ATIYAH CLASSES OF LIE ALGEBROIDS

FRANCESCO BOTTACIN

Abstract. Given a smooth morphism of analytic spaces $\pi\colon X \to Y$, we introduce the notion of a relative Lie algebroid $(\mathcal{A}, \sharp)$ over X. By replacing the relative tangent sheaf $\mathcal{T}_{X/Y}$ with the Lie algebroid $\mathcal{A}$, we define the notion of a relative $(\mathcal{A}, \sharp)$-connection on a quasi-coherent $\mathcal{O}_X$-module $\mathcal{E}$. Then, we define the $(\mathcal{A}, \sharp)$-Atiyah class of $\mathcal{E}$ as the obstruction to the existence of a holomorphic $(\mathcal{A}, \sharp)$-connection on $\mathcal{E}$. Many results of the classical theory of connections can be restated in the more general setting of Lie algebroid connections. As an application we prove the following result.

Let X be a complex manifold and $(A, \sharp)$ a Lie algebroid over X. For any quasi-coherent sheaf of commutative $\mathcal{O}_X$-algebras $\mathcal{F}$, let us write $\mathfrak{g}_i = H^{i-1}(X, A \otimes \mathcal{F})$. The $(A, \sharp)$-Atiyah class of A yields maps $\mathfrak{g}_i \otimes \mathfrak{g}_j \to \mathfrak{g}_{i+j}$. These maps define a graded Lie algebra structure on the graded vector space $\mathfrak{g}^\bullet = \bigoplus_i \mathfrak{g}_i$. In a similar way, for any holomorphic vector bundle E over X, let us write $V_j = H^{j-1}(X, E \otimes \mathcal{F})$. Then, for any i and j, the $(A, \sharp)$-Atiyah class of E yields a map $\mathfrak{g}_i \otimes V_j \to V_{i+j}$, and these maps define a structure of graded module on the graded vector space $V^\bullet = \bigoplus_j V_j$, over the graded Lie algebra $\mathfrak{g}^\bullet$. This generalizes a similar result proved by Kapranov in [K]. Similar results have been obtained by Chen, Stiénon and Xu in [CSX], by using different techniques.

Introduction

The theory of connections is a central topic in differential geometry. A rather natural generalization of the classical notion of connection on a vector or principal bundle over a differentiable manifold X is obtained by replacing the tangent bundle of X with a Lie algebroid $(A, \sharp)$ over X; this leads to the notion of a *Lie algebroid connection*.

Most of the results of the classical theory of connections (e.g., the Chern–Weil theory of characteristic classes) extend to Lie algebroid connections. We refer to [M] for an introduction to Lie algebroids and to [LF] for a detailed account on Lie algebroid connections.

While Lie algebroid connections on a smooth vector bundle over a differentiable manifold X always exist (this is a consequence of the existence of partitions of unity on X), when X is a complex manifold there is an obstruction to the existence of a global holomorphic Lie algebroid connection on a holomorphic vector bundle E over X. This obstruction is given by a cohomology class that is the analogue of the Atiyah class of E; we call it the $(A, \sharp)$-Atiyah class of E. As a special case, if we take $E = A$, we may look at the $(A, \sharp)$-Atiyah class of A itself.

Key words and phrases. Atiyah classes, connections, differential forms, jets, Lie algebroids.

As happens for their classical counterparts, the new Atiyah classes arising from Lie algebroid connections present very interesting features.

In the classical case, i.e., when the Lie algebroid $(A, \sharp)$ is the tangent bundle of a complex manifold X, M. Kapranov [K] (inspired by ideas of M. Kontsevich) discovered the fundamental role played by the Atiyah class of T_X in the construction of the topological invariants of 3-dimensional manifolds, previously introduced by L. Rozansky and E. Witten. One of the main results contained in Kapranov's paper may be restated as follows. Let $T_X[-1]$ denotes the shifted tangent sheaf of X, considered as an object in the derived category $D^+(X)$ of bounded below complexes of sheaves of $\mathcal{O}_X$-modules with coherent cohomology. Then the Atiyah class of the tangent bundle of X determines a map $T_X[-1] \otimes T_X[-1] \to T_X[-1]$, which makes $T_X[-1]$ into a Lie algebra object in $D^+(X)$.

As an application of the general theory of Lie algebroid connections, we prove that similar results hold if we replace the tangent bundle of a complex manifold X with a Lie algebroid A over X. In this case the role of the Atiyah class of T_X is played by the $(A, \sharp)$-Atiyah class of A.

More precisely, we prove that, given a Lie algebroid $(A, \sharp)$ over X and a quasi-coherent sheaf of commutative $\mathcal{O}_X$-algebras $\mathcal{F}$, there is a map

$$H^i(X, A \otimes \mathcal{F}) \otimes H^j(X, A \otimes \mathcal{F}) \to H^{i+j+1}(X, A \otimes \mathcal{F})$$

obtained by composing the cup-product of two cohomology classes with the $(A, \sharp)$-Atiyah class of A. If we set $\mathfrak{g}_i = H^{i-1}(X, A \otimes \mathcal{F})$, the collection of maps $\mathfrak{g}_i \otimes \mathfrak{g}_j \to \mathfrak{g}_{i+j}$ defines a graded Lie algebra structure on the graded vector space $\mathfrak{g}^\bullet = \bigoplus_i \mathfrak{g}_i$.

In a similar way, for any holomorphic vector bundle E over X, we can define a map

$$H^i(X, A \otimes \mathcal{F}) \otimes H^j(X, E \otimes \mathcal{F}) \to H^{i+j+1}(X, E \otimes \mathcal{F}),$$

by using the $(A, \sharp)$-Atiyah class of E. If we write $V_j = H^{j-1}(X, E \otimes \mathcal{F})$, we get a collection of maps $\mathfrak{g}_i \otimes V_j \to V_{i+j}$, for any i and j, defining a structure of graded module on the graded vector space $V^\bullet = \bigoplus_j V_j$, over the graded Lie algebra $\mathfrak{g}^\bullet$.

We remark that, in a recent paper, Z. Chen, M. Stiénon and P. Xu [CSX] developed a general theory of Atiyah classes relative to pairs consisting of a Lie algebroid A over X and a Lie subalgebroid of A, over the same base manifold. They also proved a generalization of Kapranov's results by using different techniques.

This paper is organized as follows. In Section 1 we develop the basic theory of holomorphic Lie algebroids and Lie algebroid connections in a relative setting. More precisely, we introduce the notion of a relative Lie algebroid over X, where $\pi\colon X \to Y$ is a smooth morphism of analytic spaces. Then we define relative $(\mathcal{A}, \sharp)$-connections on a quasi-coherent sheaf of $\mathcal{O}_X$-modules $\mathcal{E}$ and study their basic properties.

In Section 2 we introduce the sheaf of first $(\mathcal{A},\sharp)$-jets of $\mathcal{E}$ and define the $(\mathcal{A},\sharp)$-Atiyah class of $\mathcal{E}$ as the obstruction to the existence of a global holomorphic $(\mathcal{A},\sharp)$-connection on $\mathcal{E}$. We also prove that the $(\mathcal{A},\sharp)$-Atiyah class of $\mathcal{A}$ is symmetric.

In Sections 3 and 4 we define the sheaves of higher $(\mathcal{A},\sharp)$-jets and the sheaf of $(\mathcal{A},\sharp)$-differential operators. Then, in Section 5, we prove a version of the so-called 'cohomological Bianchi identity,' originally proved in [K] for the usual Atiyah class of a vector bundle.

