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Francesco Bottacin

Poisson structures on moduli spaces of parabolic bundles on surfaces

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Abstract. Let X be a smooth complex projective surface and D an effective divisor on X such that $H^0(X, \omega_X^{-1}(-D)) \neq 0$. Let us denote by $\mathcal{PB}$ the moduli space of stable parabolic vector bundles on X with parabolic structure over the divisor D (with fixed weights and Hilbert polynomials). We prove that the moduli space $\mathcal{PB}$ is a non-singular quasi-projective variety naturally endowed with a family of holomorphic Poisson structures parametrized by the global sections of $\omega_X^{-1}(-D)$. This result is the natural generalization to the moduli spaces of parabolic vector bundles of the results obtained in [B2] for the moduli spaces of stable sheaves on a Poisson surface. We also give, in some special cases, a detailed description of the symplectic leaf foliation of the Poisson manifold $\mathcal{PB}$.

1. Introduction

Moduli spaces of sheaves on smooth projective surfaces were studied, from the point of view of symplectic geometry, by S. Mukai in [Mu]. Mukai discovered that if a surface X has a holomorphic symplectic structure (this means that the canonical line bundle ω_X of X is trivial, i.e., X is an abelian or K3 surface) then the moduli space $\mathcal{M}$ of stable sheaves on X has a holomorphic symplectic structure too. This result was generalized to the case of Poisson structures in [B2]. More precisely, we proved that the moduli space $\mathcal{M}$ of stable sheaves on a Poisson surface X has a family of Poisson structures $\theta_s \in H^0(\mathcal{M}, \wedge^2 T\mathcal{M})$ parametrized by the global sections s of the anti-canonical line bundle ω_X^{-1}. Since a Poisson structure on X is determined by choosing such a section s, it follows that the choice of a Poisson structure on X naturally determines the Poisson structure of the moduli space $\mathcal{M}$. Similar results can be proven also for other kinds of moduli spaces, like Hilbert schemes of points of X [B3], or moduli spaces of framed vector bundles on X [B4].

In this paper we prove that similar results, concerning the existence of Poisson structures on moduli spaces, can be obtained for yet another kind of moduli spaces, namely moduli spaces of parabolic vector bundles on X. In this case too, we construct a family of holomorphic Poisson structures on the moduli space $\mathcal{PB}$ of parabolic vector bundles on X parametrized by the global sections of a certain

F. Bottacin: Dipartimento di Ingegneria, Università degli Studi di Bergamo, Viale Marconi 5, 24044 Dalmine (BG), Italy. e-mail: bottacin@math.unipd.it

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line bundle on X. By considering parabolic vector bundles with trivial parabolic structure over the divisor defined by a section $s \in H^0(X, \omega_X^{-1})$, we recover the results of [B2] as a special case of the general construction proposed in this paper.

This paper is organized as follows. In Sect. 2 we recall some basic definitions and results on parabolic sheaves and their moduli spaces. Then, in Sect. 3, we recall, for the convenience of the reader, some results of symplectic geometry that will be needed in the sequel.

In Sect. 4 we construct the Poisson structure θ_ζ on the moduli space $\mathcal{PB}$ of parabolic vector bundles on X with parabolic structure over an effective divisor D, determined by a global section ζ of the line bundle $\omega_X^{-1}(-D)$, and in Sect. 5 we prove that θ_ζ is actually a Poisson structure, i.e., that the Poisson bracket defined by θ_ζ satisfies the Jacobi identity.

Finally, in Sect. 6 we study, in some particular cases, the symplectic leaf foliation of the moduli space $\mathcal{PB}$ determined by the Poisson structure θ_ζ.

2. Parabolic sheaves

Let X be a non-singular complex projective surface and H an ample divisor on it. We shall also fix an effective Cartier divisor D on X.

Definition 2.1. *A parabolic structure over D on a coherent, torsion-free $\mathcal{O}_X$-module E is the data of a filtration*

$$F_*: \quad E = F_1(E) \supset F_2(E) \supset \cdots \supset F_l(E) \supset F_{l+1}(E) = E(-D),$$

where $E(-D)$ denotes the image of $E \otimes_{\mathcal{O}_X} \mathcal{O}_X(-D) \to E$, together with a sequence of real numbers $\alpha_* = (\alpha_1, \ldots, \alpha_l)$, called weights, such that

$$0 \le \alpha_1 < \alpha_2 < \cdots < \alpha_l < 1.$$

A parabolic sheaf is a coherent, torsion-free $\mathcal{O}_X$-module E with a parabolic structure over D.

Remark 2.2. Some authors (cf., [Bh]) define a parabolic structure over D on a sheaf E as a sequence of subsheaves of $E|_D$

$$E|_D = \mathcal{F}_D^1(E) \supset \mathcal{F}_D^2(E) \supset \cdots \supset \mathcal{F}_D^l(E) \supset \mathcal{F}_D^{l+1}(E) = 0,$$

together with a system of weights $0 \le \alpha_1 < \alpha_2 < \cdots < \alpha_l < 1$.

Our definition is related to this one by setting

$$F_i(E) = \ker(E \to E|_D / \mathcal{F}_D^i(E)).$$

All definitions related to parabolic sheaves can be stated more efficiently in terms of $\mathbb{R}$-filtered sheaves (see [Y2] for the definition).

Given a parabolic sheaf (E, F_*, α_*), we define its associated $\mathbb{R}$-filtered sheaf $E_* = (E_x)$, for $0 \le x \le 1$, by setting $E_0 = E$ and $E_x = F_i(E)$ if $\alpha_{i-1} < x \le \alpha_i$, where we have set $\alpha_0 = 0$ and $\alpha_{l+1} = 1$. The definition of E_x can be extended to all $x \in \mathbb{R}$ by setting $E_{x+1} = E_x(-D)$. Figure 2.1 illustrates the $\mathbb{R}$-filtered

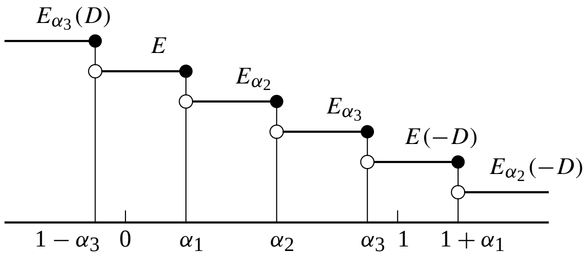

Fig. 2.1. The $\mathbb{R}$-filtered sheaf E_* associated to a parabolic sheaf (E, F_*, α_*)

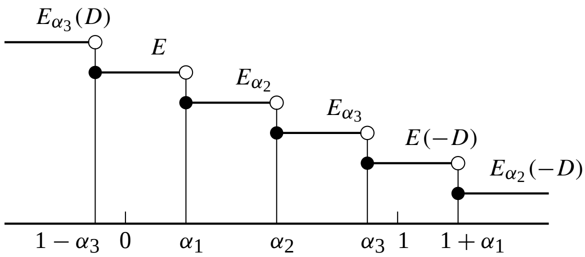

Fig. 2.2. The $\mathbb{R}$-filtered sheaf $\hat{E}_*$ associated to a parabolic sheaf (E, F_*, α_*)

sheaf corresponding to a parabolic sheaf (E, F_*, α_*) with weights $0 \le \alpha_1 < \alpha_2 < \alpha_3 < 1$.

