reader.

5. Symplectic and Poisson structures

We recall here some definitions and results of symplectic geometry that will be used in the sequel.

Let Y be a smooth algebraic variety over the complex field $\mathbb{C}$. A symplectic structure on Y is a closed nondegenerate 2-form $\omega \in H^0(Y, \Omega^2_Y)$. Given a symplectic structure ω, the Hamiltonian vector field H_f of a regular function f is defined by requiring that $\omega(H_f, v) = \langle df, v\rangle$, for every tangent vector field v. Then we define the Poisson bracket of two regular functions f and g on Y by setting

$$\{f, g\} = \langle H_f, dg\rangle = \omega(H_g, H_f).$$

The pairing $\{\cdot,\cdot\}$ on $\mathcal{O}_Y$ is a bilinear antisymmetric map that is a derivation in each entry and satisfies the Jacobi identity

$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0, \tag{5.1}$$

for any $f, g, h \in \Gamma(U, \mathcal{O}_Y)$. This implies that $[H_f, H_g] = H_{\{f,g\}}$, where $[u, v] = uv - vu$ is the commutator of the vector fields u and v.

Example 5.1. The cotangent bundle T^*Y of a smooth variety Y has a canonical symplectic structure, defined as follows. Let $\pi : T^*Y \to Y$ be the structure morphism. By restricting the cotangent morphism to π,

$$T^*\pi : \pi^*T^*Y = T^*Y \times_Y T^*Y \to T^*T^*Y,$$

to the diagonal of $T^*Y \times_Y T^*Y$, we get a map $T^*Y \to T^*T^*Y$, which is a section of the bundle $T^*T^*Y \to T^*Y$, i.e., a differential form of degree 1 on T^*Y. This is the canonical 1-form on T^*Y, denoted by α_Y. The closed 2-form $\omega = -d\alpha_Y$ is the canonical symplectic form on T^*Y.

A natural generalization of symplectic structures is given by the notion of a Poisson structure.

A Poisson structure on Y is a Lie algebra structure $\{\cdot,\cdot\}$ on $\mathcal{O}_Y$ satisfying the identity $\{f, gh\} = \{f, g\}h + g\{f, h\}$. Equivalently, this is given by an antisymmetric contravariant 2-tensor $\theta \in H^0(Y, \wedge^2 TY)$, where we set

$$\{f, g\} = \langle \theta, df \wedge dg\rangle.$$

Then θ is a Poisson structure if the bracket it defines satisfies the Jacobi identity (5.1). When θ has maximal rank everywhere, we say that the Poisson structure is symplectic. In fact, in this case, to give θ is equivalent to giving its inverse 2-form $\omega \in H^0(Y, \Omega^2_Y)$, i.e., a symplectic structure on Y.

The following construction generalizes to the Poisson case the canonical symplectic structure of the cotangent bundle of a smooth variety.

Let Y be a smooth variety and $\mathfrak{G}$ a locally free $\mathcal{O}_Y$-module endowed with a structure of a locally free sheaf of Lie algebras over $\mathbb{C}$. We shall regard $\mathfrak{G}$ as a vector bundle over Y. Let $u : \mathfrak{G} \to TY$ be a homomorphism for the structures of $\mathcal{O}_Y$-modules and of sheaves of Lie algebras, satisfying the following compatibility condition between the two structures:

$$[\xi, f\,\zeta] = f\,[\xi,\zeta] + u(\xi)(f)\,\zeta, \tag{5.2}$$

for any $f \in \Gamma(U, \mathcal{O}_Y)$ and any $\xi, \zeta \in \Gamma(U, \mathfrak{G})$, where $[\cdot,\cdot]$ denotes the Lie bracket operation on $\mathfrak{G}$. Let $\mathfrak{G}^*$ be the dual of $\mathfrak{G}$.

In this situation there is a canonical Poisson structure on $\mathfrak{G}^*$, considered as a variety over Y. First we note that $\mathcal{O}_{\mathfrak{G}^*} = \mathrm{Sym}_{\mathcal{O}_Y}(\mathfrak{G})$, the symmetric algebra of $\mathfrak{G}$ over $\mathcal{O}_Y$. Then, for any open subset $U \subset Y$ and sections $\xi, \zeta \in \Gamma(U, \mathfrak{G})$ and $f, g \in \Gamma(U, \mathcal{O}_Y)$, we set

$$\begin{aligned} \{\xi,\zeta\} &= [\xi,\zeta], \\ \{\xi, f\} &= u(\xi)(f), \\ \{f, g\} &= 0, \end{aligned} \tag{5.3}$$

and extend $\{\cdot,\cdot\}$ to all of $\mathcal{O}_{\mathfrak{G}^*}$ by linearity and by using Leibnitz rule for the product of two elements.

The following result follows easily:

Proposition 5.2. *The bracket $\{\cdot,\cdot\}$ on $\mathfrak{G}^*$ is a Poisson bracket.*

The corresponding Poisson structure on the vector bundle $\mathfrak{G}^*$ is called the canonical Poisson structure associated to the sheaf of Lie algebras $\mathfrak{G}$ and the homomorphism $u : \mathfrak{G} \to TY$.

For further details on this construction, we refer to [Bo1]. To end this section we note that, if we take as $\mathfrak{G}$ the tangent bundle TY and as u the identity morphism, the canonical Poisson structure on $\mathfrak{G}^* = T^*Y$ defined above coincides with the canonical symplectic structure on the cotangent bundle of Y.

6. Poisson structures on $\mathcal{P}(L)$

In Sect. 2 we used the map

$$B_s : R^1 q_*(\check{\mathcal{C}}^{\cdot}([\Phi,\cdot])) \to R^1 q_*(\mathcal{C}^{\cdot}([\cdot,\Phi]))$$

to define an antisymmetric contravariant 2-tensor

$$\theta_s \in H^0(\mathcal{P}(L), \wedge^2 T\mathcal{P}(L)).$$

By what we have previously seen, to prove that θ_s is a Poisson structure it remains to prove that the corresponding bracket satisfies the Jacobi identity (5.1).