Finally, in the last section, we show how Kapranov's results can be generalized to the framework of Lie algebroid connections. The proofs are obtained by following Kapranov's original argument, with suitable modifications. Note that the basic tool needed for proving that the composition with the $(A,\sharp)$-Atiyah class of A defines a graded Lie algebra structure on the graded vector space $\mathfrak{g}^\bullet = \bigoplus_i H^{i-1}(X, A\otimes\mathcal{F})$ is precisely the cohomological Bianchi identity, which implies the graded Jacobi identity for the graded Lie bracket.

1. Preliminaries

1.1. $(\mathcal{A},\sharp)$**-connections.** Let $\pi\colon X\to Y$ be a smooth morphism of analytic spaces (or a smooth morphism of schemes, defined over a field of characteristic 0). We denote by $\mathcal{T}_{X/Y} = \mathcal{H}om_{\mathcal{O}_X}(\Omega^1_{X/Y},\mathcal{O}_X)$ the relative tangent sheaf (which is locally free, since π is smooth).

Definition 1.1. A *relative Lie algebroid* over X is a locally free sheaf of $\mathcal{O}_X$-modules $\mathcal{A}$, with a $\pi^{-1}\mathcal{O}_Y$-linear morphism $[\cdot,\cdot]\colon \mathcal{A}\otimes\mathcal{A}\to\mathcal{A}$ which defines a Lie algebra structure on the spaces of sections, together with a homomorphism of $\mathcal{O}_X$-modules $\sharp\colon\mathcal{A}\to\mathcal{T}_{X/Y}$, called the *anchor map*, such that the induced map on the spaces of sections $\sharp\colon\Gamma(\mathcal{A})\to\Gamma(\mathcal{T}_{X/Y})$ is a homomorphism of Lie algebras, and for any sections $a_1,a_2\in\Gamma(\mathcal{A})$ and $f\in\Gamma(\mathcal{O}_X)$, the following Leibniz identity holds:

$$[a_1, fa_2] = f[a_1,a_2] + \sharp a_1(f)\, a_2. \tag{1.1}$$

Remark 1.2. Let us denote by X_y the fiber of $\pi\colon X\to Y$ over a point $y\in Y$. If $(\mathcal{A},\sharp)$ is a relative Lie algebroid over X we shall denote by $\mathcal{A}_y$ the restriction of $\mathcal{A}$ to X_y and by $\sharp_y\colon\mathcal{A}_y\to\mathcal{T}_{X_y}$ the map induced by $\sharp$. The previous definition implies that, for any $y\in Y$, $(\mathcal{A}_y,\sharp_y)$ is a Lie algebroid over X_y. Thus a relative Lie algebroid over X may be thought as a family of Lie algebroids over the fibers X_y, parametrized by the points $y\in Y$.

Let $\flat\colon\Omega^1_{X/Y}\to\mathcal{A}^* = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{A},\mathcal{O}_X)$ be the dual of the anchor map, and let $d_{\mathcal{A}}\colon\mathcal{O}_X\to\mathcal{A}^*$ be the $\pi^{-1}\mathcal{O}_Y$-derivation defined by $d_{\mathcal{A}} = \flat\circ d_{X/Y}$

$$\begin{tikzcd} \mathcal{O}_X \arrow[r, "d_{X/Y}"] \arrow[dr, "d_{\mathcal{A}}"'] & \Omega^1_{X/Y} \arrow[d, "\flat"] \\ & \mathcal{A}^* \end{tikzcd}$$

where $d_{X/Y}\colon\mathcal{O}_X\to\Omega^1_{X/Y}$ is the usual relative differential.

Let now $\mathcal{E}$ be a quasi-coherent $\mathcal{O}_X$-module.

Definition 1.3. A *relative* $(\mathcal{A},\sharp)$*-connection* on $\mathcal{E}$ is a $\pi^{-1}\mathcal{O}_Y$-linear morphism
$$\nabla\colon \mathcal{E}\to\mathcal{E}\otimes\mathcal{A}^*$$
such that
$$\nabla(fs)=f\nabla(s)+s\otimes d_{\mathcal{A}}(f),$$
for any local sections s of $\mathcal{E}$ and f of $\mathcal{O}_X$.

For any section $a\in\Gamma(\mathcal{A})$, we define
$$\nabla_a\colon \mathcal{E}\to\mathcal{E}$$
by setting $\nabla_a(s)=\langle\nabla s,a\rangle$. The map ∇_a is $\pi^{-1}\mathcal{O}_Y$-linear and satisfies the following identity:
$$\nabla_a(fs)=f\nabla_a(s)+\sharp a(f)\,s.$$
We also have
$$\nabla_{f_1a_1+f_2a_2}=f_1\nabla_{a_1}+f_2\nabla_{a_2}.$$
Remark 1.4. If ∇ and ∇' are two relative $(\mathcal{A},\sharp)$-connections on $\mathcal{E}$, their difference $\nabla'-\nabla$ is $\mathcal{O}_X$-linear, hence $\nabla'-\nabla\in\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E},\mathcal{E}\otimes\mathcal{A}^*)$. It follows that the space $\mathsf{Conn}_{(\mathcal{A},\sharp)}(\mathcal{E})$ of relative $(\mathcal{A},\sharp)$-connections on $\mathcal{E}$ is an affine space modeled on the vector space $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E},\mathcal{E}\otimes\mathcal{A}^*)$.

1.2. **Extension of a relative $(\mathcal{A},\sharp)$-connection.** We can extend the $\pi^{-1}\mathcal{O}_Y$-derivation $d_{\mathcal{A}}\colon \mathcal{O}_X\to\mathcal{A}^*$ to an operator
$$d_{\mathcal{A}}\colon \wedge^p\mathcal{A}^*\to\wedge^{p+1}\mathcal{A}^*$$
by setting, for any section α of $\wedge^p\mathcal{A}^*$,
$$\begin{aligned}(d_{\mathcal{A}}\alpha)(a_1,\dots,a_{p+1})&=\sum_{i=1}^{p+1}(-1)^{i+1}\sharp a_i\,\alpha(a_1,\dots,\hat{a}_i,\dots,a_{p+1})\\&\quad+\sum_{i<j}(-1)^{i+j}\alpha([a_i,a_j],a_1,\dots,\hat{a}_i,\dots,\hat{a}_j,\dots,a_{p+1})\end{aligned}$$
where $[\cdot,\cdot]\colon\mathcal{A}\otimes\mathcal{A}\to\mathcal{A}$ is the Lie bracket of the relative Lie algebroid $\mathcal{A}$ (the Leibniz identity (1.1) implies that $d_{\mathcal{A}}(\alpha)$ is actually a section of $\wedge^{p+1}\mathcal{A}^*$).

The fact that $\sharp\colon\mathcal{A}\to\mathcal{T}_{X/Y}$ induces a homomorphism of Lie algebras, together with the Jacobi identity for the Lie bracket on $\mathcal{A}$, imply that $d_{\mathcal{A}}\circ d_{\mathcal{A}}=0$, hence we have a complex
$$0\longrightarrow\mathcal{O}_X\xrightarrow{d_{\mathcal{A}}}\mathcal{A}^*\xrightarrow{d_{\mathcal{A}}}\wedge^2\mathcal{A}^*\xrightarrow{d_{\mathcal{A}}}\cdots \tag{1.2}$$
called the $(\mathcal{A},\sharp)$-de Rham complex.

Let now $\nabla\colon\mathcal{E}\to\mathcal{E}\otimes\mathcal{A}^*$ be a $(\mathcal{A},\sharp)$-connection on $\mathcal{E}$. As in the classical case, we shall extend ∇ to an operator
$$\nabla\colon\mathcal{E}\otimes\wedge^p\mathcal{A}^*\to\mathcal{E}\otimes\wedge^{p+1}\mathcal{A}^*$$
by requiring that
$$\nabla(s\otimes\alpha)=(\nabla s)\wedge\alpha+s\otimes d_{\mathcal{A}}(\alpha),$$
for any sections s of $\mathcal{E}$ and α of $\wedge^p\mathcal{A}^*$.

