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# Poisson structures on moduli spaces of sheaves over Poisson surfaces

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**Summary.** We introduce and study the notion of Poisson surface. We prove that the choice of a Poisson structure on a surface $S$ canonically determines a Poisson structure on the moduli space $\mathcal{M}$ of stable sheaves on $S$. This result generalizes previous results obtained by Mukai [14], for abelian or $K3$ surfaces, and by Tyurin [16].

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## Introduction

In [14], Mukai proved that the moduli space $\mathcal{M}$ of sheaves on an abelian or $K3$ surface $S$ has a natural symplectic structure $\omega$. However, in Mukai's paper, a symplectic structure is defined as a nowhere degenerate holomorphic 2-form, hence a natural question arises: can we prove directly that $\omega$ is closed?

We may remark that the reason for considering only abelian or $K3$ surfaces is that their canonical bundle is trivial, i.e., they are symplectic surfaces, and it is actually the choice of a symplectic structure on the surface $S$ that induces a symplectic structure on the moduli space $\mathcal{M}$.

In a later paper [16], Tyurin generalized this result by showing that the choice of a 2-form $\omega \in H^0(S, \wedge^2 T^*S)$, which he calls a 'symplectic structure' on $S$ (resp. of a bivector field $\theta \in H^0(S, \wedge^2 TS)$, which he calls a 'Poisson structure'), determines in a canonical way a 2-form $\tilde{\omega} \in H^0(\mathcal{M}, \wedge^2 T^*\mathcal{M})$ (resp. a bivector field $\tilde{\theta} \in H^0(\mathcal{M}, \wedge^2 T\mathcal{M})$), i.e., a symplectic structure (resp. a Poisson structure) on $\mathcal{M}$. In this case, again, no mention is made of the closure condition $d\tilde{\omega} = 0$ for the symplectic structure $\tilde{\omega}$, nor of the analogous condition that the Poisson bracket associated to a Poisson structure must satisfy the Jacobi identity.

In this paper we consider the general case of a Poisson surface $S$ and show that, in correspondence to the choice of a Poisson structure on $S$, there is a canonically defined Poisson structure on the moduli space $\mathcal{M}$ of stable sheaves on $S$, i.e., there is a bilinear antisymmetric bracket $\{\cdot,\cdot\}$, defined on the sheaf of regular functions on $\mathcal{M}$, that is a derivation in each entry and satisfies the Jacobi identity

$$\{f,\{g,h\}\}+\{g,\{h,f\}\}+\{h,\{f,g\}\}=0,$$

for any functions $f,g,h$ on $\mathcal{M}$.

If $S$ is an abelian or a $K3$ surface, i.e., in the symplectic case, our proof shows that the symplectic form defined by Mukai is actually closed.

This paper is organized as follows: in §1 we recall some basic definitions and results of symplectic geometry, then, in §2, we introduce and study Poisson surfaces. In §3 we collect some results on moduli spaces of sheaves on a Poisson surface $S$, and, in §4, we define the Poisson structure on the moduli space $\mathcal{M}$ canonically associated to the choice of a Poisson structure on $S$. In §5, we shall prove that $\theta$ satisfies a certain closure condition, equivalent to the Jacobi identity for the Poisson bracket defined by $\theta$. Finally, in §6, we conclude with some remarks on the rank of the Poisson structure $\theta$, i.e., on the dimension of the symplectic leaves of the Poisson variety $\mathcal{M}$.

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## 1. Symplectic and Poisson structures

We recall here some definitions and results of symplectic geometry.

Let $X$ be a smooth algebraic variety over the complex field $\mathbb{C}$. A (holomorphic) symplectic structure on $X$ is a closed nondegenerate 2-form $\omega \in H^0(X,\Omega_X^2)$. Given a symplectic structure $\omega$, we define the Hamiltonian vector field $H_f$ of a regular function $f$ by requiring that $\omega(H_f,v)=\langle df,v\rangle$, for every tangent field $v$. Then, for $f,g\in\Gamma(U,\mathcal{O}_X)$, we define the Poisson bracket $\{f,g\}$ of $f$ and $g$ by setting $\{f,g\}=\langle H_f,dg\rangle=\omega(H_g,H_f)$. The map $g\mapsto\{f,g\}$ is a derivation of $\Gamma(U,\mathcal{O}_X)$ whose corresponding vector field is precisely $H_f$. The pairing $\{\cdot,\cdot\}$ on $\mathcal{O}_X$ is a bilinear antisymmetric map that is a derivation in each entry and satisfies the Jacobi identity

$$\{f,\{g,h\}\}+\{g,\{h,f\}\}+\{h,\{f,g\}\}=0, \tag{1.1}$$

for any $f,g,h\in\Gamma(U,\mathcal{O}_X)$. This implies that $[H_f,H_g]=H_{\{f,g\}}$, where $[u,v]=uv-vu$ is the commutator of the vector fields $u$ and $v$.

A natural generalization of symplectic structures is given by the notion of Poisson structure.