Let us begin by considering the case $L = \omega_X$. In this case the variety $\mathcal{P}(L) = \mathcal{P}(\omega_X)$ coincides with the cotangent bundle of M'. As we have seen, the choice of the identity section $s = 1$ of $H^0(X, \omega_X^{-1} \otimes \omega_X)$ determines an isomorphism

$$B_1 : T^*\mathcal{P}(\omega_X) \to T\mathcal{P}(\omega_X).$$

As in the case of curves we have the following result:

Theorem 6.1. *The antisymmetric contravariant* 2*-tensor* θ_1 *on* $\mathcal{P}(\omega_X)$ *defines a Poisson structure, that is symplectic and coincides with the canonical symplectic structure on* T^*M'*, under the natural identification* $\mathcal{P}(\omega_X) \cong T^*M'$*.*

Proof. The proof is an adaptation to the higher dimensional case of the proof of Theorem 4.5.1 of [Bo1]. We only sketch here the relevant modifications.

The canonical 1-form $\alpha_{\mathcal{P}(\omega_X)} : \mathcal{P}(\omega_X) \to T^*\mathcal{P}(\omega_X)$ coincides with the global section $(\bar{\Phi}, 0)$ of $R^1 q_*(\check{\mathcal{C}}^{\cdot}([\Phi, \cdot]))$, defined as the image of $\bar{\Phi}$ by the natural map

$$\pi^*(\mathcal{H}) \to R^1 q_*(\check{\mathcal{C}}^{\cdot}([\Phi, \cdot])).$$

In terms of Čech cocycles, we can write, for any $(E, \bar{\phi}) \in \mathcal{P}(\omega_X)$,

$$\alpha_{\mathcal{P}(\omega_X)}(E, \bar{\phi}) = (\{\phi_{i_0,\dots,i_{n-1}}\}, 0),$$

where $\{\phi_{i_0,\dots,i_{n-1}}\}$ is a $(n-1)$-cocycle representing the cohomology class $\bar{\phi}$.

The canonical symplectic form on $\mathcal{P}(\omega_X)$ is then given by $\omega = -d\alpha_{\mathcal{P}(\omega_X)}$.

Let D^1 and D^2 be two tangent vector fields on $\mathcal{P}(\omega_X)$, represented respectively by the global sections (α^1, η^1) and (α^2, η^2) of $R^1 q_*(\mathcal{C}^{\cdot}([\cdot, \Phi]))$. From Proposition 4.1, it follows that there exist first order differential operators $\dot{D}_i^1$ and $\dot{D}_j^2$ such that

$$\alpha^h_{i_0,\dots,i_{n-1}} = [\dot{D}^h_{i_0}, \Phi_{i_0,\dots,i_{n-1}}]$$

and

$$\eta^h_{j_0,j_1} = \dot{D}^h_{j_1} - \dot{D}^h_{j_0},$$

where $(\{\alpha^h_{i_0,\dots,i_{n-1}}\}, \{\eta^h_{j_0,j_1}\})$, for $h = 1, 2$, are cocycles representing the global sections (α^h, η^h).

Exactly as in the case of curves, it follows that the second order differential operator $D^1 D^2$ is described by giving gluing isomorphisms of the form

$$1 + \epsilon\eta^1_{j_0,j_1} + \epsilon'\eta^2_{j_0,j_1} + \epsilon\epsilon'(\dot{D}^1_{j_1}\eta^2_{j_0,j_1} - \eta^2_{j_0,j_1}\dot{D}^1_{j_0}),$$

in terms of which the infinitesimal deformation of $\bar{\Phi}$ is locally written as

$$\Phi_{i_0,\dots,i_{n-1}} + \epsilon\alpha^1_{i_0,\dots,i_{n-1}} + \epsilon'\alpha^2_{i_0,\dots,i_{n-1}} + \epsilon\epsilon'[\dot{D}^1_{i_0}, \alpha^2_{i_0,\dots,i_{n-1}}].$$

It follows that the "infinitesimal deformation of the pair (α^2,η^2) along the vector field D^1" is given by the pair

$$(\{\alpha^2_{i_0,\dots,i_{n-1}}+\epsilon[\dot{D}^1_{i_0},\alpha^2_{i_0,\dots,i_{n-1}}]\},\{\eta^2_{j_0,j_1}+\epsilon(\dot{D}^1_{j_1}\eta^2_{j_0,j_1}-\eta^2_{j_0,j_1}\dot{D}^1_{j_0})\}).$$

Analogous considerations hold for the second order differential operator D^2D^1.

Finally, the vector field $[D^1,D^2]$ corresponds to the cocycle given by

$$(\{[[\dot{D}^1,\dot{D}^2]_{i_0},\Phi_{i_0,\dots,i_{n-1}}]\},\{[\dot{D}^1,\dot{D}^2]_{j_1}-[\dot{D}^1,\dot{D}^2]_{j_0}\}).$$

Using these expressions we are now able to compute explicitly the differential $d\alpha_{\mathcal{P}(\omega_X)}$, evaluated against the two vector fields D^1 and D^2. The computations are similar to those carried out in the proof of Theorem 4.5.1 of [Bo1]. We obtain:

$$d\alpha_{\mathcal{P}(\omega_X)}(D^1,D^2)=\{\mathrm{tr}(\eta^2_{i_0,i_1}\alpha^1_{i_1,\dots,i_n}-\eta^1_{i_0,i_1}\alpha^2_{i_1,\dots,i_n})\},\tag{6.1}$$

where this is regarded as a n-cocycle determining a cohomology class in $H^n(X,\omega_X)$. If we denote by $\cup$ the cup product, (6.1) can be written simply as follows:

$$d\alpha_{\mathcal{P}(\omega_X)}(D^1,D^2)=\mathrm{tr}(\eta^2\cup\alpha^1-\eta^1\cup\alpha^2).$$

Now we recall that, by the choice of the identity section $s=1$ of $H^0(X,\omega_X^{-1}\otimes\omega_X)$, we have defined an antisymmetric contravariant 2-tensor θ_1 on $\mathcal{P}(\omega_X)$. Its inverse 2-form ω_1 is defined by $\omega_1(D^1,D^2)=\langle D^1,B_1^{-1}(D^2)\rangle$.