From now on an $\mathbb{R}$-filtered sheaf $E_* = (E_x)_{x\in\mathbb{R}}$ associated to a parabolic sheaf (E, F_*, α_*) as above, will be simply called a parabolic sheaf.

Given a parabolic sheaf E_*, we shall define the $\mathbb{R}$-filtered sheaf $\hat{E}_*$ by setting, for any $x \in [0, 1]$,

$$\hat{E}_x = \begin{cases} E_x & \text{if } x \neq \alpha_i, \\ E_{\alpha_{i+1}} & \text{if } x = \alpha_i, \end{cases}$$

and by extending the definition to all $x \in \mathbb{R}$ by setting, as usual, $\hat{E}_{x+1} = \hat{E}_x(-D)$. Figure 2.2 illustrates the $\mathbb{R}$-filtered sheaf $\hat{E}_*$ corresponding to the parabolic sheaf E_* of Fig. 2.1.

If E_* is an $\mathbb{R}$-filtered sheaf, we shall always write E for the sheaf E_0.

Definition 2.3. *A homomorphism of $\mathbb{R}$-filtered sheaves $\phi : E_* \to E'_*$ is a homomorphism of $\mathcal{O}_X$-modules $\phi : E \to E'$ such that $\phi(E_x) \subseteq E'_x$, for any $x \in \mathbb{R}$.*

We shall denote by $\mathcal{H}om(E_*, E'_*)$ the sheaf of homomorphisms of $\mathbb{R}$-filtered sheaves from E_* to E'_*; it is a subsheaf of $\mathcal{H}om(E, E')$.

With these definitions the notion of parabolic homomorphism of two parabolic sheaves becomes very simple:

Definition 2.4. *If E_* and E'_* are two parabolic sheaves, a parabolic homomorphism $\phi : E_* \to E'_*$ is a homomorphism of $\mathbb{R}$-filtered sheaves.*

In order to construct moduli spaces of parabolic sheaves we need, as usual, a suitable notion of stability. This was introduced in [MY], where moduli spaces of semistable parabolic sheaves were constructed in great generality. We only state here the results we shall need in the sequel.

Proposition 2.5. *Let us fix a sequence of rational numbers $\alpha_* = (\alpha_1, \dots, \alpha_l)$ with $0 \le \alpha_1 < \alpha_2 < \cdots < \alpha_l < 1$, and polynomials $H, H_1, \dots, H_l$. Then there exists a quasi-projective moduli space $\mathcal{PS}$ parametrizing isomorphism classes of stable parabolic sheaves E_* having α_* as system of weights and such that the Hilbert polynomial of E is H and the Hilbert polynomial of $E/F_{i+1}(E)$ is H_i, for $i = 1, \dots, l$.*

Remark 2.6. Note that, in general, there does not exist a universal family of parabolic sheaves on the moduli space $\mathcal{PS}$, i.e., a parabolic sheaf $\mathcal{E}_*$ over $\mathcal{PS} \times X$, flat over $\mathcal{PS}$, such that $\mathcal{E}_*|_{\{E_*\}\times X} \cong E_*$, for any $E_* \in \mathcal{PS}$. However, universal families always exist locally, for the complex or étale topology, on $\mathcal{PS}$.

In the sequel we shall be particularly interested in a special class of parabolic sheaves, namely locally free parabolic sheaves (also called parabolic vector bundles).

Definition 2.7. *A parabolic sheaf E_* is said to be locally free if, for any x, E_x is a locally free $\mathcal{O}_X$-module and, for any x, y, with $x \le y < x+1$, E_x/E_y is a locally free $\mathcal{O}_D$-module.*

We shall denote by $\mathcal{PB}$ the open subset of the moduli space $\mathcal{PS}$ parametrizing isomorphism classes of locally free parabolic sheaves.

Infinitesimal deformation theory for parabolic sheaves (cf. [Y2]) yields the following result:

Proposition 2.8. *The tangent space $T_{E_*}\mathcal{PB}$ to the moduli space $\mathcal{PB}$ at a point E_* is canonically identified with the cohomology group $H^1(X, \mathcal{H}om(E_*, E_*))$ and the obstruction to the smoothness of $\mathcal{PB}$ at the point E_* lies in $H^2(X, \mathcal{H}om(E_*, E_*))$.*

By recalling the version of Serre duality for parabolic sheaves [Y2, Proposition 3.7], we have:

Corollary 2.9. *The cotangent space $T^*_{E_*}\mathcal{PB}$ to the moduli space $\mathcal{PB}$ at a point E_* is canonically identified with the cohomology group $H^1(X, \mathcal{H}om(E_*, \hat{E}_*) \otimes \omega_X(D))$.*

Remark 2.10. With the notations of Remark 2.2, if the parabolic structure of a vector bundle E is given by a filtration

$$E|_D = \mathcal{F}_D^1(E) \supset \mathcal{F}_D^2(E) \supset \cdots \supset \mathcal{F}_D^l(E) \supset \mathcal{F}_D^{l+1}(E) = 0,$$

and if E_* is the corresponding $\mathbb{R}$-filtered sheaf, then a section ϕ of $\mathcal{H}om(E_*, \hat{E}_*)$ is a homomorphism $\phi : E \to E$ such that $\phi|_D$ is nilpotent with respect to the filtration of $E|_D$ given above, i.e., such that $\phi|_D(\mathcal{F}_D^i(E)) \subseteq \mathcal{F}_D^{i+1}(E)$, for $i = 1, \dots, l$.

3. Poisson structures

In this section we recall, for the convenience of the reader, some basic definitions of symplectic geometry that will be needed in the sequel.