A Poisson structure on $X$ is a Lie algebra structure $\{\cdot,\cdot\}$ on $\mathcal{O}_X$ satisfying the identity $\{f,gh\}=\{f,g\}h+g\{f,h\}$. Equivalently, this is given by an antisymmetric contravariant 2-tensor $\theta\in H^0(X,\wedge^2TX)$, where we set $\{f,g\}=\langle\theta,df\wedge dg\rangle$. Then $\theta$ is a Poisson structure if the bracket it defines satisfies the Jacobi identity (1.1). For any $f\in\Gamma(U,\mathcal{O}_X)$, the map $g\mapsto\{f,g\}$ is a derivation of $\Gamma(U,\mathcal{O}_X)$, hence corresponds to a vector field $H_f$ on $U$, called the hamiltonian vector field associated to $f$.

Note that giving $\theta \in H^0(X, \wedge^2 TX)$ is equivalent to giving a homomorphism of vector bundles $B : T^*X \to TX$, with $\langle \theta, \alpha \wedge \beta \rangle = \langle B(\alpha), \beta \rangle$ (or $\langle \alpha, B(\beta) \rangle$, up to a sign), for 1-forms $\alpha, \beta$.

Let us define an operator $\tilde{d} : H^0(X, \wedge^2 TX) \to H^0(X, \wedge^3 TX)$ as follows:

$$\begin{aligned}\tilde{d}\theta(\alpha, \beta, \gamma) = {} & B(\alpha)\theta(\beta, \gamma) - B(\beta)\theta(\alpha, \gamma) + B(\gamma)\theta(\alpha, \beta) \\ & - \langle [B(\alpha), B(\beta)], \gamma \rangle + \langle [B(\alpha), B(\gamma)], \beta \rangle - \langle [B(\beta), B(\gamma)], \alpha \rangle,\end{aligned}$$

for 1-forms $\alpha, \beta, \gamma$, where $[\cdot, \cdot]$ denotes the usual commutator of vector fields.

By a straightforward computation (using local coordinates), it is easy to prove the following

**Proposition 1.1.** *The bracket $\{\cdot, \cdot\}$, defined by an element $\theta \in H^0(X, \wedge^2 TX)$, satisfies the Jacobi identity if and only if $\tilde{d}\theta = 0$.*

*Remark 1.2.* A condition classically known to be equivalent to the Jacobi identity for the bracket $\{\cdot, \cdot\}$ is the vanishing of the so-called Schouten bracket $[\theta, \theta] = 0$ (see [15], for example). This is equivalent to the condition expressed by the preceding proposition.

When $\theta$ has maximal rank everywhere, to give $\theta$ is equivalent to giving its inverse 2-form $\omega \in H^0(X, \Omega^2_X)$, and the condition $\tilde{d}\theta = 0$ is equivalent to $d\omega = 0$. In this case the Poisson structure induces a symplectic structure.

## 2. Poisson surfaces

In this section we consider Poisson structures on smooth algebraic surfaces. Let $S$ be a smooth algebraic surface over the complex field $\mathbb{C}$. We shall denote by $\omega_S$ its canonical line bundle, by $K_S$ its canonical divisor, and by $q = \dim H^1(S, \mathcal{O}_S)$ the irregularity of $S$. We have the following

**Proposition 2.1.** *A Poisson structure on $S$ is given by a global section $s$ of the anticanonical line bundle $\omega_S^{-1}$.*

*Proof*. A Poisson structure on $S$ is, by definition, an element $s \in H^0(S, \wedge^2 TS) = H^0(S, \omega_S^{-1})$ that satisfies the condition $\tilde{d}s = 0$. But $S$ is a surface, hence the map $\tilde{d}$ is identically zero. □

**Definition 2.2.** *A Poisson surface $S$ is a smooth algebraic surface which admits a non-zero Poisson structure, i.e., such that $H^0(S, \omega_S^{-1}) \neq 0$.*

We have the following

**Proposition 2.3.** *Let $S$ be a connected Poisson surface. Then $S$ is either a $K3$ surface (if $\omega_S \cong \mathcal{O}_S$ and $q = 0$) or an abelian surface (if $\omega_S \cong \mathcal{O}_S$ and $q = 2$) or a ruled surface (if $\omega_S$ is not trivial).*

*Proof*. Let $s$ be a non-zero section of $\omega_S^{-1}$; we have an exact sequence

$$0 \to \omega_S \xrightarrow{s} \mathcal{O}_S \to \mathcal{O}_D \to 0,$$

where $D$ is the divisor of $s$. It follows that either $H^0(S, \omega_S) = 0$ or $H^0(S, \omega_S) = \mathbb{C}$. In the second case $\omega_S$ has a non-vanishing global section, hence it is trivial; $S$ is then either a $K3$ or an abelian surface, according to the value of $q$. If $H^0(S, \omega_S) = 0$, by considering the exact sequence

$$0 \to \omega_S^{n+1} \xrightarrow{s} \omega_S^n \to \omega_S^n|_D \to 0,$$

and using induction on $n$, it follows that $H^0(S, \omega_S^n) = 0$ for all $n \geq 1$. By recalling now a theorem of Enriques (cf., for example, [3, Chap. VI]), we conclude that $S$ is a ruled surface. $\square$

*Remark 2.4.* Note that not every ruled surface is a Poisson surface. Let, for example, $S = C \times \mathbb{P}^1$; then we have, for every integer $n$,

$$H^0(S, \omega_S^n) = H^0(C, \omega_C^n) \otimes H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-2n)).$$

It follows that $H^0(S, \omega_S^{-1}) \neq 0$ if and only if $C$ is a rational or an elliptic curve.