Using the preceding notations, we have:

$$\begin{aligned}\omega_1(D^1,D^2)&=\langle(\alpha^1,\eta^1),B_1^{-1}(\alpha^2,\eta^2)\rangle\\&=\langle(\alpha^1,\eta^1),(\alpha^2,-\eta^2)\rangle\\&=\mathrm{tr}(-\alpha^1\cup\eta^2+\eta^1\cup\alpha^2),\end{aligned}$$

hence $\omega_1=-d\alpha_{\mathcal{P}(\omega_X)}$. This shows that ω_1 is precisely the canonical symplectic structure on $\mathcal{P}(\omega_X)\cong T^*M'$.

Remark 6.2. We have already seen in Remark 2.14 that, if $\deg(L)<\deg(\omega_X)$ and s is a section of $\omega_X\otimes L^{-1}$, there is a natural 2-form $\omega_s\in H^0(\mathcal{P}(L),\wedge^2T^*\mathcal{P}(L))$ defined by setting

$$\omega_s(D^1,D^2)=\langle D^1,B_s(D^2)\rangle,$$

where D^1 and D^2 are two tangent vector fields on $\mathcal{P}(L)$.

Let us consider the 1-form $\alpha_{\mathcal{P}(L)}:\mathcal{P}(L)\to T^*\mathcal{P}(L)$ determined by the global section $(s\bar{\Phi},0)$ of $R^1q_*(\check{\mathcal{C}}^\cdot([\Phi,\cdot]))$. This is the 1-form that associates to a point $(E,\bar{\phi})\in\mathcal{P}(L)$ the cohomology class of $(\{s\phi_{i_0,\dots,i_{n-1}}\},0)$ in $H^1(\check{C}^\cdot([\phi,\cdot]))$.

The computations carried out in the proof of Theorem 6.1 can be repeated, almost unchanged, to prove that $\omega_s = -d\alpha_{\mathcal{P}(L)}$. It follows, in particular, that ω_s is a *closed* 2-form on $\mathcal{P}(L)$. However, it is not a symplectic form, in general, because it may be degenerate.

Now we come to the general case. Let us suppose that $L = \omega_X(D_s)$, where D_s is an effective divisor defined by the non-zero section s. The variety $\mathcal{P}(L)$ is the total space of the vector bundle $\mathcal{H} = R^{n-1}q_*(\mathcal{E}nd(\mathcal{E}) \otimes p^*L)$, whose (relative) dual is $\mathcal{H}^* = R^1q_*(\mathcal{E}nd(\mathcal{E}) \otimes p^*(L^{-1} \otimes \omega_X))$, which, by abuse of notation, we shall denote simply by $R^1q_*(\mathcal{E}nd(\mathcal{E}) \otimes \mathcal{O}_X(-D_s))$.

From the discussion made at the end of Sect. 5, it follows that the data of a structure of sheaf of Lie algebras on $\mathcal{H}^*$ plus a homomorphism $u : \mathcal{H}^* \to TM'$ satisfying the compatibility condition (5.2), determine a Poisson structure on $\mathcal{P}(L)$.

If $L = \omega_X$ this is easy to define. In fact, in this case we have $\mathcal{H}^* = R^1q_*(\mathcal{E}nd(\mathcal{E}))$, and we have seen that for every section of $R^1q_*(\mathcal{E}nd(\mathcal{E}))$, represented by a cocycle $\{\eta_{j_0,j_1}\}$, there exist differential operators $\dot{D}_i$ such that $\eta_{j_0,j_1} = \dot{D}_{j_1} - \dot{D}_{j_0}$.

By recalling the canonical isomorphism $\mathcal{H}^* = R^1q_*(\mathcal{E}nd(\mathcal{E})) \cong TM'$, which we shall take as the homomorphism u, the Lie algebra structure of TM' can be transferred to $\mathcal{H}^*$. It follows that, if $\eta^1_{j_0,j_1} = \dot{D}^1_{j_1} - \dot{D}^1_{j_0}$ and $\eta^2_{j_0,j_1} = \dot{D}^2_{j_1} - \dot{D}^2_{j_0}$ represent two sections of $\mathcal{H}^*$, their Lie bracket is expressed by

$$\begin{aligned} [\{\eta^1_{j_0,j_1}\}, \{\eta^2_{j_0,j_1}\}] &= \{[\dot{D}^1, \dot{D}^2]_{j_1} - [\dot{D}^1, \dot{D}^2]_{j_0}\} \\ &= \{[\eta^1_{j_0,j_1}, \dot{D}^2_{j_1}] + [\dot{D}^1_{j_0}, \eta^2_{j_0,j_1}]\}. \end{aligned} \tag{6.2}$$

Needless to say, the Poisson structure we obtain in this way on $\mathcal{P}(\omega_X)$ is precisely the canonical symplectic structure on the cotangent bundle $T^*M' \cong \mathcal{P}(\omega_X)$, hence coincides with the one defined by the antisymmetric contravariant 2-tensor θ_1 (cf. Theorem 6.1).

In the general case $L = \omega_X(D_s)$, the Lie algebra structure on $\mathcal{H}^*$ can be defined exactly as in [Bo1]. Let us recall briefly this construction here.

From the exact sequences

$$0 \to \mathcal{E}nd(\mathcal{E}) \to \mathcal{D}^1_X(\mathcal{E}) \to q^*TM' \to 0$$

and

$$0 \to \mathcal{O}_X(-D_s) \xrightarrow{s} \mathcal{O}_X \to \mathcal{O}_{D_s} \to 0,$$

where the first one is the analogue of (4.2) for the moduli space M', we obtain a commutative diagram

$$\begin{tikzcd} TM' \otimes \mathcal{O}_X(-D_s) \arrow[r] \arrow[d, "s"] & R^1 q_*(\mathcal{E}nd(\mathcal{E}) \otimes \mathcal{O}_X(-D_s)) \arrow[d, "s"] \\ TM' \arrow[r, "\cong"] & R^1 q_*(\mathcal{E}nd(\mathcal{E})). \end{tikzcd}$$

If η^1 and η^2 are two global sections of $\mathcal{H}^*$, represented by the cocycles $\{\eta^1_{j_0,j_1}\}$ and $\{\eta^2_{j_0,j_1}\}$ with values in $R^1 q_*(\mathcal{E}nd(\mathcal{E}) \otimes \mathcal{O}_X(-D_s))$, we have $s\eta^1_{j_0,j_1} = \dot{D}^1_{j_1} - \dot{D}^1_{j_0}$ and $s\eta^2_{j_0,j_1} = \dot{D}^2_{j_1} - \dot{D}^2_{j_0}$, for some differential operators $\dot{D}^1_i$ and $\dot{D}^2_i$.