Definition 3.1. *A (holomorphic) Poisson structure on a non-singular complex variety X is a Lie algebra structure $\{\cdot,\cdot\}$ on the sheaf of regular functions $\mathcal{O}_X$ which is a derivation in each entry, i.e., satisfies $\{f, gh\} = \{f, g\}h + g\{f, h\}$.*

It is easy to see that to give a Poisson structure on X is equivalent to giving an antisymmetric contravariant 2-tensor $\theta \in H^0(X, \wedge^2 TX)$, defined by setting

$$\{f, g\} = \langle\theta, df \wedge dg\rangle. \tag{3.1}$$

Given $\theta \in H^0(X, \wedge^2 TX)$, the bracket defined by the formula above satisfies all the properties required to be a Poisson structure, except for the Jacobi identity

$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0,$$

hence we must impose an additional condition on θ. In order to express this condition in a suitable way, we have to introduce one more piece of notation: we denote by $B_\theta : T^*X \to TX$ the homomorphism of vector bundles defined by setting $\langle\theta, \alpha \wedge \beta\rangle = \langle B_\theta(\alpha), \beta\rangle$, for 1-forms α and β. The homomorphism B_θ is called the Hamiltonian morphism. In fact, for any regular function f on X, the vector field $X_f = B_\theta(df)$ is precisely the Hamiltonian vector field of f, defined by $X_f(g) = \{f, g\}$, for any regular function g.

We can now define an operator $\tilde{d} : H^0(X, \wedge^2 TX) \to H^0(X, \wedge^3 TX)$ by setting

$$\begin{aligned}\tilde{d}\theta(\alpha, \beta, \gamma) &= B_\theta(\alpha)\theta(\beta, \gamma) - B_\theta(\beta)\theta(\alpha, \gamma) + B_\theta(\gamma)\theta(\alpha, \beta)\\ &\quad -\langle[B_\theta(\alpha), B_\theta(\beta)], \gamma\rangle + \langle[B_\theta(\alpha), B_\theta(\gamma)], \beta\rangle - \langle[B_\theta(\beta), B_\theta(\gamma)], \alpha\rangle,\end{aligned} \tag{3.2}$$

for 1-forms α, β, γ, where $[\cdot,\cdot]$ denotes the usual commutator of vector fields. We have the following result, whose proof consists in a straightforward computation using local coordinates.

Proposition 3.2. *The bracket $\{\cdot,\cdot\}$ defined by an element $\theta \in H^0(X, \wedge^2 TX)$ as in (3.1) is a Poisson structure, i.e., satisfies the Jacobi identity, if and only if $\tilde{d}\theta = 0$.*

Remark 3.3. The element $\tilde{d}\theta \in H^0(X, \wedge^3 TX)$ coincides (up to a factor of 2) with the so-called Schouten bracket $[\theta, \theta]$ (see [V] for the definition). However, the expression given in (3.2) is more convenient for our computations.

Remark 3.4. When θ has maximal rank everywhere, i.e., when $B_\theta : T^*X \to TX$ is an isomorphism, to give θ is equivalent to giving its inverse 2-form $\omega \in \Omega^2_X$, which corresponds to the inverse isomorphism $B_\theta^{-1} : TX \to T^*X$. It is easy to check that, in this situation, the condition $\tilde{d}\theta = 0$ is equivalent to $d\omega = 0$, i.e., to the closure of the 2-form ω. In this case we say that ω defines a symplectic structure on X, or simply, that the Poisson structure is symplectic. Note that a necessary condition for the existence of a symplectic structure on X is that the dimension of X be even.

Remark 3.5. In the case of surfaces, the map $\tilde{d}$ is identically 0 ($\wedge^3 TX = 0$, since $\dim X = 2$), hence a Poisson structure on a smooth surface X is given by a section of $\wedge^2 TX \cong \omega_X^{-1}$, i.e., by a global section θ of the anti-canonical line bundle.

Let now X be a Poisson variety, and $\theta \in H^0(X, \wedge^2 TX)$ its Poisson structure. For any $x \in X$, we set $\mathcal{D}(x) = \operatorname{Im}(B_\theta(x)) \subseteq T_x X$. The collection $\mathcal{D} = (\mathcal{D}(x))_{x \in X}$ of subspaces of the tangent spaces of X is called the characteristic distribution of the Poisson variety (X, θ).

It turns out that, for any Poisson variety, the characteristic distribution is completely integrable, and the Poisson structure of X determines a symplectic structure on the integral leaves of this distribution. These integral leaves are then called the symplectic leaves of the Poisson variety (X, θ). For more details on the structure of Poisson varieties, we refer to [V].

4. Poisson structures on moduli spaces

From now on we shall assume that the surface X and the divisor D are such that $H^0(X, \omega_X^{-1}(-D)) \neq 0$, where ω_X is the canonical line bundle on X. We shall also fix a non-zero global section ζ of the line bundle $\omega_X^{-1}(-D)$ and denote by D_ζ the divisor defined by ζ.

Remark 4.1. From the standard exact sequence

$$0 \to \mathcal{O}_X(-D) \to \mathcal{O}_X \to \mathcal{O}_D \to 0$$

we obtain, by tensoring with ω_X^{-1}, a canonical injection

$$H^0(X, \omega_X^{-1}(-D)) \hookrightarrow H^0(X, \omega_X^{-1}).$$

It follows that the choice of ζ determines a natural Poisson structure on X.

In [B2] we proved that the moduli space $\mathcal{M}$ of stable vector bundles on a Poisson surface is always non-singular. For the special class of surfaces we are considering here, the same result holds also for the moduli space of parabolic vector bundles.

Proposition 4.2. *Let X be a non-singular projective surface and D an effective divisor such that $H^0(X, \omega_X^{-1}(-D)) \neq 0$. Then the moduli space $\mathcal{PB}$ of stable parabolic vector bundles on X is a non-singular quasi-projective variety.*

Proof. The obstruction to the smoothness of the moduli space $\mathcal{PB}$ at a point E_* lies in $H^2(X, \mathcal{H}om(E_*, E_*))$. By the Serre duality theorem for parabolic bundles (cf. [Y2, Proposition 3.7]), the dual of this vector space is canonically identified with $H^0(X, \mathcal{H}om(E_*, \hat{E}_*) \otimes \omega_X(D))$. Finally, the existence of a non-zero global section of $\omega_X^{-1}(-D)$, together with the hypothesis of stability of E_*, implies that $H^0(X, \mathcal{H}om(E_*, \hat{E}_*) \otimes \omega_X(D)) = 0$. $\square$