*Remark 2.5.* If $S$ is a Poisson surface and $\omega_S^{-1}$ is ample, then it follows that either $S = \mathbb{P}^1 \times \mathbb{P}^1$ or $S$ is obtained from $\mathbb{P}^2$ by blowing-up $n$ distinct points in general position, with $n \leq 8$. It follows, in particular, that $S$ is rational ([3, Ch. V, Ex. 1]).

*Remark 2.6.* Let $s \in H^0(S, \omega_S^{-1})$ be a Poisson structure on $S$, and denote by $D$ the divisor of $s$. Then the rank of the Poisson structure is 2 on the open subset $S \setminus D$, and is 0 on $D$. Hence the Poisson structure induces a symplectic structure on $S \setminus D$.

*Example 2.7.* Let $S = \mathbb{P}^2$ and take as the anticanonical divisor $D$ a triple line. We may fix homogeneous coordinates $(x_0, x_1, x_2)$ on $\mathbb{P}^2$ such that the section $s$ defining the divisor $D$ is given by $s = x_0^3$. The Poisson structure defined by $s$ on $\mathbb{P}^2$ induces a symplectic structure on $\mathbb{C}^2 = \mathbb{P}^2 \setminus D$. If we consider the coordinates $(X, Y)$ on $\mathbb{C}^2$ given by $X = x_1/x_0$ and $Y = x_2/x_0$, it is immediate to see that this symplectic structure is given by the 2-form $dX \wedge dY$. In other words, the Poisson structure determined by $s$ on $\mathbb{P}^2$ induces on $\mathbb{C}^2$ the standard symplectic structure.

## 3. The moduli space of semistable sheaves on $S$

Let $S$ be a connected Poisson surface, and let us fix a Poisson structure $s \in H^0(S, \omega_S^{-1})$ on $S$. Let us denote by $D$ the divisor of $s$, $D \in |-K_S|$.

We recall now some basic definitions and results on moduli spaces of semistable sheaves on surfaces.

Let $H$ be a very ample divisor on $S$. For every coherent torsion-free $\mathcal{O}_S$-module $E$, we set

$$p_E(n) = \frac{\chi(E(n))}{\mathrm{rk}(E)},$$

where $\mathrm{rk}(E)$ is the rank at the generic point of $S$, and $E(n) = E \otimes \mathcal{O}_S(nH)$.

From now on, by a sheaf on $S$ we will always mean an $\mathcal{O}_S$-module.

**Definition 3.1.** *A sheaf $E$ on $S$ is said to be $H$-stable (resp. $H$-semistable) if it is a coherent torsion-free $\mathcal{O}_S$-module and, for every proper coherent subsheaf $F$ of $E$, we have*

$$p_F(n) < p_E(n), \quad (\text{resp. } p_F(n) \leq p_E(n)),$$

*for all sufficiently large integers $n$.*

Let us denote by $r$, $c_1$ and $c_2$, respectively the rank and the Chern classes of a coherent torsion-free sheaf on $S$. We have the following well-known result:

**Theorem 3.2.** *For fixed $r, c_1, c_2$, there exists a coarse moduli space $\overline{\mathcal{M}} = \overline{\mathcal{M}}(r, c_1, c_2)$ parametrizing $S$-equivalence classes of $H$-semistable sheaves of rank $r$ and Chern classes $c_1$ and $c_2$ on $S$. $\overline{\mathcal{M}}$ is a projective variety and it contains an open subset $\mathcal{M} = \mathcal{M}(r, c_1, c_2)$ parametrizing isomorphism classes of $H$-stable sheaves.*

In the sequel we shall denote by $E$ either an $H$-semistable (resp. $H$-stable) sheaf on $S$, or the point of $\overline{\mathcal{M}}$ (resp. $\mathcal{M}$) corresponding to the $S$-equivalence class (resp. isomorphism class) of $E$.

From infinitesimal deformation theory, it follows that there is a canonical isomorphism

$$T_E\overline{\mathcal{M}} \cong \mathrm{Ext}^1(E, E), \tag{3.1}$$

where $T_E\overline{\mathcal{M}}$ denotes the tangent space to $\overline{\mathcal{M}}$ at $E$.