Then we can define the $(\mathcal{A},\sharp)$-curvature of ∇ by setting

$$R = \nabla \circ \nabla \colon \mathcal{E} \to \mathcal{E} \otimes \wedge^2 \mathcal{A}^*.$$

It is immediate to check that R is $\mathcal{O}_X$-linear, hence it is a section of $\mathcal{E}nd(\mathcal{E}) \otimes \wedge^2\mathcal{A}^*$. A $(\mathcal{A},\sharp)$-connection is called *flat* if its $(\mathcal{A},\sharp)$-curvature vanishes. The $(\mathcal{A},\sharp)$-curvature R satisfies an analogue of the classical Bianchi identity.

2. $(\mathcal{A},\sharp)$-jets and Atiyah classes.

For a quasi-coherent $\mathcal{O}_X$-module $\mathcal{E}$ let us consider the standard 1-jet exact sequence (also called Atiyah sequence)

$$0 \longrightarrow \mathcal{E} \otimes \Omega^1_{X/Y} \longrightarrow J^1_{X/Y}(\mathcal{E}) \longrightarrow \mathcal{E} \longrightarrow 0 \tag{2.1}$$

(which is split as a sequence of $\pi^{-1}\mathcal{O}_Y$-modules but not, in general, as a sequence of $\mathcal{O}_X$-modules).

We can define the sheaf of first $(\mathcal{A},\sharp)$-jets of $\mathcal{E}$ by pushing forward the previous exact sequence via the map $\mathrm{id}_\mathcal{E} \otimes \flat \colon \mathcal{E} \otimes \Omega^1_{X/Y} \to \mathcal{E} \otimes \mathcal{A}^*$. Hence, by definition, we have a commutative diagram (morphism of extensions)

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{E} \otimes \Omega^1_{X/Y} \arrow[r] \arrow[d] & J^1_{X/Y}(\mathcal{E}) \arrow[r] \arrow[d] & \mathcal{E} \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \mathcal{E} \otimes \mathcal{A}^* \arrow[r] & J^1_{(\mathcal{A},\sharp)}(\mathcal{E}) \arrow[r] & \mathcal{E} \arrow[r] & 0 \end{tikzcd} \tag{2.2}$$

Note that the exact sequence

$$0 \longrightarrow \mathcal{E} \otimes \mathcal{A}^* \longrightarrow J^1_{(\mathcal{A},\sharp)}(\mathcal{E}) \longrightarrow \mathcal{E} \longrightarrow 0 \tag{2.3}$$

is split as a sequence of $\pi^{-1}\mathcal{O}_Y$-modules but not, in general, as a sequence of $\mathcal{O}_X$-modules.

Remark 2.1. As sheaves of $\pi^{-1}\mathcal{O}_Y$-modules, we have

$$J^1_{(\mathcal{A},\sharp)}(\mathcal{E}) = \mathcal{E} \oplus (\mathcal{E} \otimes \mathcal{A}^*).$$

Note that $J^1_{(\mathcal{A},\sharp)}(\mathcal{E})$ has two structures of $\mathcal{O}_X$-module: one is given by

$$f \cdot (s,\sigma) = (fs, f\sigma),$$

for sections $f \in \Gamma(\mathcal{O}_X)$, $s \in \Gamma(\mathcal{E})$ and $\sigma \in \Gamma(\mathcal{E} \otimes \mathcal{A}^*)$; we shall call this the *left* $\mathcal{O}_X$-module structure.

The other one is defined by setting

$$(s,\sigma) \cdot f = (fs, f\sigma + s \otimes d_\mathcal{A} f),$$

and is called the *right* $\mathcal{O}_X$-module structure.

Unless otherwise stated, we shall always consider $J^1_{(\mathcal{A},\sharp)}(\mathcal{E})$ as an $\mathcal{O}_X$-module with its right module structure.

It is well known that the data of a relative connection on $\mathcal{E}$ is equivalent to a splitting of the exact sequence (2.1). A similar result holds for relative $(\mathcal{A},\sharp)$-connections:

Lemma 2.2. *A splitting of the sequence* (2.3) *is equivalent to a relative* $(\mathcal{A}, \sharp)$*-connection on* $\mathcal{E}$*.*

Proof. As sheaves of $\pi^{-1}\mathcal{O}_Y$-modules, we have

$$J^1_{(\mathcal{A},\sharp)}(\mathcal{E}) = \mathcal{E} \oplus (\mathcal{E} \otimes \mathcal{A}^*),$$

hence a splitting of (2.3) is given by a homomorphism of $\mathcal{O}_X$-modules

$$\phi\colon \mathcal{E} \to J^1_{(\mathcal{A},\sharp)}(\mathcal{E}), \qquad s \mapsto \phi(s) = \big(s, \nabla(s)\big),$$

for some map $\nabla\colon \mathcal{E} \to \mathcal{E} \otimes \mathcal{A}^*$. Since ϕ is $\mathcal{O}_X$-linear, we have $\phi(fs) = \phi(s)f$, for sections $f \in \Gamma(\mathcal{O}_X)$ and $s \in \Gamma(\mathcal{E})$. But $\phi(fs) = \big(fs, \nabla(fs)\big)$ and $\phi(s)f = \big(s, \nabla(s)\big)f = (fs, f\nabla(s) + s \otimes d_{\mathcal{A}}(f))$, hence the map ∇ must satisfy the identity

$$\nabla(fs) = f\nabla(s) + s \otimes d_{\mathcal{A}}(f).$$

So, requiring that ϕ be a homomorphism of $\mathcal{O}_X$-modules is equivalent to requiring that ∇ be a $(\mathcal{A}, \sharp)$-connection on $\mathcal{E}$. ☐

Definition 2.3. The $(\mathcal{A}, \sharp)$-Atiyah class of $\mathcal{E}$ is the class

$$a_{(\mathcal{A},\sharp)}(\mathcal{E}) \in \operatorname{Ext}^1(\mathcal{E}, \mathcal{E} \otimes \mathcal{A}^*)$$

corresponding to the extension (2.3).

From Lemma 2.2 we obtain the following result:

Corollary 2.4. *A relative* $(\mathcal{A}, \sharp)$*-connection on* $\mathcal{E}$ *exists if and only if the* $(\mathcal{A}, \sharp)$*-Atiyah class of* $\mathcal{E}$ *vanishes.*

Let us now compare the $(\mathcal{A}, \sharp)$-Atiyah class of $\mathcal{E}$ with its usual Atiyah class. The usual Atiyah class of $\mathcal{E}$ is the class $a(\mathcal{E}) \in \operatorname{Ext}^1(\mathcal{E}, \mathcal{E} \otimes \Omega^1_{X/Y})$ corresponding to the extension (2.1). The morphism of extensions (2.2) induces a morphism

$$\operatorname{Ext}^1(\mathcal{E}, \mathcal{E} \otimes \Omega^1_{X/Y}) \to \operatorname{Ext}^1(\mathcal{E}, \mathcal{E} \otimes A^*).$$

It is now immediate to verify that the $(\mathcal{A}, \sharp)$-Atiyah class of $\mathcal{E}$ is the image of the usual Atiyah class $a(\mathcal{E})$ under the previous map.

Remark 2.5. Exactly as the usual Atiyah class can be used to define the Chern classes of $\mathcal{E}$, we could use the $(\mathcal{A}, \sharp)$-Atiyah class to define what we may call $(\mathcal{A}, \sharp)$-Chern classes.