By recalling the formula (6.2), we are led to consider the cocycle $\{[s\eta^1_{j_0,j_1}, \dot{D}^2_{j_1}] + [\dot{D}^1_{j_0}, s\eta^2_{j_0,j_1}]\}$, with values in $R^1 q_*(\mathcal{E}nd(\mathcal{E}))$. Since the differential operators are $\mathcal{O}_X$-linear, it follows that $\{[s\eta^1_{j_0,j_1}, \dot{D}^2_{j_1}] + [\dot{D}^1_{j_0}, s\eta^2_{j_0,j_1}]\} = s\{[\eta^1_{j_0,j_1}, \dot{D}^2_{j_1}] + [\dot{D}^1_{j_0}, \eta^2_{j_0,j_1}]\}$, for a well-defined cocycle $\{[\eta^1_{j_0,j_1}, \dot{D}^2_{j_1}] + [\dot{D}^1_{j_0}, \eta^2_{j_0,j_1}]\}$ with values in $R^1 q_*(\mathcal{E}nd(\mathcal{E}) \otimes \mathcal{O}_X(-D_s))$. We thus define the Lie bracket of $\{\eta^1_{j_0,j_1}\}$ and $\{\eta^2_{j_0,j_1}\}$ by setting

$$[\eta^1, \eta^2] = \{[\eta^1_{j_0,j_1}, \dot{D}^2_{j_1}] + [\dot{D}^1_{j_0}, \eta^2_{j_0,j_1}]\},$$

a formula which is formally analogous to (6.2). Since the multiplication by s is injective at the level of cocycles, it follows that this defines a Lie algebra structure on $\mathcal{H}^*$. Finally, we take as $u : \mathcal{H}^* \to TM'$ the composition of $s : \mathcal{H}^* \to R^1 q_*(\mathcal{E}nd(\mathcal{E}))$ with the canonical isomorphism $R^1 q_*(\mathcal{E}nd(\mathcal{E})) \cong TM'$. It is trivial to verify that u is a homomorphism of sheaves of Lie algebras and satisfies the compatibility condition (5.2).

From this construction we thus obtain a Poisson structure $\{\cdot,\cdot\}$ on $\mathcal{P}(L)$. If we denote by $\{\cdot,\cdot\}_s$ the bracket defined, at the end of Sect. 2, by the antisymmetric contravariant 2-tensor θ_s, we have the following result, whose proof is similar to the proof of Theorem 4.6.3 of [Bo1]:

Theorem 6.3. *The bracket $\{\cdot,\cdot\}_s$ coincides with the bracket $\{\cdot,\cdot\}$, hence it defines a Poisson structure on $\mathcal{P}(L)$.*

Remark 6.4. The family of Poisson structures $\{\cdot,\cdot\}_s$, parametrized by the global sections of $\omega_X^{-1} \otimes L$, is compatible, in the sense that the sum of two Poisson structures in this family is again a Poisson structure. Precisely, we have $\{\cdot,\cdot\}_{s_1} + \{\cdot,\cdot\}_{s_2} = \{\cdot,\cdot\}_{s_1+s_2}$.

7. Higgs bundles and $(n-1)$-cohomology Higgs pairs

As we have already remarked, if X is a curve and $L = \omega_X$, the definition of a $(n-1)$-cohomology L-twisted Higgs pair coincides with the definition of a Higgs bundle, introduced by Hitchin in [Hi]. For higher dimensional varieties, however, this is not the case. In this section we shall analyze the relationships between the usual definition of Higgs bundles, as given for example in [S1], and our definition of $(n-1)$-cohomology pairs.

Let us begin by recalling the definition of Higgs bundles on higher dimensional varieties.

Definition 7.1. A Higgs bundle on an n-dimensional variety X is a pair (E, θ) consisting of a vector bundle E and a homomorphism of vector bundles $\theta : E \to E \otimes \Omega^1_X$, such that $\theta \wedge \theta = 0$, considered as a homomorphism from E to $E \otimes \Omega^2_X$.

The integrability condition $\theta \wedge \theta = 0$ implies that to a Higgs bundle (E, θ) there is associated the following *Dolbeault complex*:

$$0 \longrightarrow E \xrightarrow{\wedge\theta} E \otimes \Omega^1_X \xrightarrow{\wedge\theta} E \otimes \Omega^2_X \longrightarrow \cdots ,$$

whose hypercohomology is called the *Dolbeault cohomology* with coefficients in E, denoted by $H^*_{\mathrm{Dol}}(X, E)$.

There is an obvious notion of stability for Higgs bundles, obtained by considering, in the usual definition of stability, only subsheaves F of E that are fixed by the Higgs field θ, i.e. such that $\theta(F) \subset F \otimes \Omega^1_X$. In particular, this implies that if E is a stable vector bundle, then (E, θ) is a stable Higgs bundle, for any Higgs field θ. This definition of stability leads to the construction of moduli spaces of (semi)stable Higgs bundles on X (see [S2]).

Let us now fix some "moduli data" and denote by $\mathcal{M}$ the moduli space of Higgs bundles (E, θ) with E stable. If we drop the integrability condition $\theta \wedge \theta = 0$ in the definition of a Higgs bundle, we may construct another moduli space $\mathcal{N}$, parametrizing all pairs (E, θ) with E stable and without any condition on θ, containing $\mathcal{M}$ as a closed subset.