Then, from Grothendieck-Serre duality, it follows that the cotangent space to $\overline{\mathcal{M}}$ at $E$ is given by

$$T_E^*\overline{\mathcal{M}} \cong \mathrm{Ext}^1(E, E \otimes \omega_S). \tag{3.2}$$

We now turn to the problem of smoothness of moduli spaces. We have the following result:

**Proposition 3.3.** *The moduli space $\mathcal{M} = \mathcal{M}(r, c_1, c_2)$ is a smooth quasi-projective variety of dimension $(1-r)c_1^2 + 2rc_2 - r^2\chi(\mathcal{O}_S) + 1$.*

*Proof*. If $S$ is a symplectic surface ($\omega_S \cong \mathcal{O}_S$), this is proved in [14, Theorem 0.1]. If $S$ is a Poisson surface and $\omega_S$ is not trivial, then the divisor $D \in |-K_S|$ of a Poisson structure $s$ is effective. It follows that $(D \cdot H) > 0$, hence $(K_S \cdot H) < 0$. Under this condition, the smoothness of $\mathcal{M}$ is proved in [13, Corollary 6.7.3]. The computation of the dimension of $\mathcal{M}$ is an easy application of the Riemann-Roch theorem, and is done in [13, Proposition 6.9]. □

Since $\mathcal{M}$ is a smooth variety, we may give global versions of (3.1) and (3.2).

First we need a definition:

**Definition 3.4.** *Let $f : X \to T$ be a $T$-scheme, and $E$, $F$ two coherent $\mathcal{O}_X$-modules. The $i$-th relative* Ext*-sheaf $\mathcal{E}xt^i_{\mathcal{O}_T}(E, F)$ is the sheaf associated to the presheaf $U \mapsto \mathrm{Ext}^i_{f^{-1}(U)}(E_U, F_U)$ for every open subset $U$ of $T$.*

Now we note that, in general, a universal family $\mathcal{E}$ on $\mathcal{M}$ does not exist (not even on any Zariski open subset of $\mathcal{M}$); however it does exist locally in the étale topology (or in the complex topology). As shown by Mukai, this is enough to ensure that the $i$-th relative Ext-sheaves $\mathcal{E}xt^i_{\mathcal{O}_\mathcal{M}}(\mathcal{E},\mathcal{E})$ on $\mathcal{M}$ are well defined, for any integer $i$.

Then we have:

**Proposition 3.5.** *Let $p : \mathcal{M}\times S \to \mathcal{M}$ and $q : \mathcal{M}\times S \to S$ be the canonical projections. There are canonical isomorphisms*

$$T\mathcal{M} \cong \mathcal{E}xt^1_{\mathcal{O}_\mathcal{M}}(\mathcal{E},\mathcal{E}), \tag{3.3}$$

*and*

$$T^*\mathcal{M} \cong \mathcal{E}xt^1_{\mathcal{O}_\mathcal{M}}(\mathcal{E},\mathcal{E}\otimes q^*(\omega_S)). \tag{3.4}$$

Let us denote by $\mathcal{M}^0$ the open subset of $\mathcal{M}$ parametrizing isomorphism classes of $H$-stable locally free sheaves on $S$. Then, for $E\in\mathcal{M}^0$, we have

$$T_E\mathcal{M}^0 \cong H^1(S,\mathcal{E}nd(E)), \tag{3.5}$$

and

$$T^*_E\mathcal{M}^0 \cong H^1(S,\mathcal{E}nd(E)\otimes\omega_S). \tag{3.6}$$

Now, even if a universal family $\mathcal{E}$ on $\mathcal{M}$ does not exist, the sheaf $\mathcal{E}nd(\mathcal{E})$ on $\mathcal{M}\times S$ is well defined, and we have

$$\mathcal{E}nd(\mathcal{E})_{|\{E\}\times S} \cong \mathcal{E}nd(E), \quad \forall E\in\mathcal{M}.$$

Hence we may rewrite (3.3) and (3.4) for $\mathcal{M}^0$ as follows:

$$T\mathcal{M}^0 \cong R^1p_*(\mathcal{E}nd(\mathcal{E})), \tag{3.7}$$

and

$$T^*\mathcal{M}^0 \cong R^1p_*(\mathcal{E}nd(\mathcal{E})\otimes q^*(\omega_S)). \tag{3.8}$$

This may also be proved directly by noting that, since $\mathcal{M}^0$ is smooth, the function $E\mapsto \dim H^1(S,\mathcal{E}nd(E))$ is constant on $\mathcal{M}^0$. It follows that $R^1p_*(\mathcal{E}nd(\mathcal{E}))$ is locally free and we have isomorphisms $R^1p_*(\mathcal{E}nd(\mathcal{E}))\otimes\mathbb{C}(E)\cong H^1(S,\mathcal{E}nd(E))$, for every $E\in\mathcal{M}^0$. A similar argument works for the cotangent bundle.