By recalling the identifications of the tangent and cotangent spaces to the moduli space $\mathcal{PB}$ described above, we can define a map

$$B_\zeta(E_*) : T^*_{E_*}\mathcal{PB} \to T_{E_*}\mathcal{PB}$$

as follows:

$$B_\zeta(E_*) : H^1(X, \mathcal{H}om(E_*, \hat{E}_*) \otimes \omega_X(D)) \xrightarrow{\zeta} H^1(X, \mathcal{H}om(E_*, E_*)), \qquad (4.1)$$

where this is the map induced on the cohomology groups by the following map of sheaves (where ζ denotes multiplication by the section ζ):

$$\mathcal{H}om(E_*, \hat{E}_*) \otimes \omega_X(D) \xrightarrow{\zeta} \mathcal{H}om(E_*, \hat{E}_*) \hookrightarrow \mathcal{H}om(E_*, E_*).$$

From the smoothness of the moduli space $\mathcal{PB}$, it follows that the family of maps $B_\zeta(E_*)$, as E_* varies in $\mathcal{PB}$, defines a global homomorphism

$$B_\zeta : T^*\mathcal{PB} \to T\mathcal{PB}.$$

From this we obtain a global section $\theta_\zeta \in H^0(\mathcal{PB}, \otimes^2 T\mathcal{PB})$ defined by setting

$$\theta_\zeta(\alpha, \beta) = \langle B_\zeta(\alpha), \beta\rangle,$$

for any two 1-forms α, β on $\mathcal{PB}$.

By recalling the expression of the Serre duality and the definition of B_ζ, it is not difficult to see that, for any $E_* \in \mathcal{PB}$, the map

$$\theta_\zeta(E_*) : T^*_{E_*}\mathcal{PB} \times T^*_{E_*}\mathcal{PB} \to \mathbb{C}$$

is identified to the map

$$\theta_\zeta(E_*) : H^1(X, \mathcal{H}om(E_*, \hat{E}_*) \otimes \omega_X(D)) \times H^1(X, \mathcal{H}om(E_*, \hat{E}_*) \otimes \omega_X(D)) \\ \to H^2(X, \omega_X) \cong \mathbb{C},$$

given by the composition of the following maps: first we take the cup-product of two cohomology classes followed by the composition of the homomorphisms of filtered sheaves, then we compose with the map induced by the multiplication by ζ, and finally we apply the trace map

$$\mathrm{tr} : H^2(X, \mathcal{H}om(E_*, \hat{E}_*) \otimes \omega_X(D)) \to H^2(X, \omega_X).$$

From this explicit description of θ_ζ it follows, by recalling the graded commutativity property of the cup-product, that θ_ζ is skew-symmetric, hence we have actually defined a bivector field $\theta_\zeta \in H^0(\mathcal{PB}, \wedge^2 T\mathcal{PB})$. This is our candidate to define a Poisson structure on the moduli space $\mathcal{PB}$.

Remark 4.3. If the divisor D is non-singular, the definition of the bivector field θ_ζ on the moduli space $\mathcal{PB}$ can be extended to the moduli space $\mathcal{PS}$ of torsion-free coherent parabolic sheaves. In order to do that it is enough to modify the construction of θ_ζ given above by simply replacing the cohomology groups $H^1(X, \mathcal{H}om(E_*, E_*))$ and $H^1(X, \mathcal{H}om(E_*, \hat{E}_*) \otimes \omega_X(D))$ with appropriate "parabolic Ext"-groups (defined in [Y2]), that are canonically identified with the tangent and cotangent spaces to the moduli space $\mathcal{PS}$ at the point corresponding to a parabolic sheaf E_*. The assumption of non-singularity of D is needed in order to be able to apply the Serre duality theorem for non-locally free parabolic sheaves (cf. [Y2, Proposition 3.7]).

5. The closure of θ_ζ

In this section we shall prove that θ_ζ satisfies the closure condition $\tilde{d}\theta_\zeta = 0$, i.e., it defines a Poisson structure on the moduli space $\mathcal{PB}$. The proof of this result is a generalization to the case of parabolic bundles of the proof of the analogous result for the moduli space of vector bundles given in [B2]. We shall describe only the relevant modifications.

First of all let us recall some preliminary results. Let $\pi : X \to Y$ be a morphism (locally of finite presentation) of schemes, and E, F two locally free sheaves on X. We denote by $\mathcal{D}iff^1_{X/Y}(E, F)$ the sheaf of relative differential operators from E to F of order ≤ 1. We have the following short exact sequence

$$0 \to \mathcal{H}om_X(E, F) \to \mathcal{D}iff^1_{X/Y}(E, F) \xrightarrow{\sigma} \mathcal{D}er_Y(\mathcal{O}_X) \otimes \mathcal{H}om_X(E, F) \to 0,$$

where σ is the symbol morphism. From this, if $E = F$ and we restrict to differential operators with scalar symbol, written $\mathcal{D}^1_{X/Y}(E)$, we obtain the exact sequence

$$0 \to \mathcal{E}nd_X(E) \to \mathcal{D}^1_{X/Y}(E) \xrightarrow{\sigma} \mathcal{D}er_Y(\mathcal{O}_X) \to 0.$$

Now, if E_* and F_* are two parabolic bundles on X, we denote by $\mathcal{D}iff^1_{X/Y}(E_*, F_*)$ the subsheaf of $\mathcal{D}iff^1_{X/Y}(E, F)$ of differential operators $D : E \to F$ such that $D(E_x) \subset F_x$, for any $x \in \mathbb{R}$. We call it the sheaf of relative parabolic differential operators from E_* to F_* of order ≤ 1. Analogously, if $E_* = F_*$ we can define the sheaf $\mathcal{D}^1_{X/Y}(E_*)$ of relative parabolic differential operators with scalar symbol. We have the following exact sequence:

$$0 \to \mathcal{H}om_X(E_*, E_*) \to \mathcal{D}^1_{X/Y}(E_*) \xrightarrow{\sigma} \mathcal{D}er_Y(\mathcal{O}_X) \to 0.$$

Let us denote by $p : \mathcal{PB} \times X \to \mathcal{PB}$ and $q : \mathcal{PB} \times X \to X$ the canonical projections. We would like to apply the preceding results to the map $q : \mathcal{PB} \times X \to X$ and to the 'universal parabolic bundle' $\mathcal{E}_*$ on $\mathcal{PB} \times X$, but unfortunately such a universal parabolic bundle does not exist in general (cf. Remark 2.6). So let us consider an open subset U (for the complex or étale topology, not for the Zariski topology) of the moduli space $\mathcal{PB}$ such that there exists a universal parabolic family $\mathcal{E}_* = \mathcal{E}^U_*$ on $U \times X$. We still denote by p and q the restrictions of the canonical projections to $U \times X$. By applying the preceding results to the map $q : U \times X \to X$ and to the (local) universal parabolic bundle $\mathcal{E}_*$, we obtain the following exact sequence

$$0 \to \mathcal{H}om(\mathcal{E}_*, \mathcal{E}_*) \to \mathcal{D}^1_X(\mathcal{E}_*) \to p^*TU \to 0,$$

where $\mathcal{D}^1_X(\mathcal{E}_*) = \mathcal{D}^1_{U\times X/X}(\mathcal{E}_*)$ denotes the sheaf of first-order parabolic differential operators on $\mathcal{E}_*$ with scalar symbol, that are $q^*\mathcal{O}_X$-linear.