If we consider the morphism $\operatorname{Ext}^1(\mathcal{E}, \mathcal{E} \otimes \Omega^1_{X/Y}) \to \operatorname{Ext}^1(\mathcal{E}, \mathcal{E} \otimes \mathcal{A}^*)$, induced by the map $\flat\colon \Omega^1_{X/Y} \to \mathcal{A}^*$, and we apply the trace maps, we obtain a commutative diagram

$$\begin{tikzcd}
\operatorname{Ext}^1(\mathcal{E}, \mathcal{E} \otimes \Omega^1_{X/Y}) \arrow[r, "\mathrm{id}_{\mathcal{E}} \otimes \flat"] \arrow[d, "\mathrm{tr}"'] & \operatorname{Ext}^1(\mathcal{E}, \mathcal{E} \otimes \mathcal{A}^*) \arrow[d, "\mathrm{tr}"] \\
H^1(X, \Omega^1_{X/Y}) \arrow[r, "\flat"'] & H^1(X, \mathcal{A}^*)
\end{tikzcd}$$

Since the first Chern class of $\mathcal{E}$ is given by $c_1(\mathcal{E}) = \operatorname{tr}(a(\mathcal{E}))$, we find that $\flat(c_1(\mathcal{E})) = \operatorname{tr}(a_{(\mathcal{A},\sharp)}(\mathcal{E}))$ (and a similar statement holds for all higher Chern classes). It follows that the $(\mathcal{A}, \sharp)$-Chern classes that we could define using a $(\mathcal{A},\sharp)$-connection on $\mathcal{E}$ are not particularly interesting because they are the image of the usual Chern classes of $\mathcal{E}$ under the maps $H^i(X,\Omega^i_{X/Y}) \to H^i(X,\wedge^i\mathcal{A}^*)$ induced by the morphism $\flat\colon \Omega^1_{X/Y} \to \mathcal{A}^*$.

2.1. **$(\mathcal{A},\sharp)$-connections on $\mathcal{A}$.** Let us consider now the special case $\mathcal{E}=\mathcal{A}$. Let $\nabla\colon \mathcal{A}\to\mathcal{A}\otimes\mathcal{A}^*$ be a $(\mathcal{A},\sharp)$-connection on $\mathcal{A}$.

For any section $a\in\Gamma(\mathcal{A})$ we define the derivation $\nabla_a\colon\mathcal{A}\to\mathcal{A}$ by setting

$$\nabla_a(b)=\langle\nabla b,a\rangle.$$

Then we define the $(\mathcal{A},\sharp)$-torsion of ∇ by setting

$$T(a,b)=\nabla_a(b)-\nabla_b(a)-[a,b],$$

for sections a, b of $\mathcal{A}$. It is easy to see that $T\in\operatorname{Hom}_{\mathcal{O}_X}(\wedge^2\mathcal{A},\mathcal{A})$. A $(\mathcal{A},\sharp)$-connection on $\mathcal{A}$ is said to be *torsion-free* if its $(\mathcal{A},\sharp)$-torsion vanishes.

The following result, proved in [K], carries over into this more general setting (with a similar proof).

Theorem 2.6. *Let $(\mathcal{A},\sharp)$ be a relative Lie algebroid over X and let*

$$a_{(\mathcal{A},\sharp)}(\mathcal{A})\in\operatorname{Ext}^1(\mathcal{A},\mathcal{A}\otimes\mathcal{A}^*)=\operatorname{Ext}^1(\mathcal{A}\otimes\mathcal{A},\mathcal{A})$$

be its $(\mathcal{A},\sharp)$-Atiyah class. Then $a_{(\mathcal{A},\sharp)}(\mathcal{A})$ is symmetric, i.e., it belongs to $\operatorname{Ext}^1(\mathsf{S}^2\mathcal{A},\mathcal{A})$.

Proof. Let $\mathsf{Conn}_{(\mathcal{A},\sharp)}(\mathcal{A})$ be the sheaf whose sections over $U\subset X$ are the holomorphic $(\mathcal{A},\sharp)$-connections defined on $\mathcal{A}|_U$. As seen in Remark 1.4, this is an affine space over $\Gamma(U,\mathcal{E}nd(\mathcal{A})\otimes\mathcal{A}^*)$. Then $\mathsf{Conn}_{(\mathcal{A},\sharp)}(\mathcal{A})$ is a sheaf of torsors over $\mathcal{E}nd(\mathcal{A})\otimes\mathcal{A}^*$. Sheaves of torsors over $\mathcal{E}nd(\mathcal{A})\otimes\mathcal{A}^*$ are classified by elements of $H^1(X,\mathcal{E}nd(\mathcal{A})\otimes\mathcal{A}^*)=\operatorname{Ext}^1(\mathcal{A},\mathcal{A}\otimes\mathcal{A}^*)$, and $a_{(\mathcal{A},\sharp)}(\mathcal{A})$ is precisely the element that classifies $\mathsf{Conn}_{(\mathcal{A},\sharp)}(\mathcal{A})$.

Similarly, let $\mathsf{Conn}^{\mathrm{tf}}_{(\mathcal{A},\sharp)}(\mathcal{A})$ be the sheaf whose sections over $U\subset X$ are the torsion free $(\mathcal{A},\sharp)$-connections on $\mathcal{A}|_U$. Then $\mathsf{Conn}^{\mathrm{tf}}_{(\mathcal{A},\sharp)}(\mathcal{A})$ is a sheaf of torsors over $\mathsf{S}^2(\mathcal{A}^*)\otimes\mathcal{A}$. Since the sheaf of torsors $\mathsf{Conn}_{(\mathcal{A},\sharp)}(\mathcal{A})$ is obtained from $\mathsf{Conn}^{\mathrm{tf}}_{(\mathcal{A},\sharp)}(\mathcal{A})$ by "change of scalars" (i.e., from $\mathsf{S}^2(\mathcal{A}^*)\otimes\mathcal{A}$ to $\mathcal{A}^*\otimes\mathcal{A}^*\otimes\mathcal{A}$), it follows that the classifying element $a_{(\mathcal{A},\sharp)}(\mathcal{A})\in H^1(X,\mathcal{A}^*\otimes\mathcal{A}^*\otimes\mathcal{A})$ actually belongs to the summand $H^1(X,\mathsf{S}^2(\mathcal{A}^*)\otimes\mathcal{A})=\operatorname{Ext}^1(\mathsf{S}^2\mathcal{A},\mathcal{A})$. □

3. Higher $(\mathcal{A},\sharp)$-jets.

In this section we shall briefly describe how it is possible to define sheaves of higher order $(\mathcal{A},\sharp)$-jets.

For a quasi-coherent $\mathcal{O}_X$-module $\mathcal{E}$ we have already seen that $J^1_{(\mathcal{A},\sharp)}(\mathcal{E})=\mathcal{E}\oplus(\mathcal{E}\otimes\mathcal{A}^*)$, with the structure of $\mathcal{O}_X$-module (on the right) given by

$$(s,\sigma)\cdot f=(fs,f\sigma+s\otimes d_{\mathcal{A}}f).$$

We can now define the sheaf of 2^{nd} $(\mathcal{A},\sharp)$-jets of $\mathcal{E}$ by setting

$$J^2_{(\mathcal{A},\sharp)}(\mathcal{E})=J^1_{(\mathcal{A},\sharp)}(\mathcal{E})\oplus(\mathcal{E}\otimes\mathsf{S}^2\mathcal{A}^*)=\mathcal{E}\oplus(\mathcal{E}\otimes\mathcal{A}^*)\oplus(\mathcal{E}\otimes\mathsf{S}^2\mathcal{A}^*),$$

as $\pi^{-1}\mathcal{O}_Y$-modules, where $\mathsf{S}^2\mathcal{A}^*$ denotes the symmetric square of $\mathcal{A}^*$.