## 4. Poisson structures on $\mathcal{M}$

Let $S$ be a Poisson surface and choose a Poisson structure $s\in H^0(S,\omega_S^{-1})$ on $S$. We shall define an element $\theta=\theta_s\in H^0(\mathcal{M},\otimes^2 T\mathcal{M})$ as follows: for any $E\in\mathcal{M}$, $\theta(E): T^*_E\mathcal{M}\times T^*_E\mathcal{M}\to\mathbb{C}$ is defined by

$$\begin{aligned}\theta(E):\operatorname{Ext}^1(E,E\otimes\omega_S)\times\operatorname{Ext}^1(E,E\otimes\omega_S)&\xrightarrow{\circ}\\ \operatorname{Ext}^2(E,E\otimes\omega_S^2)&\xrightarrow{s}\operatorname{Ext}^2(E,E\otimes\omega_S)\xrightarrow{\operatorname{Tr}}\mathbb{C},\end{aligned} \tag{4.1}$$

where the first map is the composition map, the second is induced by the multiplication by the section $s$, and the third is the trace map.

Note that, by Grothendieck-Serre duality and the stability hypothesis on $E$, it follows that the trace map $\operatorname{Tr}:\operatorname{Ext}^2(E,E\otimes\omega_S)\to\mathbb{C}$ is an isomorphism.

A method analogous to the one used by Mukai in [14] may be used to prove the following

**Proposition 4.1.** *The sheaf $\mathcal{L} = \mathcal{E}xt^2_{\mathcal{O}_{\mathcal{M}}}(\mathcal{E}, \mathcal{E} \otimes q^*(\omega_S))$ is a trivial invertible sheaf on $\mathcal{M}$, and there is a bilinear map*

$$\theta : \mathcal{E}xt^1_{\mathcal{O}_{\mathcal{M}}}(\mathcal{E}, \mathcal{E} \otimes q^*(\omega_S)) \otimes \mathcal{E}xt^1_{\mathcal{O}_{\mathcal{M}}}(\mathcal{E}, \mathcal{E} \otimes q^*(\omega_S)) \xrightarrow{\circ} \mathcal{E}xt^2_{\mathcal{O}_{\mathcal{M}}}(\mathcal{E}, \mathcal{E} \otimes q^*(\omega_S^2)) \xrightarrow{s} \mathcal{L}$$

*such that, for every $E \in \mathcal{M}$, $\theta \otimes \mathbb{C}(E)$ coincides with the map $\theta(E)$ defined in (4.1).*

As we have previously stated, giving $\theta$ is equivalent to giving a homomorphism of vector bundles

$$B : T^*\mathcal{M} \to T\mathcal{M},$$

where we set $\theta(\alpha \otimes \beta) = \langle B(\alpha), \beta\rangle$.

It is easy to see that in this situation the homomorphism $B$ is the map induced on Ext-sheaves by the multiplication by the section $s$. On the fibers over a point $E \in \mathcal{M}$, we have

$$B(E) : \mathrm{Ext}^1(E, E \otimes \omega_S) \xrightarrow{s} \mathrm{Ext}^1(E, E). \tag{4.2}$$

From now on we shall restrict ourselves to the open subset $\mathcal{M}^0$ of $\mathcal{M}$, and we shall use indifferently the expressions 'locally-free sheaf' or 'vector bundle'.

If $E$ is an $H$-stable locally-free sheaf, the map $\theta(E)$ may be written as

$$\begin{aligned} \theta(E) :& H^1(S, \mathcal{E}nd(E) \otimes \omega_S) \times H^1(S, \mathcal{E}nd(E) \otimes \omega_S) \xrightarrow{\circ} \\ & H^2(S, \mathcal{E}nd(E) \otimes \omega_S^2) \xrightarrow{s} H^2(S, \mathcal{E}nd(E) \otimes \omega_S) \xrightarrow{\mathrm{Tr}} \mathbb{C}. \end{aligned} \tag{4.3}$$

This is essentially the cup-product of two cohomology classes, followed by the multiplication by $s$. From the graded commutativity of the usual cup-product it follows that $\theta(E)$ is skew-symmetric (cf., for example, [10, p. 707]), hence to prove that $\theta$ defines a Poisson structure on $\mathcal{M}^0$ we have only to prove that it satisfies the closure condition $\tilde{d}\theta = 0$. Note that if we prove that $\theta$ defines a Poisson structure on the open subset $\mathcal{M}^0$, then the same holds on the closure $\overline{\mathcal{M}}^0$.