By applying p_*, and noting that $p_*p^*TU \cong TU$ since $p : U \times X \to U$ is a proper morphism, we get a long exact sequence, a piece of which is

$$\cdots \to TU \to R^1 p_*(\mathcal{H}om(\mathcal{E}_*, \mathcal{E}_*)) \to R^1 p_*(\mathcal{D}^1_X(\mathcal{E}_*)) \to \cdots . \tag{5.1}$$

It is not difficult to see that the map

$$TU \to R^1 p_*(\mathcal{H}om(\mathcal{E}_*, \mathcal{E}_*))$$

is the global version of the canonical isomorphism

$$T_{E_*}\mathcal{PB} \xrightarrow{\sim} H^1(X, \mathcal{H}om(\mathcal{E}_*, \mathcal{E}_*)),$$

for $E_* \in U$, hence it is an isomorphism.

From the exact sequence (5.1) it follows that the map

$$R^1 p_*(\mathcal{H}om(\mathcal{E}_*, \mathcal{E}_*)) \to R^1 p_*(\mathcal{D}^1_X(\mathcal{E}_*))$$

factors through 0, i.e., the image of a global section $\{\eta_{ij}\}$ of $R^1 p_*(\mathcal{H}om(\mathcal{E}_*, \mathcal{E}_*))$ is zero in $R^1 p_*(\mathcal{D}^1_X(\mathcal{E}_*))$. This means that there exist sections $\dot{D}_i$ of $R^1 p_*(\mathcal{D}^1_X(\mathcal{E}_*))$ over suitable open subsets V_i, such that, on $V_i \cap V_j$, we have

$$\eta_{ij} = \dot{D}_j - \dot{D}_i.$$

Note that this equation is formally the same as equation (5.2) of [B2]. From now on all the subsequent discussion carried out in [B2, Sect. 5] can be repeated, almost literally, for the moduli space $\mathcal{PB}$. The proof of the closure condition $\tilde{d}\theta_\zeta = 0$ is now practically identical to the proof of [B2, Theorem 5.1]. Finally, note that the preceding discussion applies to any open subset of a suitable open covering of the moduli space $\mathcal{PB}$.

We have thus proved the following result:

Theorem 5.1. *For any non-zero section $\zeta \in H^0(X, \omega_X^{-1}(-D))$, the antisymmetric contravariant* 2*-tensor $\theta_\zeta \in H^0(\mathcal{PB}, \wedge^2 T\mathcal{PB})$ defines a Poisson structure on the moduli space $\mathcal{PB}$.*

Remark 5.2. Let X, D and $\zeta \in H^0(X, \omega_X^{-1}(-D))$ be as above. We have already observed (cf. Remark 4.1) that ζ determines a Poisson structure on X, hence, by the results of [B2], it also defines a Poisson structure on the moduli space $\mathcal{M}$ of stable vector bundles on X. If we restrict to the open subscheme $\mathcal{PB}^o$ of $\mathcal{PB}$ parametrizing parabolic bundles E_* such that the vector bundle $E = E_0$ is stable, and consider the natural projection map $\pi : \mathcal{PB}^o \to \mathcal{M}$ sending a parabolic bundle E_* to E, it is easy to prove that π is a Poisson morphism, i.e., it is compatible with the Poisson structures of $\mathcal{PB}^o$ and $\mathcal{M}$ determined by ζ.

Remark 5.3. Let X be a Poisson surface, with Poisson structure given by a non-zero section $s \in H^0(X, \omega_X^{-1})$ (cf. [B2]) and let D be the divisor defined by s. In this situation it is natural to consider parabolic vector bundles on X with parabolic structure over D. Let us denote by $\mathcal{PB}$ the corresponding moduli space.

According to Theorem 5.1, to define a Poisson structure on $\mathcal{PB}$ we need a non-zero global section ζ of $\omega_X^{-1}(-D)$, but in this case we have $\omega_X^{-1}(-D) \cong \mathcal{O}_X$, hence there is a canonical choice for the section ζ, namely $\zeta = 1 \in H^0(X, \mathcal{O}_X)$. The corresponding Hamiltonian morphism

$$B = B_1 : T^*\mathcal{PB} \to \mathcal{PB}$$

is given, at any point $E_* \in \mathcal{PB}$, by the map

$$H^1(X, \mathcal{H}om(E_*, \hat{E}_*)) \to H^1(X, \mathcal{H}om(E_*, E_*))$$

induced by the natural injection of sheaves

$$\mathcal{H}om(E_*, \hat{E}_*) \hookrightarrow \mathcal{H}om(E_*, E_*).$$

Note that this Hamiltonian morphism is not an isomorphism, in general, hence the Poisson structure on $\mathcal{PB}$ corresponding to the choice $\zeta = 1$ is, in general, not a symplectic structure.

For instance, if we consider parabolic vector bundles with trivial parabolic structure over D, i.e., vector bundles E with parabolic structure given by

$$E = F_1(E) \supset F_2(E) = E(-D)$$

(and weight $\alpha_1 = 0$), then the moduli space $\mathcal{PB}$ coincides with the moduli space $\mathcal{M}$ of stable vector bundles on X. We also have the following identifications:

$$\mathcal{H}om(E_*, E_*) = \mathcal{H}om(E, E)$$

and

$$\mathcal{H}om(E_*, \hat{E}_*) = \mathcal{H}om(E, E) \otimes \mathcal{O}_X(-D) = \mathcal{H}om(E, E) \otimes \omega_X.$$

From this it follows that the Hamiltonian map

$$B : T^*\mathcal{M} \to T\mathcal{M}$$

is given, at any point $E \in \mathcal{M}$, by the map

$$H^1(X, \mathcal{H}om(E, E) \otimes \omega_X) \to H^1(X, \mathcal{H}om(E, E))$$

induced on the cohomology groups by the map of sheaves

$$\mathcal{H}om(E, E) \otimes \omega_X \xrightarrow{s} \mathcal{H}om(E, E)$$

given by the multiplication by the section $s \in H^0(X, \omega_X^{-1})$. But this is precisely the Poisson structure on the moduli space $\mathcal{M}$ constructed in [B2]. In this way we can recover the results obtained in [B2] as a special case of the general construction of Poisson structures on moduli spaces of parabolic vector bundles on X.