Let $d^{(2)}_{X/Y} \colon \mathcal{O}_X \to \mathsf{S}^2\,\Omega^1_{X/Y}$ be the quadratic differential expressed, in suitable local coordinates z_i, by

$$d^{(2)}_{X/Y} f = \frac{1}{2!} \sum_{i,j} \frac{\partial^2 f}{\partial z_i \partial z_j}\, dz_i \odot dz_j,$$

where $\odot$ denotes the symmetric product.

Let us define the quadratic derivation $d^{(2)}_{\mathcal{A}} \colon \mathcal{O}_X \to \mathsf{S}^2\,\mathcal{A}^*$ as the composition $d^{(2)}_{\mathcal{A}} = (\flat \odot \flat) \circ d^{(2)}_{X/Y}$

$$\begin{tikzcd} \mathcal{O}_X \arrow[r, "d^{(2)}_{X/Y}"] \arrow[dr, "d^{(2)}_{\mathcal{A}}"'] & \mathsf{S}^2\,\Omega^1_{X/Y} \arrow[d, "\flat\odot\flat"] \\ & \mathsf{S}^2\,\mathcal{A}^* \end{tikzcd}$$

The structure of (right) $\mathcal{O}_X$-module on $J^2_{(\mathcal{A},\sharp)}(\mathcal{E})$ is defined by setting

$$(s,\sigma,\tau)\cdot f = (fs, f\sigma + s\otimes d_{\mathcal{A}} f, f\tau + \sigma\otimes d_{\mathcal{A}} f + s \otimes d^{(2)}_{\mathcal{A}} f),$$

for sections $f \in \Gamma(\mathcal{O}_X)$, $s \in \Gamma(\mathcal{E})$, $\sigma \in \Gamma(\mathcal{E}\otimes\mathcal{A}^*)$ and $\tau \in \Gamma(\mathcal{E}\otimes \mathsf{S}^2\,\mathcal{A}^*)$ (here, by $\sigma\otimes d_{\mathcal{A}} f$ we mean the image of $\sigma \otimes d_{\mathcal{A}} f \in \mathcal{E}\otimes\mathcal{A}^*\otimes\mathcal{A}^*$ in $\mathcal{E}\otimes\mathsf{S}^2\,\mathcal{A}^*$ under the symmetrisation map $\mathcal{E}\otimes\mathcal{A}^*\otimes\mathcal{A}^* \to \mathcal{E}\otimes\mathsf{S}^2\,\mathcal{A}^*$).

There is an exact sequence of $\mathcal{O}_X$-modules

$$0 \to \mathcal{E}\otimes\mathsf{S}^2\,\mathcal{A}^* \to J^2_{(\mathcal{A},\sharp)}(\mathcal{E}) \to J^1_{(\mathcal{A},\sharp)}(\mathcal{E}) \to 0 \tag{3.1}$$

(which is split as a sequence of $\pi^{-1}\mathcal{O}_Y$-modules but not, in general, as a sequence of $\mathcal{O}_X$-modules).

More generally, for any $r \geq 1$ we can define inductively the sheaf of r-th $(\mathcal{A},\sharp)$-jets of $\mathcal{E}$ by setting

$$J^r_{(\mathcal{A},\sharp)}(\mathcal{E}) = J^{r-1}_{(\mathcal{A},\sharp)}(\mathcal{E}) \oplus (\mathcal{E}\otimes\mathsf{S}^r\,\mathcal{A}^*) = \bigoplus_{i=0}^{r} (\mathcal{E}\otimes\mathsf{S}^i\,\mathcal{A}^*),$$

as $\pi^{-1}\mathcal{O}_Y$-modules, where $\mathsf{S}^i\,\mathcal{A}^*$ denotes the i-th symmetric power of $\mathcal{A}^*$. The (right) $\mathcal{O}_X$-module structure of $J^r_{(\mathcal{A},\sharp)}(\mathcal{E})$ is defined as follows. For any $j \geq 0$, let $d^{(j)}_{\mathcal{A}} \colon \mathcal{O}_X \to \mathsf{S}^j\,\mathcal{A}^*$ be the composition $d^{(j)}_{\mathcal{A}} = (\mathsf{S}^j\,\flat)\circ d^{(j)}_{X/Y}$

$$\begin{tikzcd} \mathcal{O}_X \arrow[r, "d^{(j)}_{X/Y}"] \arrow[dr, "d^{(j)}_{\mathcal{A}}"'] & \mathsf{S}^j\,\Omega^1_{X/Y} \arrow[d, "\mathsf{S}^j\,\flat"] \\ & \mathsf{S}^j\,\mathcal{A}^* \end{tikzcd}$$

where $d^{(j)}_{X/Y} \colon \mathcal{O}_X \to \mathsf{S}^j\,\Omega^1_{X/Y}$ is given locally by

$$d^{(j)}_{X/Y} f = \frac{1}{j!} \sum_{i_1,\dots,i_j} \frac{\partial^j f}{\partial z_{i_1}\cdots\partial z_{i_j}}\, dz_{i_1}\odot\cdots\odot dz_{i_j}.$$

Let $(s_0, s_1, \dots, s_r)$ be a section of $J^r_{(\mathcal{A},\sharp)}(\mathcal{E})$, with $s_i \in \Gamma(\mathcal{E} \otimes \mathsf{S}^i \mathcal{A}^*)$. Then, for any $f \in \Gamma(\mathcal{O}_X)$, we set $(s_0, s_1, \dots, s_r) \cdot f = (t_0, t_1, \dots, t_r)$ where, for each $h = 0, \dots, r$, the section $t_h \in \Gamma(\mathcal{E} \otimes \mathsf{S}^h \mathcal{A}^*)$ is given by the following expression:

$$t_h = \sum_{j=0}^{h} s_j \otimes d_{\mathcal{A}}^{(h-j)} f.$$

There is an exact sequence

$$0 \to \mathcal{E} \otimes \mathsf{S}^r \mathcal{A}^* \to J^r_{(\mathcal{A},\sharp)}(\mathcal{E}) \to J^{r-1}_{(\mathcal{A},\sharp)}(\mathcal{E}) \to 0 \tag{3.2}$$

(which is split as a sequence of $\pi^{-1}\mathcal{O}_Y$-modules but not, in general, as a sequence of $\mathcal{O}_X$-modules).

Finally, note that, for any r, there is a homomorphism of sheaves of abelian groups

$$d^r_{(\mathcal{A},\sharp),\mathcal{E}} \colon \mathcal{E} \to J^r_{(\mathcal{A},\sharp)}(\mathcal{E})$$

that is $\mathcal{O}_X$-linear for the *right* $\mathcal{O}_X$-module structure of $J^r_{(\mathcal{A},\sharp)}(\mathcal{E})$. All the verifications are left as exercises.

4. $(\mathcal{A}, \sharp)$-differential operators.

Let us recall that $\mathcal{D} = \mathcal{D}_{X/Y}$, the sheaf of rings of finite-order (holomorphic) differential operators on X over Y, is generated, as an algebra, by $\mathcal{O}_X$ and by $\mathcal{T}_{X/Y}$.