As a final remark note that, for $E \in \mathcal{M}^0$, the global map $\theta$ of Proposition 4.1 may be written as

$$\begin{aligned} \theta : R^1p_*(\mathcal{E}nd(\mathcal{E}) \otimes q^*(\omega_S)) \otimes R^1p_*(\mathcal{E}nd(\mathcal{E}) \otimes q^*(\omega_S)) \xrightarrow{\circ} \\ R^2p_*(\mathcal{E}nd(\mathcal{E}) \otimes q^*(\omega_S^2)) \xrightarrow{s} \mathcal{L}, \end{aligned}$$

where the trivial invertible sheaf $\mathcal{L}$ is given by $\mathcal{L} = R^2p_*(\mathcal{E}nd(\mathcal{E}) \otimes q^*(\omega_S))$, while the map $B$ is the map induced on cohomology by the multiplication by the section $s$:

$$B(E) : H^1(S, \mathcal{E}nd(E) \otimes \omega_S) \xrightarrow{\mathrm{id} \otimes s} H^1(S, \mathcal{E}nd(E)). \tag{4.4}$$

## 5. The closure of $\theta$

In this section we shall prove that $\tilde{d}\theta = 0$, thus proving that $\theta$ defines a Poisson structure on $\mathcal{M}^0$. We note that this proof is original even in the symplectic case.

We start by recalling some preliminaries (see [5] for a detailed description of what follows).

Let $\mathbb{C}[\epsilon]/(\epsilon^2)$ be the ring of dual numbers over $\mathbb{C}$. By convenience of notations, in the sequel it will be denoted simply by $\mathbb{C}[\epsilon]$. Let us denote by $S_\epsilon$ the fiber product $S \times \operatorname{Spec}(\mathbb{C}[\epsilon])$. If $p_\epsilon : S_\epsilon \to S$ is the natural morphism and $F$ is a vector bundle on $S$, we shall denote by $F[\epsilon]$ its trivial infinitesimal deformation, $F[\epsilon] = p_\epsilon^*(F)$.

Let $\pi : X \to Y$ be a morphism (locally of finite presentation) of schemes, and $F, G$ two locally free sheaves on $X$. We denote by $\mathcal{D}\!\mathit{iff}^{\,1}_{X/Y}(F, G)$ the sheaf of relative differential operators from $F$ to $G$ of order $\leq 1$. There is an exact sequence ([11, Ch. IV, §16.8])

$$0 \to \mathcal{H}om_X(F, G) \to \mathcal{D}\!\mathit{iff}^{\,1}_{X/Y}(F, G) \xrightarrow{\sigma} \mathcal{D}er_Y(\mathcal{O}_X) \otimes \mathcal{H}om_X(F, G) \to 0,$$

where $\sigma$ is the *symbol* morphism. Then, if $F = G$ and we restrict to differential operators with 'scalar symbol', written $\mathcal{D}^1_{X/Y}(F)$, we get the exact sequence

$$0 \to \mathcal{E}nd_X(F) \to \mathcal{D}^1_{X/Y}(F) \xrightarrow{\sigma} \mathcal{D}er_Y(\mathcal{O}_X) \to 0. \tag{5.1}$$

Let $p : \mathcal{M}^0 \times S \to \mathcal{M}^0$ and $q : \mathcal{M}^0 \times S \to S$ denote the canonical projections. To apply (5.1) to $q : \mathcal{M}^0 \times S \to S$, note that, as previously said, even if there is no Poincaré bundle $\mathcal{E}$ on $\mathcal{M}^0 \times S$, the sheaf $\mathcal{E}nd(\mathcal{E})$ is well defined. By a similar argument it follows easily that the sheaf $\mathcal{D}^1_S(\mathcal{E}) = \mathcal{D}^1_{\mathcal{M}^0 \times S/S}(\mathcal{E})$ of first-order differential operators with scalar symbols that are $q^*(\mathcal{O}_S)$-linear, is also well defined. Then we have an exact sequence

$$0 \to \mathcal{E}nd(\mathcal{E}) \to \mathcal{D}^1_S(\mathcal{E}) \to p^* T\mathcal{M}^0 \to 0.$$

By applying $p_*$, and noting that $p_* p^* T\mathcal{M}^0 \cong T\mathcal{M}^0$ since $p$ is a proper morphism, we get a long exact cohomology sequence, a piece of which is

$$\cdots \to T\mathcal{M}^0 \to R^1 p_*(\mathcal{E}nd(\mathcal{E})) \to R^1 p_*(\mathcal{D}^1_S(\mathcal{E})) \to \cdots.$$

It is easy to prove that the map $T\mathcal{M}^0 \to R^1 p_*(\mathcal{E}nd(\mathcal{E}))$ is precisely the isomorphism (3.7), hence the image of $R^1 p_*(\mathcal{E}nd(\mathcal{E}))$ in $R^1 p_*(\mathcal{D}^1_S(\mathcal{E}))$ is zero. It follows that, for every section $\{\eta_{ij}\}$ of $R^1 p_*(\mathcal{E}nd(\mathcal{E}))$, there exist sections $\dot{D}_i$ of $\mathcal{D}^1_S(\mathcal{E})$ over suitable open subsets $V_i$, such that

$$\eta_{ij} = \dot{D}_j - \dot{D}_i, \tag{5.2}$$

where, by simplicity of notations, we have not explicitly indicated the restrictions to the intersection $V_{ij} = V_i \cap V_j$.