Let us consider the following complexes:

$$C^{\cdot}_\theta : \quad 0 \longrightarrow \mathcal{E}nd(E) \xrightarrow{[\cdot,\theta]} \mathcal{E}nd(E) \otimes \Omega^1_X \xrightarrow{[\cdot,\theta]} \mathcal{E}nd(E) \otimes \Omega^2_X \xrightarrow{[\cdot,\theta]} \cdots ,$$

and

$$D^{\cdot}_\theta : \quad 0 \longrightarrow \mathcal{E}nd(E) \xrightarrow{[\cdot,\theta]} \mathcal{E}nd(E) \otimes \Omega^1_X \longrightarrow 0.$$

By standard infinitesimal deformation computations, analogous to the ones carried out in [Bo1], we can prove that there are canonical identifications

$$T_{(E,\theta)}\mathcal{M} \cong \mathbb{H}^1(C^{\cdot}_\theta) \tag{7.1}$$

and

$$T_{(E,\theta)}\mathcal{N} \cong \mathbb{H}^1(D^{\cdot}_\theta). \tag{7.2}$$

Moreover, the morphism of complexes $C^{\cdot}_\theta \to D^{\cdot}_\theta$ defined by

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{E}nd(E) \arrow[r, "{[\cdot,\theta]}"] \arrow[d, "\cong"] & \mathcal{E}nd(E)\otimes\Omega^1_X \arrow[r, "{[\cdot,\theta]}"] \arrow[d, "\cong"] & \mathcal{E}nd(E)\otimes\Omega^2_X \arrow[r, "{[\cdot,\theta]}"] \arrow[d] & \cdots \\
0 \arrow[r] & \mathcal{E}nd(E) \arrow[r, "{[\cdot,\theta]}"] & \mathcal{E}nd(E)\otimes\Omega^1_X \arrow[r] & 0 \arrow[r] & \cdots,
\end{tikzcd}$$

induces a homomorphism on the hypercohomology groups

$$\mathbb{H}^1(C^{\cdot}_\theta) \to \mathbb{H}^1(D^{\cdot}_\theta),$$

which, under the identifications (7.1) and (7.2), coincides with the differential of the natural inclusion $\mathcal{M} \hookrightarrow \mathcal{N}$.

Let us now come to the relationships between the moduli spaces $\mathcal{N}$ and $\mathcal{P}(\omega_X)$. The fundamental result is contained in the following proposition:

Proposition 7.2. *Let X be an n-dimensional variety and let us fix an ample class $\xi \in H^1(X,\Omega^1_X)$. If E is a polystable vector bundle on X with $c_1(E) = c_2(E) = 0$, then, for every $i,j \geq 0$ with $i+j \leq n$, the cup-product with $\xi^{n-(i+j)}$ determines an isomorphism*

$$H^i(X, E\otimes\Omega^j_X) \xrightarrow{\sim} H^{n-j}(X, E\otimes\Omega^{n-i}_X).$$

Proof. This result follows easily from the Lemma 2.6 of [S1]. The vector bundle E is, in fact, a harmonic bundle, since it is polystable (i.e. a direct sum of stable bundles of the same slope) and has vanishing first and second Chern classes. Hence we may apply Simpson's lemma to the Higgs bundle $(E,0)$, obtaining isomorphisms

$$H^p_{\mathrm{Dol}}(X,E) \xrightarrow{\sim} H^{2n-p}_{\mathrm{Dol}}(X,E) \tag{7.3}$$

giving by cupping with ξ^{n-p}.

Since the Higgs field θ is zero, the Dolbeault cohomology decomposes as follows:

$$H^p_{\mathrm{Dol}}(X,E) = \bigoplus_{\substack{0\leq i,j\leq n \\ i+j=p}} H^i(X, E\otimes\Omega^j_X),$$

for $p = 0,\ldots,2n$. By combining this decomposition with the isomorphisms (7.3), we conclude our proof.

We recall now that a stable vector bundle E on X admits a Hermitian-Einstein metric. The metric induced on $\mathcal{E}nd(E)$ is again Hermitian-Einstein, hence the vector bundle $\mathcal{E}nd(E)$ is polystable, obviously with vanishing first Chern class. From now on we shall also suppose that $c_2(\mathcal{E}nd(E)) = 0$, in order to be able to apply Proposition 7.2 to this vector bundle. We thus obtain an isomorphism

$$H^0(X, \mathcal{E}nd(E) \otimes \Omega_X^1) \xrightarrow{\sim} H^{n-1}(X, \mathcal{E}nd(E) \otimes \omega_X), \tag{7.4}$$

given by cupping with ξ^{n-1}. This, in turn, determines an isomorphism of moduli spaces

$$\mathcal{N} \xrightarrow{\sim} \mathcal{P}(\omega_X), \tag{7.5}$$

defined by sending a pair (E, θ) to the $(n-1)$-cohomology Higgs pair $(E, \theta \cup \xi^{n-1})$.

On the moduli space $\mathcal{N}$ there is a symplectic structure, depending on the choice of the ample class $\xi \in H^1(X, \Omega_X^1)$: this is constructed in [Bi] only for the moduli space $\mathcal{M}$ of Higgs bundles, but it is easy to see that an analogous symplectic structure can be defined on $\mathcal{N}$ (needless to say, the symplectic structure constructed by Biswas on $\mathcal{M}$ is then the restriction of the analogous one defined on $\mathcal{N}$). It is now easy to see that this symplectic structure on $\mathcal{N}$ coincides, under the isomorphism (7.5), with the canonical symplectic structure previously constructed on $\mathcal{P}(\omega_X)$, which, in turn, coincides with the canonical symplectic structure on the cotangent bundle T^*M' (see Theorem 6.1).

8. Parabolic bundles

In this section we discuss the relations between the moduli space $\mathcal{P}(L)$, where $L \cong \omega_X(D)$ for an effective divisor D, and the moduli space of parabolic vector bundles, with parabolic structure over D.

Let X be, as usual, a smooth n-dimensional projective variety defined over $\mathbb{C}$, with a very ample invertible sheaf $\mathcal{O}_X(1)$, and let $D \subset X$ be an effective divisor. Let $L \cong \omega_X(D)$ be an invertible sheaf and $s \in H^0(X, \omega_X^{-1} \otimes L)$ a section defining the divisor D.