In a similar way, we define $\mathcal{D}_{(\mathcal{A},\sharp)}$ to be the algebra generated by $\mathcal{O}_X$ and $\mathcal{A}$, with the commutation relations given by

$$af = \sharp(a)(f) + fa \quad \text{and} \quad a_1 a_2 = a_2 a_1 + [a_1, a_2], \tag{4.1}$$

where a, a_1, a_2 are sections of $\mathcal{A}$ and f is a section of $\mathcal{O}_X$.

The sheaf of non-commutative rings $\mathcal{D}_{(\mathcal{A},\sharp)}$ is endowed with a filtration

$$0 \subset \mathcal{O}_X = \mathcal{D}^{\leq 0}_{(\mathcal{A},\sharp)} \subset \mathcal{D}^{\leq 1}_{(\mathcal{A},\sharp)} \subset \cdots \subset \mathcal{D}^{\leq r}_{(\mathcal{A},\sharp)} \subset \cdots \subset \mathcal{D}_{(\mathcal{A},\sharp)}$$

such that

$$\mathcal{D}_{(\mathcal{A},\sharp)} = \bigcup_{r \geq 0} \mathcal{D}^{\leq r}_{(\mathcal{A},\sharp)},$$

where, for each r, the $\mathcal{O}_X$-module $\mathcal{D}^{\leq r}_{(\mathcal{A},\sharp)}$ is the dual of the sheaf of r-th $(\mathcal{A}, \sharp)$-jets $J^r_{(\mathcal{A},\sharp)}(\mathcal{O}_X)$,

$$\mathcal{D}^{\leq r}_{(\mathcal{A},\sharp)} = \mathcal{H}om_{\mathcal{O}_X}(J^r_{(\mathcal{A},\sharp)}(\mathcal{O}_X), \mathcal{O}_X).$$

If $\mathcal{D}$ is the usual ring of differential operators on X over Y, the anchor map $\sharp \colon \mathcal{A} \to \mathcal{T}_{X/Y}$ induces a homomorphism of filtered rings

$$\sharp \colon \mathcal{D}_{(\mathcal{A},\sharp)} \to \mathcal{D}.$$

The map $\sigma \colon \mathcal{D}^{\leq r}_{(\mathcal{A},\sharp)} \to \mathsf{S}^r \mathcal{A}$, that associates to a $(\mathcal{A}, \sharp)$-differential operator its highest order term, is well defined and is called the *principal symbol* map. For every $r > 0$, there is an exact sequence

$$0 \to \mathcal{D}^{\leq r-1}_{(\mathcal{A},\sharp)} \to \mathcal{D}^{\leq r}_{(\mathcal{A},\sharp)} \to \mathsf{S}^r \mathcal{A} \to 0$$

which is the dual of

$$0 \to \mathsf{S}^r \mathcal{A}^* \to J^r_{(\mathcal{A},\sharp)}(\mathcal{O}_X) \to J^{r-1}_{(\mathcal{A},\sharp)}(\mathcal{O}_X) \to 0.$$

The associated graded ring of the filtered ring $\mathcal{D}_{(\mathcal{A},\sharp)}$ is isomorphic to the symmetric algebra over $\mathcal{A}$

$$\operatorname{gr}\left(\mathcal{D}_{(\mathcal{A},\sharp)}\right) \cong \mathsf{S}^{\cdot}(\mathcal{A}).$$

Let us recall that a relative flat connection on a coherent sheaf of $\mathcal{O}_X$-modules $\mathcal{E}$ is equivalent to a structure of $\mathcal{D}$-module on $\mathcal{E}$. In a similar way it is easy to prove that a relative flat $(\mathcal{A},\sharp)$-connection on $\mathcal{E}$ is equivalent to a structure of $\mathcal{D}_{(\mathcal{A},\sharp)}$-module on $\mathcal{E}$.

5. Kapranov's 'cohomological Bianchi identity.'

In this section we shall generalize the so-called 'cohomological Bianchi identity,' proved by Kapranov in [K], to the setting of Lie algebroid connections.

Let X be a complex manifold, $(A,\sharp)$ a Lie algebroid over X, and $\mathcal{D}_{(A,\sharp)}$ the sheaf of rings of $(A,\sharp)$-differential operators.

Remark 5.1. Let E be a vector bundle over X and E^* its dual bundle. The exact sequence

$$0 \longrightarrow E^* \otimes A^* \longrightarrow J^1_{(A,\sharp)}(E^*) \longrightarrow E^* \longrightarrow 0$$

computes the $(A,\sharp)$-Atiyah class of E^*, $a_{(A,\sharp)}(E^*) = -a_{(A,\sharp)}(E)$.

By dualizing, we obtain the exact sequence

$$0 \longrightarrow E \longrightarrow \mathcal{D}^{\leq 1}_{(A,\sharp)}(E) \longrightarrow E \otimes A \longrightarrow 0, \tag{5.1}$$

where $\mathcal{D}^{\leq 1}_{(A,\sharp)}(E) = \mathcal{D}^{\leq 1}_{(A,\sharp)} \otimes_{\mathcal{O}_X} E$, that also computes the class

$$-a_{(A,\sharp)}(E) \in \operatorname{Ext}^1(E \otimes A, E) = \operatorname{Ext}^1(E, E \otimes A^*).$$

Let M be a locally free left $\mathcal{D}_{(A,\sharp)}$-module, endowed with a good filtration M_i by vector bundles. The $\mathcal{D}_{(A,\sharp)}$-module structure on M is equivalent to a flat $(A,\sharp)$-connection $\nabla\colon M \to M \otimes A^*$. It follows that, for any j, we have an induced map $\nabla_j\colon M_j \to M_{j+1} \otimes A^*$.

The following result is a generalization of a similar statement, proved in [AL]:

Lemma 5.2 ([AL], n. (4.1.2.3))**.** *Let M be as before. Then:*

(a) *The class $-a_{(A,\sharp)}(M_i)$ is given by the following composition of maps*

$$M_i \xrightarrow{\pi_i} M_i/M_{i-1} \xrightarrow{\nabla_i} (M_{i+1}/M_i) \otimes A^* \xrightarrow{\alpha_i \otimes 1_{A^*}} M_i \otimes A^*[1],$$

where $\pi_i\colon M_i \to M_i/M_{i-1}$ is the projection, $\nabla_i\colon M_i/M_{i-1} \to (M_{i+1}/M_i)\otimes A^*$ is induced by the $(A,\sharp)$-connection ∇, and α_i is the element of $\operatorname{Ext}^1(M_{i+1}/M_i, M_i)$ *that corresponds to the exact sequence*

$$0 \longrightarrow M_i \longrightarrow M_{i+1} \longrightarrow M_{i+1}/M_i \longrightarrow 0.$$

(b) *The class* $-a_{(A,\sharp)}(M_i/M_{i-1})$ *is equal to the difference between the composition of morphisms*

$$M_i/M_{i-1} \xrightarrow{\nabla_i} (M_{i+1}/M_i) \otimes A^* \xrightarrow{\alpha_i \otimes 1_{A^*}} M_i \otimes A^*[1] \xrightarrow{\pi_i[1]} (M_i/M_{i-1}) \otimes A^*[1],$$

and the composition

$$M_i/M_{i-1} \xrightarrow{\alpha_{i-1}} M_{i-1}[1] \xrightarrow{\pi_{i-1}[1]} (M_{i-1}/M_{i-2})[1] \xrightarrow{\nabla_{i-1}[1]} (M_i/M_{i-1}) \otimes A^*[1].$$

Proof. The proof is the same as in [AL], since it follows from purely formal properties of extension classes. $\square$