Remark 5.4. Let us consider here another special case: $X = \mathbb{P}^2$ and $D = \ell$ a line in $\mathbb{P}^2$. In [B4] we proved that the moduli space $\mathcal{FB}$ of stable framed vector bundles on X, i.e., the moduli space parametrizing isomorphism classes of pairs (E, η), where E is a vector bundle of rank r on X with $c_1 = 0$ and $\eta : E|_D \xrightarrow{\sim} \mathcal{O}_D^{\oplus r}$ is an isomorphism of vector bundles, has a canonical holomorphic Poisson structure, determined by the section $s \in H^0(X, \mathcal{O}_X(1))$ defining the divisor D (this Poisson structure is actually everywhere non-degenerate, hence it defines a holomorphic symplectic structure on $\mathcal{FB}$). Note that, by the work of Donaldson [D], the moduli space $\mathcal{FB}$ is isomorphic to the moduli space of framed $\mathrm{SU}(r)$-instantons on $\mathbb{R}^4 \cup \{\infty\}$.

In [M2] Maruyama proved that the moduli space $\mathcal{FB}$ of framed vector bundles is isomorphic to an open subset of a certain moduli space $\mathcal{PB}$ of stable parabolic vector bundles on X with parabolic structure over D. The isomorphism can be described as follows: to a vector bundle E with a trivialization $\eta : E|_D \xrightarrow{\sim} \mathcal{O}_D^{\oplus r}$ we can associate a homomorphism of vector bundles $\phi : E|_D \to \mathcal{O}_D(r-1)$ inducing an isomorphism on the spaces of global sections. The kernel of ϕ is then isomorphic to $\mathcal{O}_D(-1)^{\oplus r-1}$, hence we obtain the following quasi-parabolic structure on E over D:

$$E|_D = \mathcal{F}_D^1(E) \supset \mathcal{O}_D(-1)^{\oplus r-1} = \mathcal{F}_D^2(E) \supset 0 = \mathcal{F}_D^3(E)$$

(we are using here the alternative definition of parabolic structure given in Remark 2.2). To obtain a parabolic structure it remains only to fix two weights, α_1 and α_2. The choice of weights influences the parabolic stability of the resulting parabolic vector bundle E_*, and Maruyama proved that there is a special choice for the weights, namely $\alpha_1 = \frac{1}{3}$ and $\alpha_2 = \frac{1}{2}$, such that the parabolic vector bundle E_* corresponding to a framed vector bundle (E, η) as above is always parabolic stable (see [M2] for more details).

From Theorem 5.1 it follows that a Poisson structure on $\mathcal{PB}$ is determined by a global section ζ of $\omega_X^{-1}(-D) \cong \mathcal{O}_X(2)$. Obviously, there is now a natural choice for ζ, namely $\zeta = s^2$. It is now easy to check, by using the explicit description of the Hamiltonian morphism B, that the Poisson structure defined by $\zeta = s^2$ on $\mathcal{PB}$ coincides with the Poisson structure defined by s on $\mathcal{FB}$.

6. The symplectic leaf foliation

In this section we shall investigate the structure of the symplectic leaf foliation of the moduli space $\mathcal{PB}$ in some special cases. We start by considering the simpler case of the moduli space $\mathcal{M}$ of vector bundles (with no parabolic structure) on a Poisson surface X.

Let X be a smooth projective surface with the Poisson structure determined by $s \in H^0(X, \omega_X^{-1})$. Let us fix an ample divisor H on X, and let $\mathcal{M} = \mathcal{M}(r, c_1, c_2)$ denote the moduli space of H-stable vector bundles on X of rank r and Chern classes c_1 and c_2. In [B2] we proved that the moduli space $\mathcal{M}$ has a canonical Poisson structure θ_s determined by the section s. The Poisson structure θ_s can be described by giving its Hamiltonian morphism $B_s : T^*\mathcal{M} \to T\mathcal{M}$. For any $E \in \mathcal{M}$ we define $B_s(E)$ to be the map

$$B_s(E) : H^1(X, \mathcal{E}nd(E) \otimes \omega_X) \to H^1(X, \mathcal{E}nd(E))$$

induced on cohomology by the homomorphism of sheaves

$$\mathcal{E}nd(E) \otimes \omega_X \xrightarrow{s} \mathcal{E}nd(E)$$

given by the multiplication by the section s.

It is not difficult to see that the kernel of $B_s(E)$ can be canonically identified with $H^0(D_s, \mathcal{E}nd(E|_{D_s}))/\mathbb{C}$, where D_s is the divisor defined by the section s. It follows that the codimension of the symplectic leaf passing through a point $E \in \mathcal{M}$ is equal to $h^0(D_s, \mathcal{E}nd(E|_{D_s})) - 1$. We refer to [B2] for further details.

Let us now fix a vector bundle F on D_s and consider the moduli space $\mathcal{FB}_F$ parametrizing framed vector bundles on X whose restriction to D_s is isomorphic to F. More precisely, a point in $\mathcal{FB}_F$ is an isomorphism class of pairs (E, η), where E is a stable vector bundle (of rank r and Chern classes c_1 and c_2) on X and $\eta : E|_{D_s} \xrightarrow{\sim} F$ is an isomorphism of vector bundles. We refer to [B4] or [HL] for more details on moduli spaces of framed vector bundles.

There is an obvious map $\pi : \mathcal{FB}_F \to \mathcal{M}$ which "forgets the framing", i.e., it sends a framed vector bundle (E, η) to E (remember that we are considering only framed vector bundles (E, η) such that E is a stable vector bundle). We shall denote by $\mathcal{M}_F$ the image of this map:

$$\mathcal{M}_F = \{E \in \mathcal{M} \mid E|_{D_s} \cong F\}.$$

Remark 6.1. Two framed vector bundles (E, η) and (E', η') are isomorphic (as framed vector bundles) if there exists an isomorphism $\phi : E \xrightarrow{\sim} E'$ such that its restriction to the divisor D_s satisfies $\eta' \circ \phi|_{D_s} = \lambda\eta$, for some constant $\lambda \in \mathbb{C}^*$, hence, for every $\lambda \in \mathbb{C}^*$, the framed vector bundles (E, η) and $(E, \lambda\eta)$ determine the same point in the moduli space $\mathcal{FB}_F$. It follows that there is a natural action of the group $G_F = \mathbb{P}\,\mathrm{Aut}_{\mathcal{O}_{D_s}}(F)$ on $\mathcal{FB}_F$ defined by $\bar{g} \cdot (E, \eta) = (E, g \circ \eta)$, where $g \in \mathrm{Aut}_{\mathcal{O}_{D_s}}(F)$ is any representative of the class $\bar{g} \in G_F$. The quotient $\mathcal{FB}_F/G_F$ is naturally identified with the subscheme $\mathcal{M}_F$ of the moduli space $\mathcal{M}$.