We briefly recall here the definition of a parabolic sheaf, as given in [MY]:

Definition 8.1. A parabolic structure over D on a coherent, torsion-free, $\mathcal{O}_X$-module E is the data of a filtration

$$F_* : \quad E = F_1(E) \supset F_2(E) \supset \cdots \supset F_l(E) \supset F_{l+1}(E) = E(-D),$$

where $E(-D)$ is the image of $E \otimes_{\mathcal{O}_X} \mathcal{O}_X(-D) \to E$, and a sequence of real numbers $\alpha_* = (\alpha_1, \ldots, \alpha_l)$, called weights, such that

$$0 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_l < 1.$$

A parabolic sheaf is a coherent, torsion-free, $\mathcal{O}_X$-module E with a parabolic structure over D.

Given a parabolic sheaf (E, F_*, α_*), we define a filtered sheaf E_x, for $0 \leq x \leq 1$, by setting $E_0 = E$ and $E_x = F_i(E)$ if $\alpha_{i-1} < x \leq \alpha_i$, where we have set $\alpha_0 = 0$ and $\alpha_{l+1} = 1$. The definition of E_x can also be extended to all $x \in \mathbb{R}$ by setting $E_{x+1} = E_x(-D)$.

The filtered sheaf $E_* = (E_x)_{x\in\mathbb{R}}$ contains all the data necessary to recover the original parabolic sheaf (E, F_*, α_*), hence, in the sequel, it will be convenient to denote a parabolic sheaf simply by E_*. This notation will be particularly useful in the definition of homomorphisms of parabolic sheaves.

Remark 8.2. Some authors (cf. [B]) define a parabolic structure over D on a sheaf E as a sequence of subsheaves of $E|_D$

$$E|_D = \mathcal{F}_D^1(E) \supset \mathcal{F}_D^2(E) \supset \cdots \supset \mathcal{F}_D^l(E) \supset \mathcal{F}_D^{l+1}(E) = 0,$$

together with a system of weights $0 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_l < 1$.

Our definition is related to this one by setting

$$F_i(E) = \ker(E \to E|_D/\mathcal{F}_D^i(E)).$$

We come now to the definition of homomorphisms of parabolic sheaves.

Definition 8.3. A homomorphism of parabolic sheaves $\phi : E_* \to F_*$ is a homomorphism of $\mathcal{O}_X$-modules $\phi : E \to F$ such that $\phi(E_x) \subset F_x$, for any $x \in [0, 1]$.

We shall denote by $\mathcal{P}ar\mathcal{H}om(E_*, F_*)$ the sheaf of homomorphisms of parabolic sheaves from E_* to F_*; it is a subsheaf of $\mathcal{H}om(E, F)$.

In order to construct moduli spaces of parabolic sheaves we need, as usual, a suitable notion of stability. This was introduced in [MY], where moduli spaces of semistable parabolic sheaves were constructed in great generality. We only state here the result we shall need in the sequel.

Proposition 8.4. *Let us fix a sequence of real numbers $\alpha_* = (\alpha_1, \ldots, \alpha_l)$ with $0 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_l < 1$, and polynomials $H, H_1, \ldots, H_l$. Then there exists a quasi-projective moduli space $\mathcal{M}_{par}$ parametrizing isomorphism classes of stable parabolic vector bundles E_* having α_* as system of weights and such that the Hilbert polynomial of E is H and the Hilbert polynomial of $E/F_{i+1}(E)$ is H_i, for $i = 1, \ldots, l$.*

By infinitesimal deformation theory (cf. [Y]), it follows that the tangent space to the moduli space $\mathcal{M}_{par}$ at a point E_* is canonically isomorphic to the first cohomology group $H^1(X, \mathcal{P}ar\mathcal{H}om(E_*, E_*))$. From this, by applying the version of Serre duality for parabolic vector bundles, proved in [Y, Proposition 3.7], it follows that the cotangent space to the moduli space $\mathcal{M}_{par}$ at a point E_* is canonically isomorphic to $H^{n-1}(X, \mathcal{P}ar\mathcal{H}om(E_*, \hat{E}_*) \otimes \omega_X(D))$, where $\hat{E}_*$ is the filtered sheaf defined by setting, for any $x \in [0,1]$,

$$\hat{E}_x = \begin{cases} E_x & \text{if } x \neq \alpha_i, \\ E_{\alpha_{i+1}} & \text{if } x = \alpha_i. \end{cases}$$

Remark 8.5. With the notations of Remark 8.2, if the parabolic structure of a vector bundle E is given by a filtration

$$E|_D = \mathcal{F}_D^1(E) \supset \mathcal{F}_D^2(E) \supset \cdots \supset \mathcal{F}_D^l(E) \supset \mathcal{F}_D^{l+1}(E) = 0,$$

and if E_* is the corresponding filtered sheaf, then a section ϕ of $\mathcal{P}ar\mathcal{H}om(E_*, \hat{E}_*)$ is a homomorphism $\phi : E \to E$ such that $\phi|_D$ is nilpotent with respect to the filtration of $E|_D$ given above, i.e., such that $\phi|_D(\mathcal{F}_D^i(E)) \subset \mathcal{F}_D^{i+1}(E)$, for $i = 1, \dots, l$.

Unfortunately, the moduli space $\mathcal{M}_{par}$ is, in general, not smooth hence we shall restrict to consider its smooth locus $\mathcal{M}_{par}^{sm}$. Let us denote by $\mathcal{T}_{par}$ the cotangent bundle to the smooth locus of $\mathcal{M}_{par}$: $\mathcal{T}_{par} = T^*\mathcal{M}_{par}^{sm}$. By what we have previously seen, $\mathcal{T}_{par}$ can be described as the set of isomorphism classes of pairs $(E_*, \bar{\phi})$, where $E_* \in \mathcal{M}_{par}^{sm}$ and $\bar{\phi} \in H^{n-1}(X, \mathcal{P}ar\mathcal{H}om(E_*, \hat{E}_*) \otimes \omega_X(D))$.

Being the cotangent bundle to a smooth variety, $\mathcal{T}_{par}$ has a canonical symplectic structure (cf. Example 5.1). This can be described explicitly in a way very similar to the description of the Poisson structure of $\mathcal{P}(L)$ (we refer to [Bo1, Sect. 5] for the study of the symplectic structure of $\mathcal{T}_{par}$ when X is a curve).