Remark 5.3. The map $\nabla_j \colon M_j/M_{j-1} \to M_{j+1}/M_j \otimes A^*$ induced by the $(A,\sharp)$-connection ∇ on M, corresponds to the so-called "symbol multiplication map"

$$\mu_j \colon A \otimes M_j/M_{j-1} \to M_{j+1}/M_j.$$

If we denote by $f_i \in \operatorname{Ext}^1(M_{i+1}/M_i, M_i/M_{i-1})$ the composition

$$M_{i+1}/M_i \xrightarrow{\alpha_i} M_i[1] \xrightarrow{\pi_i[1]} (M_i/M_{i-1})[1],$$

then part (b) of the previous lemma can be restated by saying that the class $-a_{(A,\sharp)}(M_i/M_{i-1})$ is given by the difference between the following two compositions of morphisms:

$$A \otimes M_i/M_{i-1} \xrightarrow{\mu_i} M_{i+1}/M_i \xrightarrow{f_i} (M_i/M_{i-1})[1],$$

and

$$A \otimes M_i/M_{i-1} \xrightarrow{1_A \otimes f_{i-1}} A \otimes (M_{i-1}/M_{i-2})[1] \xrightarrow{\mu_{i-1}[1]} (M_i/M_{i-1})[1],$$

i.e., we can write

$$-a_{(A,\sharp)}(M_i/M_{i-1}) = f_i \circ \mu_i - \mu_{i-1}[1] \circ (1_A \otimes f_{i-1}). \tag{5.2}$$

If E is a vector bundle over X, we can consider the $\mathcal{D}_{(A,\sharp)}$-module $M = \mathcal{D}_{(A,\sharp)} \otimes_{\mathcal{O}_X} E$, with the filtration given by $M_i = \mathcal{D}^{\leq i}_{(A,\sharp)} \otimes E$. The exact sequence

$$0 \longrightarrow \frac{M_1}{M_0} \longrightarrow \frac{M_2}{M_0} \longrightarrow \frac{M_2}{M_1} \longrightarrow 0$$

becomes

$$0 \longrightarrow A \otimes E \longrightarrow \frac{\mathcal{D}^{\leq 2}_{(A,\sharp)} \otimes E}{E} \longrightarrow \mathsf{S}^2(A) \otimes E \longrightarrow 0.$$

Let us denote by $\xi \in \operatorname{Ext}^1(\mathsf{S}^2(A) \otimes E, A \otimes E)$ the corresponding extension class. Let $\sigma : A \otimes A \to \mathsf{S}^2(A)$ be the symmetrization map. From Lemma 5.2 and the subsequent remark, it follows that:

Lemma 5.4. *With the above notations, we have*

$$a_{(A,\sharp)}(A \otimes E) = -\xi \circ (\sigma \otimes 1) - 1 \otimes a_{(A,\sharp)}(E).$$

Proof. Since $M_i = \mathcal{D}^{\leq i}_{(A,\sharp)} \otimes E$, we have $M_1/M_0 = A \otimes E$ and $M_2/M_1 = \mathsf{S}^2(A) \otimes E$. From (5.2) we know that $-a_{(A,\sharp)}(M_1/M_0) = -a_{(A,\sharp)}(A \otimes E)$ is the difference between the following composition of morphisms:

$$A \otimes A \otimes E \xrightarrow{\sigma \otimes 1_E} \mathsf{S}^2(A) \otimes E \xrightarrow{\xi} A \otimes E[1]$$

and

$$A \otimes A \otimes E \xrightarrow{1 \otimes a_{(A,\sharp)}(E)} A \otimes E[1] \xrightarrow{-\mathrm{id}} A \otimes E[1].$$

Hence $-a_{(A,\sharp)}(A \otimes E) = \xi \circ (\sigma \otimes 1) + 1 \otimes a_{(A,\sharp)}(E)$. □

Now we introduce some notation in order to state the main result.

Let $a,\, b \in H^1(X, \mathcal{E}nd(E) \otimes A^*)$. Their cup-product is

$$a \smile b \in H^2(X, \mathcal{E}nd(E) \otimes \mathcal{E}nd(E) \otimes A^* \otimes A^*).$$

Consider the map

$$\begin{aligned} \mathcal{E}nd(E) \otimes \mathcal{E}nd(E) \otimes A^* \otimes A^* &\to \mathcal{E}nd(E) \otimes \mathsf{S}^2(A^*) \\ \phi \otimes \psi \otimes \alpha \otimes \beta &\mapsto [\phi, \psi] \otimes (\alpha \odot \beta) \end{aligned}$$

We denote by $[a \smile b] \in H^2(X, \mathcal{E}nd(E) \otimes \mathsf{S}^2(A^*))$ the image of $a \smile b$ under the induced map in cohomology.

Let $a \in H^1(X, \mathcal{E}nd(E) \otimes A^*) = \mathrm{Ext}^1(E, E \otimes A^*) = \mathrm{Ext}^1(A \otimes E, E)$, and let $c \in \mathrm{Ext}^1(A \otimes A, A)$. Let us consider the composition

$$\mathsf{S}^2(A) \otimes E \hookrightarrow A \otimes A \otimes E \xrightarrow{c \otimes 1} A \otimes E[1] \xrightarrow{a} E[2]$$

We denote by

$$\begin{aligned} a * c \in \mathrm{Hom}(\mathsf{S}^2(A) \otimes E, E[2]) &= \mathrm{Ext}^2(\mathsf{S}^2(A) \otimes E, E) \\ &= H^2(X, \mathcal{E}nd(E) \otimes \mathsf{S}^2(A^*)) \end{aligned}$$

the corresponding element.

Theorem 5.5 (Cohomological Bianchi identity)**.** *Let $a_{(A,\sharp)}(E) \in \mathrm{Ext}^1(E, E \otimes A^*) = H^1(X, \mathcal{E}nd(E) \otimes A^*)$ be the $(A, \sharp)$-Atiyah class of a vector bundle E. Let $a_{(A,\sharp)}(A) \in \mathrm{Ext}^1(A, A \otimes A^*) = H^1(X, \mathcal{E}nd(A) \otimes A^*)$ be the $(A, \sharp)$-Atiyah class of A. Then we have the identity*

$$2\,[a_{(A,\sharp)}(E) \smile a_{(A,\sharp)}(E)] + a_{(A,\sharp)}(E) * a_{(A,\sharp)}(A) = 0$$

in $H^2(X, \mathcal{E}nd(E) \otimes \mathsf{S}^2(A^*))$.

Proof. Let $M = \mathcal{D}_{(A,\sharp)} \otimes E$, with the filtration

$$0 \subset M_0 = E \subset M_1 = \mathcal{D}^{\leq 1}_{(A,\sharp)} \otimes E \subset M_2 = \mathcal{D}^{\leq 2}_{(A,\sharp)} \otimes E \subset \cdots$$

The exact sequence $0 \to M_0 \to M_1 \to M_1/M_0 \to 0$ is

$$0 \to E \to \mathcal{D}^{\leq 1}_{(A,\sharp)} \otimes E \to A \otimes E \to 0,$$

whose extension class is $-a_{(A,\sharp)}(E) \in \mathrm{Ext}^1(A \otimes E, A)$. The next exact sequence $0 \to M_1/M_0 \to M_2/M_0 \to M_2/M_1 \to 0$ is

$$0 \to A \otimes E \to M_2/M_0 \to \mathsf{S}^2(A) \otimes E \to 0,$$

whose extension class we have denoted by ξ.