Infinitesimal deformation theory provides a canonical identification between the tangent space $T_{(E,\eta)}\mathcal{FB}_F$ to the moduli space $\mathcal{FB}_F$ at a point (E, η) and the first cohomology group $H^1(X, \mathcal{E}nd(E) \otimes \mathcal{O}_X(-D_s))$. In the present situation, we have $\mathcal{O}_X(D_s) \cong \omega_X^{-1}$, hence we have a canonical identification

$$T_{(E,\eta)}\mathcal{FB}_F \cong H^1(X, \mathcal{E}nd(E) \otimes \omega_X).$$

The tangent map to $\pi : \mathcal{FB}_F \to \mathcal{M}$ at a point (E, η) is then canonically identified with the map

$$H^1(X, \mathcal{E}nd(E) \otimes \omega_X) \xrightarrow{s} H^1(X, \mathcal{E}nd(E))$$

induced on cohomology by the homomorphism of sheaves

$$\mathcal{E}nd(E) \otimes \omega_X \xrightarrow{s} \mathcal{E}nd(E)$$

given by the multiplication by the section $s \in H^0(X, \omega_X^{-1})$. But this coincides with the map B_s defining the Poisson structure of the moduli space $\mathcal{M}$.

By recalling that the symplectic leaves of a Poisson variety are the integral leaves of the characteristic distribution, which is the distribution defined by the image of the Hamiltonian morphism B_s, and by using the identification between the map B_s and the tangent map to the morphism $\pi : \mathcal{FB}_F \to \mathcal{M}$, it is now easy to see that the symplectic leaf foliation of the moduli space $\mathcal{M}$ is determined by the images of the various moduli spaces $\mathcal{FB}_F$ in $\mathcal{M}$, as we vary the vector bundle F on D_s. More precisely, we have:

Proposition 6.2. *The symplectic leaves of the moduli space $\mathcal{M}$, endowed with the Poisson structure θ_s corresponding to the global section s of ω_X^{-1}, are the (connected components of the) subschemes $\mathcal{M}_F = \{E \in \mathcal{M} \mid E|_{D_s} \cong F\}$, for any choice of a vector bundle F on the divisor D_s defined by the section s.*

Remark 6.3. This description of the symplectic leaves of $\mathcal{M}$, together with the isomorphism $\mathcal{M}_F \cong \mathcal{FB}_F/G_F$, agrees with the previous computation of the rank of the Poisson structure θ_s, expressed as $\dim \mathcal{M} - (h^0(D_s, \mathcal{E}nd(E|_{D_s})) - 1)$. In fact, we have: $\dim \mathcal{FB}_F = \dim \mathcal{M}$, and $h^0(D_s, \mathcal{E}nd(E|_{D_s})) - 1$ is precisely the dimension of the group G_F.

Remark 6.4. From the identification

$$T_{(E,\eta)}\mathcal{FB}_F \cong H^1(X, \mathcal{E}nd(E) \otimes \omega_X) \cong T_E^*\mathcal{M}$$

it follows that the Hamiltonian morphism $B_s : T^*\mathcal{M} \to T\mathcal{M}$ determines a morphism $b_s : T\mathcal{FB}_F \to T^*\mathcal{FB}_F$. The same proof given in [B2] to show that $B_s : T^*\mathcal{M} \to T\mathcal{M}$ defines a Poisson structure on $\mathcal{M}$, can be used to show that the morphism $b_s : T\mathcal{FB}_F \to T^*\mathcal{FB}_F$ defines a closed, holomorphic 2-form $\omega_s \in H^0(\mathcal{FB}_F, \wedge^2 T^*\mathcal{FB}_F)$ on the moduli space $\mathcal{FB}_F$. This is not a symplectic structure because, in general, it will be degenerate. We shall call it a quasi-symplectic structure.

The action of the group G_F is immediately seen to be a quasi-symplectic action (with the obvious meaning of the word), and an analogue of the Marsden-Weinstein symplectic reduction can be performed in this quasi-symplectic case. The corresponding moment map $\mu : \mathcal{FB}_F \to \mathfrak{g}_F^*$ is identically zero, hence the reduced space is $\mu^{-1}(0)/G_F = \mathcal{FB}_F/G_F \cong \mathcal{M}_F$.

Note that, even if the 2-form ω_s on $\mathcal{FB}_F$ is only quasi-symplectic, the induced 2-form on the reduced moduli space $\mathcal{M}_F \cong \mathcal{FB}_F/G_F$ is actually a symplectic structure. This is precisely the symplectic structure induced on the symplectic leaf $\mathcal{M}_F$ by the Poisson structure θ_s of $\mathcal{M}$.

The next case we are going to consider is the one described in Remark 5.3. Using the same notations as before, X will be a Poisson surface with Poisson structure determined by the section $s \in H^0(X, \omega_X^{-1})$ and $D = D_s$ will denote the divisor defined by s. We shall now consider the moduli space $\mathcal{PB}$ of parabolic vector bundles with parabolic structure over D. In this situation there is a canonical Poisson structure on the moduli space $\mathcal{PB}$ corresponding to the choice of the section $\zeta = 1$ of $\omega_X^{-1}(-D) \cong \mathcal{O}_X$. The corresponding Hamiltonian morphism $B = B_1 : T^*\mathcal{PB} \to T\mathcal{PB}$ is given by the maps

$$B(E_*) : H^1(X, \mathcal{H}om(E_*, \hat{E}_*)) \to H^1(X, \mathcal{H}om(E_*, E_*)) \tag{6.1}$$

induced by the natural inclusion of sheaves $\mathcal{H}om(E_*, \hat{E}_*) \hookrightarrow \mathcal{H}om(E_*, E_*)$, for any $E_* \in \mathcal{PB}$.

Since the symplectic leaves of the Poisson manifold $\mathcal{PB}$ are the integral leaves of the characteristic distribution, which is the distribution defined by the images of the maps $B(E_*)$, and recalling the results obtained in the previous special case, we shall now look for a moduli space whose tangent space at some point can be identified with the cohomology group $H^1(X, \mathcal{H}om(E_*, \hat{E}_*))$.