If $E \in \mathcal{M}^0$, and we consider restrictions to $\{E\} \times S$, it follows that (5.2) is valid with $\{\eta_{ij}\}$ being a 1-cocycle with values in $\mathcal{E}nd(E)$, and $\dot{D}_i$ being sections of $\mathcal{D}^1_S(E)$.

Let now $X$ be a $k$-scheme. A tangent vector field on $X$ is a $k$-linear map of sheaves $D : \mathcal{O}_X \to \mathcal{O}_X$ such that the induced map $D(U) : \Gamma(U, \mathcal{O}_X) \to \Gamma(U, \mathcal{O}_X)$ is a $k$-derivation, for every open subset $U$ of $X$. Equivalently, a vector field on $X$ can be expressed by an automorphism over $\operatorname{Spec} k[\epsilon]$

$$\begin{array}{ccc} X\times \operatorname{Spec}(k[\epsilon]) & \xrightarrow{\tilde{D}} & X\times \operatorname{Spec}(k[\epsilon]) \\ & \searrow \quad \swarrow & \\ & \operatorname{Spec}(k[\epsilon]) & \end{array}$$

that restricts to the identity morphism of $X$ when one looks at the fibers over $\operatorname{Spec} k$.

Over an open affine subset $U = \operatorname{Spec} A$ of $X$ the tangent field $D : \mathcal{O}_X \to \mathcal{O}_X$ is given equivalently by a $k$-derivation $D(U) : A \to A$. In this situation the automorphism $\tilde{D}$ is determined by the $k$-algebra homomorphism $\tilde{D}(U) : A[\epsilon] \to A[\epsilon]$ given by $\tilde{D}(U) = 1 + \epsilon D(U)$.

Let now $D : \mathcal{O}_{\mathcal{M}^0} \to \mathcal{O}_{\mathcal{M}^0}$ be a tangent vector field on $\mathcal{M}^0$ and denote by $\tilde{D}$ the corresponding automorphism of $\mathcal{M}^0 \times \operatorname{Spec}\mathbb{C}[\epsilon]$. Let $\mathcal{E}$ be a local universal family for stable vector bundles (the local existence of $\mathcal{E}$ in the étale topology is enough for our purposes), and $\mathcal{E}[\epsilon]$ be its pull-back to $\mathcal{M}^0 \times \operatorname{Spec}\mathbb{C}[\epsilon]$. The vector field $D$ (or the automorphism $\tilde{D}$) may be described locally by giving the infinitesimal deformation $\mathcal{E}_\epsilon = (\tilde{D} \times \mathrm{id}_S)^*\mathcal{E}[\epsilon]$ of the local universal family $\mathcal{E}$. At a point $\mathcal{E} \in \mathcal{M}^0$ the corresponding tangent vector is given by $E_\epsilon = \mathcal{E}_\epsilon|_{\{E\}\times S_\epsilon}$, which is an infinitesimal deformation of the vector bundle $E$.

From what we have previously seen, the tangent field $\mathcal{E}_\epsilon$ corresponds to a global section $\eta = \{\eta_{ij}\}$ of $R^1p_*(\mathcal{E}nd(\mathcal{E}))$, which can be described in terms of first-order differential operators. Let us give another useful interpretation of this fact.

Let $D : \mathcal{O}_{\mathcal{M}^0} \to \mathcal{O}_{\mathcal{M}^0}$ be the derivation corresponding to the infinitesimal deformation $\mathcal{E}_\epsilon = (\tilde{D} \times \mathrm{id}_S)^*\mathcal{E}$ determined by the global section $\eta = \{\eta_{ij}\}$ of $R^1p_*(\mathcal{E}nd(E))$. Let $(V_i)_{i\in I}$, $V_i = \operatorname{Spec}(A_i)$, be an open affine covering of $\mathcal{M}^0 \times S$. The vector field $D$ is locally described by giving, for each $i \in I$, a $\mathbb{C}[\epsilon]$-automorphism of $A_i[\epsilon]$ of the form $1 + \epsilon D_i$, where $D_i : A_i \to A_i$ is the $\mathbb{C}$-derivation determined by the restriction of $D$ to $V_i$. Let $M_i = \Gamma(V_i, \mathcal{E})$ and $M_i[\epsilon] = \Gamma(V_i, \mathcal{E}[\epsilon])$. The infinitesimal deformation $\mathcal{E}_\epsilon = (\tilde{D} \times \mathrm{id}_S)^*\mathcal{E}[\epsilon]$ may be described as obtained by gluing the sheaves $\widetilde{M_i[\epsilon]}$ by means of suitable isomorphisms.