Standard results (cf., for instance, [BB]) tell us that the composition (Yoneda product) of these two extensions is zero: $a_{(A,\sharp)}(E) \circ \xi = 0$

$$\mathsf{S}^2(A) \otimes E \xrightarrow{\xi} A \otimes E[1] \xrightarrow{a_{(A,\sharp)}(E)} E[2].$$

From Lemma 5.4 we have

$$a_{(A,\sharp)}(A \otimes E) = -\xi \circ (\sigma \otimes 1) - 1 \otimes a_{(A,\sharp)}(E).$$

The $(A, \sharp)$-Atiyah class of a tensor product of vector bundles is given by

$$a_{(A,\sharp)}(A \otimes E) = a_{(A,\sharp)}(A) \otimes 1 + 1 \otimes a_{(A,\sharp)}(E),$$

hence

$$2\big(1 \otimes a_{(A,\sharp)}(E)\big) + a_{(A,\sharp)}(A) \otimes 1 = -\xi \circ (\sigma \otimes 1).$$

Now we take the Yoneda product of the previous expression with $a_{(A,\sharp)}(E)$ (on the left), and we recall that $a_{(A,\sharp)}(E) \circ \xi = 0$.

We get

$$2\,[a_{(A,\sharp)}(E) \smile a_{(A,\sharp)}(E)] + a_{(A,\sharp)}(E) * a_{(A,\sharp)}(A) = 0.$$

□

6. The Lie algebra structure

Let X and $(A, \sharp)$ be as before, and let $\mathcal{F}$ be a quasi-coherent sheaf of commutative $\mathcal{O}_X$-algebras. We consider the composition of the following maps: first we take the cup-product

$$H^i(X, A \otimes \mathcal{F}) \otimes H^j(X, A \otimes \mathcal{F}) \to H^{i+j}(X, A \otimes A \otimes \mathcal{F} \otimes \mathcal{F})$$

followed by the map

$$H^{i+j}(X, A \otimes A \otimes \mathcal{F} \otimes \mathcal{F}) \to H^{i+j}(X, A \otimes A \otimes \mathcal{F})$$

induced by the commutative multiplication $\mathcal{F} \otimes \mathcal{F} \to \mathcal{F}$.

Then we take the Yoneda product with $a_{(A,\sharp)}(A) \in H^1(X, \mathrm{Hom}(\mathsf{S}^2(A), A))$:

$$H^{i+j}(X, A \otimes A \otimes \mathcal{F}) \to H^{i+j+1}(X, A \otimes \mathcal{F}).$$

So, for any i and j, we obtain maps

$$H^i(X, A \otimes \mathcal{F}) \otimes H^j(X, A \otimes \mathcal{F}) \to H^{i+j+1}(X, A \otimes \mathcal{F}).$$

Let us set $\mathfrak{g}_i = H^{i-1}(X, A \otimes \mathcal{F})$. Then we can rewrite the previous maps as follows:

$$\mathfrak{g}_i \otimes \mathfrak{g}_j \to \mathfrak{g}_{i+j}.$$

Theorem 6.1. *The maps above define a graded Lie algebra structure on the graded vector space* $\mathfrak{g}^\bullet = \bigoplus_i \mathfrak{g}_i$.

Proof. Let $\alpha_i \in \mathfrak{g}_i$, $\alpha_j \in \mathfrak{g}_j$, and let us denote the bracket by $[\alpha_i, \alpha_j] \in \mathfrak{g}_{i+j}$. The bilinearity of the bracket is obvious. The (graded) antisymmetry is given by the following expression:

$$[\alpha_j, \alpha_i] = -(-1)^{ij}[\alpha_i, \alpha_j].$$

This follows immediately from the graded commutativity of the cup-product. It remains only to prove the (graded) Jacobi identity:

$$(-1)^{ik}[\alpha_i,[\alpha_j,\alpha_k]] + (-1)^{ij}[\alpha_j,[\alpha_k,\alpha_i]] + (-1)^{jk}[\alpha_k,[\alpha_i,\alpha_j]] = 0.$$

Let us denote the left-hand side by $\theta(\alpha_i,\alpha_j,\alpha_k)$. This defines an element $\theta \in \mathrm{Hom}(\wedge^3\mathfrak{g}^\bullet,\mathfrak{g}^\bullet)$, and we can check that $\theta(\alpha_i,\alpha_j,\alpha_k)$ is obtained by taking the cup-product

$$\alpha_i \smile \alpha_j \smile \alpha_k \in H^{i+j+k-3}(X, A\otimes A\otimes A\otimes \mathcal{F})$$

followed by the Yoneda composition with an element of $H^2(X,\mathcal{H}om(\mathsf{S}^3(A),A))$. This element turns out to be the symmetrization of

$$[a_{(A,\sharp)}(A) \smile a_{(A,\sharp)}(A)] \in H^2(X,\mathcal{H}om(A\otimes \mathsf{S}^2(A),A)).$$

Now we use the cohomological Bianchi identity (for $E = A$):

$$2[a_{(A,\sharp)}(A) \smile a_{(A,\sharp)}(A)] + a_{(A,\sharp)}(A) * a_{(A,\sharp)}(A) = 0.$$

From the definition, it follows that the symmetrization of $a_{(A,\sharp)}(A) * a_{(A,\sharp)}(A)$ is 0, hence the same is true for the symmetrization of $[a_{(A,\sharp)}(A) \smile a_{(A,\sharp)}(A)]$. This finally means that $\theta = 0$, which proves the Jacobi identity. □

Let X, $(A,\sharp)$, $\mathcal{F}$ be as before, and let E be a holomorphic vector bundle over X. We consider now the composition of the following maps: first we take the cup-product

$$H^i(X,A\otimes\mathcal{F})\otimes H^j(X,E\otimes\mathcal{F}) \to H^{i+j}(X,A\otimes E\otimes\mathcal{F})$$

(where we have used the multiplication $\mathcal{F}\otimes\mathcal{F}\to\mathcal{F}$, as before). Then we take the Yoneda product with $a_{(A,\sharp)}(E) \in H^1(X,\mathrm{Hom}(A\otimes E,E))$:

$$H^{i+j}(X,A\otimes E\otimes\mathcal{F}) \to H^{i+j+1}(X,E\otimes\mathcal{F}).$$

If we set $\mathfrak{g}_i = H^{i-1}(X,A\otimes\mathcal{F})$ and $V_j = H^{j-1}(X,E\otimes\mathcal{F})$, for any i and j, we have maps $\mathfrak{g}_i\otimes V_j \to V_{i+j}$. We can now prove the following result:

Theorem 6.2. *The maps above define a structure of graded module on the graded vector space $V^\bullet = \bigoplus_j V_j$, over the graded Lie algebra $\mathfrak{g}^\bullet$.*

Proof. Let $\alpha_i \in \mathfrak{g}_i$, $\alpha_j\in\mathfrak{g}_j$ and $v_k \in V_k$. We must prove that

$$[\alpha_i,\alpha_j]v_k - \alpha_i(\alpha_j v_k) + (-1)^{ij}\alpha_j(\alpha_i v_k) = 0.$$

The left-hand side defines an element $\phi \in \mathrm{Hom}(\wedge^2\mathfrak{g}^\bullet\otimes V^\bullet, V^\bullet)$, and we can check that ϕ is obtained by taking the cup-product

$$\alpha_i \smile \alpha_j \smile v_k \in H^{i+j+k-3}(X,A\otimes A\otimes E\otimes\mathcal{F})$$

followed by the Yoneda composition with an element of $H^2(X,\mathcal{H}om(\mathsf{S}^2(A)\otimes E,E))$. This element is precisely

$$2[a_{(A,\sharp)}(E) \smile a_{(A,\sharp)}(E)] + a_{(A,\sharp)}(E) * a_{(A,\sharp)}(A),$$

which vanishes by the cohomological Bianchi identity. □

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Dipartimento di Matematica, Università degli Studi di Padova, Via Trieste, 63, I–35121, Padova, Italy

E-mail address: `bottacin@math.unipd.it`