For a parabolic vector bundle $E_* \in \mathcal{PB}$, let us denote by

$$E = F_1(E) \supset F_2(E) \supset \cdots \supset F_l(E) \supset F_{l+1}(E) = E(-D)$$

its parabolic structure over D (with some fixed weights α_*). Let us choose now, for any $i = 1, \ldots, l$, a vector bundle G_i on D and set $\mathcal{G} = (G_1, \ldots, G_l)$. We shall now denote by $\mathcal{FPB}_{\mathcal{G}}$ the moduli space parametrizing isomorphism classes of $(l+1)$-tuples $(E_*, \eta_1, \ldots, \eta_l)$, where $E_* \in \mathcal{PB}$ and $\eta_i : F_i(E)/F_{i+1}(E) \xrightarrow{\sim} G_i$ is an isomorphism of vector bundles, for $i = 1, \ldots, l$. Then we have:

Proposition 6.5. *The tangent space at a point $(E_*, \eta_1, \ldots, \eta_l)$ to the moduli space $\mathcal{FPB}_{\mathcal{G}}$ is canonically identified to the first cohomology group $H^1(X, \mathcal{H}om(E_*, \hat{E}_*))$.*

Proof. We shall give a sketch of the proof in the special case of parabolic vector bundles with parabolic structure of length 2. The general case is only notationally more complicated. Moreover, since in this case it is more convenient to use the definition of parabolic structure as given in Remark 2.2, we shall denote the parabolic structure of E by

$$E|_D = \mathcal{F}_D^1(E) \supset \mathcal{F}_D^2(E) \supset \mathcal{F}_D^3(E) = 0$$

(with some fixed weights α_1 and α_2).

Let $(E_*, \eta_1, \eta_2) \in \mathcal{FPB}_{\mathcal{G}}$. If we choose a suitable open covering $\mathcal{U} = (U_i)_{i \in I}$ of X, the rank-r vector bundle E can be described by giving a collection of transition functions

$$g_{ij} : U_i \cap U_j \to \mathrm{GL}(r, \mathbb{C}),$$

satisfying the usual cocycle conditions. Since $\mathcal{F}_D^2(E)$ is a sub-vector bundle of $E|_D$, it is possible to choose the open covering $\mathcal{U}$ and the transition functions g_{ij} such that the restriction of g_{ij} to $(U_i \cap U_j) \cap D$ has the following form

$$g_{ij}|_D = \begin{pmatrix} f_{ij} & h_{ij} \\ 0 & k_{ij} \end{pmatrix}$$

where f_{ij} and k_{ij} are transition functions for the sub-bundle $\mathcal{F}_D^2(E)$ and for the quotient bundle $\mathcal{F}_D^1(E)/\mathcal{F}_D^2(E)$, respectively.

To construct a tangent vector to the moduli space $\mathcal{FPB}_{\mathcal{G}}$ at the point (E_*, η_1, η_2), let us consider a curve

$$(-\epsilon, \epsilon) \ni t \mapsto (E_*(t), \eta_1(t), \eta_2(t)) \in \mathcal{FPB}_{\mathcal{G}}$$

with $(E_*(0), \eta_1(0), \eta_2(0)) = (E_*, \eta_1, \eta_2)$. The transition functions $g_{ij}(t)$ of the vector bundle $E(t)$ can be choosen so that their restrictions to D have the following form

$$g_{ij}(t)|_D = \begin{pmatrix} f_{ij}(t) & h_{ij}(t) \\ 0 & k_{ij}(t) \end{pmatrix}.$$

The tangent vector to $\mathcal{FPB}_{\mathcal{G}}$ at the point (E_*, η_1, η_2) determined by this curve is identified with the cohomology class determined by the cocycle $\dot{g}_{ij} = \frac{dg_{ij}}{dt}|_{t=0}$. Since the isomorphism classes of the vector bundles $\mathcal{F}_D^2(E(t))$ and

$$\mathcal{F}_D^1(E(t))/\mathcal{F}_D^2(E(t))$$

are fixed, it is possible to choose the transition functions $g_{ij}(t)$ such that the transition functions $f_{ij}(t)$ and $k_{ij}(t)$ do not actually depend on t. It follows that the functions $\dot{g}_{ij}$ have the property that their restriction to D have the following form

$$\dot{g}_{ij}|_D = \begin{pmatrix} 0 & \dot{h}_{ij} \\ 0 & 0 \end{pmatrix}.$$

This means that the functions $\dot{g}_{ij}$ determine a cocycle with values in the sheaf $\mathcal{H}om(E_*, \hat{E}_*)$ of homomorphisms $\phi : E \to E$ such that the restriction $\phi|_D : E|_D \to E|_D$ is nilpotent with respect to the parabolic structure of E, i.e., $\phi|_D(\mathcal{F}_D^i(E)) \subseteq \mathcal{F}_D^{i+1}(E)$. It is now easy to deduce that the tangent vectors to $\mathcal{FPB}_{\mathcal{G}}$ at a point (E_*, η_1, η_2) are naturally identified with the elements of the cohomology group $H^1(X, \mathcal{H}om(E_*, \hat{E}_*))$. $\square$

There is a natural map $\pi : \mathcal{FPB}_{\mathcal{G}} \to \mathcal{PB}$ sending a point $(E_*, \eta_1, \dots, \eta_l)$ to E_*. In view of the preceding result, the tangent map to π at a point $(E_*, \eta_1, \dots, \eta_l)$ is naturally identified with the map (6.1), but these are precisely the maps defining the characteristic distribution of the Poisson variety $\mathcal{PB}$. It follows that the images of the moduli spaces $\mathcal{FPB}_{\mathcal{G}}$ in $\mathcal{PB}$ (as we vary the collection of vector bundles $\mathcal{G}$ on D) determine the symplectic leaf foliation of $\mathcal{PB}$. Precisely, we have:

Proposition 6.6. *The symplectic leaves of the moduli space $\mathcal{PB}$ endowed with its canonical Poisson structure are the (connected components of the) subschemes*

$$\mathcal{PB}_{\mathcal{G}} = \{E_* \in \mathcal{PB} \mid F_i(E)/F_{i+1}(E) \cong G_i, \text{ for } i = 1, \dots, l\},$$

for any collection of vector bundles $\mathcal{G} = (G_1, \dots, G_l)$ on the divisor D.

Remark 6.7. Note that if the parabolic structure over the divisor D is the trivial one (cf. Remark 5.3), we recover the description of the symplectic leaf foliation of the moduli space $\mathcal{M}$ of stable vector bundles on the Poisson surface X obtained in the first part of this section.

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