Let $(E_*, \bar{\phi}) \in \mathcal{T}_{par}$. As in Sect. 2, we shall use infinitesimal deformation theory to describe the tangent space to $\mathcal{T}_{par}$ at $(E_*, \bar{\phi})$.

Let $\mathcal{U} = (U_i)_{i \in I}$ be a suitable affine open covering of X and let $\{\phi_{i_0,\dots,i_{n-1}}\}$ be a Čech $(n-1)$-cocycle in $C^{n-1}(\mathcal{U}, \mathcal{P}ar\mathcal{H}om(E_*, \hat{E}_*) \otimes \omega_X(D))$ representing the cohomology class $\bar{\phi}$. We define a complex $C^{\cdot}_{par}([\cdot, \phi])$ by setting

$$C^i_{par}([\cdot, \phi]) = C^i(\mathcal{P}ar\mathcal{H}om(E_*, E_*)) \oplus C^{i+n-2}(\mathcal{P}ar\mathcal{H}om(E_*, \hat{E}_*) \otimes \omega_X(D)),$$

with coboundary $d^i : C^i_{par}([\cdot, \phi]) \to C^{i+1}_{par}([\cdot, \phi])$ given by

$$d^i = \begin{pmatrix} \delta^i & 0 \\ [\cdot, \phi] & -\delta^{i+n-2} \end{pmatrix}, \tag{8.1}$$

where

$$[\cdot,\phi] : C^i(\mathcal{P}ar\mathcal{H}om(E_*,E_*)) \to C^{i+n-1}(\mathcal{P}ar\mathcal{H}om(E_*,\hat{E}_*)\otimes\omega_X(D))$$

is defined as in (2.1).

Now we can state the following result, whose proof is analogous to the proof of Corollary 2.5:

Proposition 8.6. *The tangent space $T_{(E_*,\bar{\phi})}\mathcal{T}_{par}$ to $\mathcal{T}_{par}$ at the point $(E_*,\bar{\phi})$ is canonically isomorphic to $H^1(C^{\cdot}_{par}([\cdot,\phi]))$.*

In order to describe the cotangent spaces to $\mathcal{T}_{par}$ we have to "dualize" the preceding construction. The foundamental tool is the version of Serre duality for parabolic bundles, proved in [Y, Proposition 3.7]. From this it follows that the "dual complex" of $C^{\cdot}_{par}([\cdot,\phi])$ is the complex $C^{\cdot}_{par}([\phi,\cdot])$ defined by setting

$$\begin{aligned} C^i_{par}([\phi,\cdot]) = C^i(&\mathcal{P}ar\mathcal{H}om(E_*,E_*)) \\ &\oplus C^{i+n-2}(\mathcal{P}ar\mathcal{H}om(E_*,\hat{E}_*)\otimes\omega_X(D)), \end{aligned}$$

with coboundary $d^i : C^i_{par}([\phi,\cdot]) \to C^{i+1}_{par}([\phi,\cdot])$ given by (cf. Remark 2.7)

$$d^i = \begin{pmatrix} \delta^i & 0 \\ [\phi,\cdot] & -\delta^{i+n-2} \end{pmatrix}. \tag{8.2}$$

Then we have:

Proposition 8.7. *The cotangent space $T^*_{(E_*,\bar{\phi})}\mathcal{T}_{par}$ to $\mathcal{T}_{par}$ at the point $(E_*,\bar{\phi})$ is canonically isomorphic to $H^1(C^{\cdot}_{par}([\phi,\cdot]))$.*

The isomorphism of complexes $(-1,1) : C^{\cdot}_{par}([\phi,\cdot]) \xrightarrow{\sim} C^{\cdot}_{par}([\cdot,\phi])$, determined by plus and minus the identity map respectively on $\mathcal{P}ar\mathcal{H}om(E_*,\hat{E}_*)\otimes\omega_X(D)$ and on $\mathcal{P}ar\mathcal{H}om(E_*,E_*)$, determines an isomorphism of cohomology groups $(-1,1) : H^1(C^{\cdot}_{par}([\phi,\cdot])) \xrightarrow{\sim} H^1(C^{\cdot}_{par}([\cdot,\phi]))$, hence an isomorphism $(-1,1) : T^*_{(E_*,\bar{\phi})}\mathcal{T}_{par} \xrightarrow{\sim} T_{(E_*,\bar{\phi})}\mathcal{T}_{par}$. This is the Hamiltonian isomorphism corresponding to the canonical symplectic structure of $\mathcal{T}_{par}$. Precisely, we have the following result, which generalizes to the higher dimensional case the result proved in [Bo1, Theorem 5.2.4], and whose proof is analogous to the proof of Theorem 6.1:

Theorem 8.8. *The isomorphisms $(-1,1) : T^*_{(E_*,\bar{\phi})}\mathcal{T}_{par} \xrightarrow{\sim} T_{(E_*,\bar{\phi})}\mathcal{T}_{par}$, $(E_*,\bar{\phi}) \in \mathcal{T}_{par}$, define a global isomorphism $(-1,1) : T^*\mathcal{T}_{par} \xrightarrow{\sim} T\mathcal{T}_{par}$, which, in turn, defines a symplectic structure on $\mathcal{T}_{par}$. This symplectic structure is precisely the canonical symplectic structure of $\mathcal{T}_{par}$, considered as the cotangent bundle to $\mathcal{M}^{sm}_{par}$.*

Remark 8.9. The section $s \in H^0(X, \omega_X^{-1} \otimes L)$ defining the divisor D determines an isomorphism $s : \mathcal{O}_X(D) \xrightarrow{\sim} \omega_X^{-1} \otimes L$ from the sheaf of meromorphic functions on X with poles at D to $\omega_X^{-1} \otimes L$, given by multiplication by s. This, in turn, determines the following isomorphisms of sheaves:

$$s : \mathcal{P}ar\mathcal{H}om(E_*, \hat{E}_*) \otimes \omega_X(D) \xrightarrow{\sim} \mathcal{P}ar\mathcal{H}om(E_*, \hat{E}_*) \otimes L$$

and

$$s : \mathcal{P}ar\mathcal{H}om(E_*, E_*) \otimes L^{-1} \otimes \omega_X(D) \xrightarrow{\sim} \mathcal{P}ar\mathcal{H}om(E_*, E_*).$$

By using these isomorphisms, we obtain two isomorphisms of complexes

$$(1, s) : C^{\cdot}_{par}([\cdot, \phi]) \xrightarrow{\sim} C^{\cdot}_{par,L}([\cdot, \phi]),$$

and

$$(s, 1) : \check{C}^{\cdot}_{par,L}([\phi, \cdot]) \xrightarrow{\sim} C^{\cdot}_{par}([\phi, \cdot]),$$

where $C^{\cdot}_{par,L}([\cdot, \phi])$ and $\check{C}^{\cdot}_{par,L}([\phi, \cdot])$ are the complexes defined by setting

$$C^i_{par,L}([\cdot, \phi]) = C^i(\mathcal{P}ar\mathcal{H}om(E_*, E_*)) \oplus C^{i+n-2}(\mathcal{P}ar\mathcal{H}om(E_*, \hat{E}_*) \otimes L),$$

and

$$\begin{aligned} \check{C}^i_{par,L}([\phi, \cdot]) = C^i(&\mathcal{P}ar\mathcal{H}om(E_*, E_*) \otimes L^{-1} \otimes \omega_X(D)) \\ &\oplus C^{i+n-2}(\mathcal{P}ar\mathcal{H}om(E_*, \hat{E}_*) \otimes \omega_X(D)), \end{aligned}$$

with the coboundaries defined as in (8.1) and (8.2) respectively.

It follows that we can identity the tangent and cotangent spaces to $\mathcal{T}_{par}$ to the first cohomology groups of $C^{\cdot}_{par,L}([\cdot, \phi])$ and $\check{C}^{\cdot}_{par,L}([\phi, \cdot])$ respectively: $T_{(E_*, \bar{\phi})}\mathcal{T}_{par} \cong H^1(C^{\cdot}_{par,L}([\cdot, \phi]))$ and $T^*_{(E_*, \bar{\phi})}\mathcal{T}_{par} \cong H^1(\check{C}^{\cdot}_{par,L}([\phi, \cdot]))$.

Finally, using these identifications, it is immediate to see that the Hamiltonian isomorphism of Theorem 8.8, $T^*\mathcal{T}_{par} \xrightarrow{\sim} T\mathcal{T}_{par}$, corresponding to the canonical symplectic structure of $\mathcal{T}_{par}$, coincides with the isomorphism induced on the cohomology groups by the isomorphism of complexes

$$(-s, s) : \check{C}^{\cdot}_{par,L}([\phi, \cdot]) \xrightarrow{\sim} C^{\cdot}_{par,L}([\cdot, \phi])$$

given by multiplication by $-s$ and s respectively on $\mathcal{P}ar\mathcal{H}om(E_*, E_*) \otimes L^{-1} \otimes \omega_X(D)$ and $\mathcal{P}ar\mathcal{H}om(E_*, \hat{E}_*) \otimes \omega_X(D)$.

Let us denote now by $\mathcal{T}^o_{par}$ the open subset of $\mathcal{T}_{par}$ consisting of pairs $(E_*, \bar{\phi})$ such that the underlying vector bundle E to the parabolic bundle E_* belongs to the smooth locus M of the moduli space of stable vector bundles on X.

To a pair $(E_*, \bar{\phi}) \in \mathcal{T}^o_{par}$ we can associate the pair $(E, \bar{\phi}) \in \mathcal{P}(L)$, where, by a slight abuse of notation, we have denoted by the same symbol $\bar{\phi}$ the image of $\bar{\phi} \in H^{n-1}(X, \mathcal{P}ar\mathcal{H}om(E_*, \hat{E}_*) \otimes \omega_X(D))$ in $H^{n-1}(X, \mathcal{E}nd(E) \otimes L)$ by the map induced on cohomology by the inclusion

$$\mathcal{P}ar\mathcal{H}om(E_*, \hat{E}_*) \otimes \omega_X(D) \subset \mathcal{E}nd(E) \otimes \omega_X(D) \cong \mathcal{E}nd(E) \otimes L.$$

In this way we obtain a morphism $f : \mathcal{T}^o_{par} \to \mathcal{P}(L)$. The main result, relating the canonical symplectic structure of $\mathcal{T}^o_{par}$ to the Poisson structure of $\mathcal{P}(L)$ corresponding to the section s defining the divisor D, is the following:

Proposition 8.10. *The morphism $f : \mathcal{T}^o_{par} \to \mathcal{P}(L)$ is a Poisson morphism of Poisson varieties, i.e., it is compatible with the Poisson structures of $\mathcal{T}^o_{par}$ and $\mathcal{P}(L)$.*

Proof. It is easy to see that, in terms of the identifications of the tangent spaces $T_{(E_*,\bar{\phi})}\mathcal{T}_{par} \cong H^1(C^{\cdot}_{par,L}([\cdot,\phi]))$ and $T_{(E,\bar{\phi})}\mathcal{P}(L) \cong H^1(C^{\cdot}([\cdot,\phi]))$, the tangent map to f at a point $(E_*, \bar{\phi})$,

$$T_{(E_*,\bar{\phi})}f : T_{(E_*,\bar{\phi})}\mathcal{T}^o_{par} \to T_{(E,\bar{\phi})}\mathcal{P}(L),$$

is given by the map $H^1(C^{\cdot}_{par,L}([\cdot,\phi])) \to H^1(C^{\cdot}([\cdot,\phi]))$ induced by the map of complexes $C^{\cdot}_{par,L}([\cdot,\phi]) \to C^{\cdot}([\cdot,\phi])$ determined by the inclusions $\mathcal{P}ar\mathcal{H}om(E_*, E_*) \hookrightarrow \mathcal{E}nd(E)$ and $\mathcal{P}ar\mathcal{H}om(E_*, \hat{E}_*) \otimes L \hookrightarrow \mathcal{E}nd(E) \otimes L$. From this, by recalling the explicit description of the canonical symplectic structure of $\mathcal{T}^o_{par}$ given in Remark 8.9 and of the Poisson structure of $\mathcal{P}(L)$ given at the end of Sect. 2, it follows immediately that f is a Poisson morphism of Poisson varieties.

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