Let us denote by

$$\widetilde{1 + \epsilon \dot{D}_i} : \mathcal{E}_\epsilon|_{V_i \times \operatorname{Spec}\mathbb{C}[\epsilon]} \xrightarrow{\sim} \widetilde{M_i[\epsilon]}$$

the trivialization isomorphisms, where $\dot{D}_i : M_i \to M_i$ is a first-order differential operator with associated $\mathbb{C}$-derivation $D_i : A_i \to A_i$. By what we have previously seen, the gluing isomorphism on the intersection $V_i \cap V_j$ is given by $1 + \epsilon\eta_{ij} = (1 + \epsilon\dot{D}_j)(1 + \epsilon\dot{D}_i)^{-1} = 1 + \epsilon(\dot{D}_j - \dot{D}_i)$. It follows that $\eta_{ij} = \dot{D}_j - \dot{D}_i$, which is precisely (5.2).

Now let us consider two tangent vector fields $D^1, D^2 : \mathcal{O}_{\mathcal{M}^0} \to \mathcal{O}_{\mathcal{M}^0}$, corresponding to the global sections $\eta^1 = \{\eta^1_{ij}\}$ and $\eta^2 = \{\eta^2_{ij}\}$ of $R^1p_*(\mathcal{E}nd(E))$. By what we have seen, there exist first-order differential operators $\dot{D}^1_i$ and $\dot{D}^2_i$ such that

$$\eta^1_{ij} = \dot{D}^1_j - \dot{D}^1_i,$$
$$\eta^2_{ij} = \dot{D}^2_j - \dot{D}^2_i.$$

Let us denote by

$$\begin{array}{ccc}\mathcal{M}^0\times\operatorname{Spec}\mathbb{C}[\epsilon,\epsilon']\times S & \xrightarrow{\tilde{D}^h\times\mathrm{id}_S} & \mathcal{M}^0\times\operatorname{Spec}\mathbb{C}[\epsilon,\epsilon']\times S\\ & \searrow\qquad\swarrow & \\ & \operatorname{Spec}\mathbb{C}[\epsilon,\epsilon'] & \end{array}$$

the automorphism corresponding to $D^h$, for $h=1,2$. The vector field $D^h$ is given equivalently by the infinitesimal deformation $\mathcal{E}^h_\epsilon=(\tilde{D}^h\times\mathrm{id}_S)^*\mathcal{E}[\epsilon]$ of $\mathcal{E}$, described by the global section $\{\eta^h_{ij}\}$ of $R^1p_*(\mathcal{E}nd(E))$. Let us recall that if $f^h_i:\mathcal{E}^h_\epsilon|_{V_i}\to\widetilde{M^h_i[\epsilon]}$ are isomorphisms, then the sheaf $\mathcal{E}^h_\epsilon$ is constructed by gluing the sheaves $\widetilde{M^h_i[\epsilon]}$ and $\widetilde{M^h_j[\epsilon]}$ along the open sets $V_{ij}$ by means of the isomorphisms $\widetilde{1+\epsilon\eta^h_{ij}}=f^h_j|_{V_{ij}}\circ {f^h_i}^{-1}|_{V_{ij}}$.

By applying the same reasoning, the second-order differential operator $D^1D^2$ is equivalent to $(\tilde{D}^1\times\mathrm{id}_S)^*(\tilde{D}^2\times\mathrm{id}_S)^*\mathcal{E}$. We have isomorphisms

$$(\tilde{D}^1\times\mathrm{id}_S)^*(\tilde{D}^2\times\mathrm{id}_S)^*\mathcal{E}|_{V_i\times\operatorname{Spec}\mathbb{C}[\epsilon,\epsilon']}\xrightarrow{(1+\epsilon\dot{D}^1_i)\circ(1+\epsilon'\dot{D}^2_i)}\widetilde{M_i[\epsilon,\epsilon']},$$

hence the gluing isomorphisms are given by $((1+\epsilon\dot{D}^1_j)\circ(1+\epsilon'\dot{D}^2_j))\circ((1+\epsilon\dot{D}^1_i)\circ(1+\epsilon'\dot{D}^2_i))^{-1}=1+\epsilon(\dot{D}^1_j-\dot{D}^1_i)+\epsilon'(\dot{D}^2_j-\dot{D}^2_i)+\epsilon\epsilon'(\dot{D}^1_j\dot{D}^2_j-\dot{D}^2_j\dot{D}^1_i-\dot{D}^1_j\dot{D}^2_i+\dot{D}^2_i\dot{D}^1_i)$, that we can write in a simpler form as $1+\epsilon\eta^1_{ij}+\epsilon'\eta^2_{ij}+\epsilon\epsilon'(\dot{D}^1_j\eta^2_{ij}-\eta^2_{ij}\dot{D}^1_i)$.

In conclusion, we have proved that the second-order differential operator $D^1D^2$, or, in other words the 'infinitesimal deformation' of $\eta^2=\{\eta^2_{ij}\}$ in the direction given by $\eta^1=\{\eta^1_{ij}\}$, is described by giving gluing isomorphisms of the form

$$1+\epsilon\eta^1_{ij}+\epsilon'\eta^2_{ij}+\epsilon\epsilon'(\dot{D}^1_j\eta^2_{ij}-\eta^2_{ij}\dot{D}^1_i). \tag{5.3}$$

Analogously, we find that $D^2D^1$ is equivalent to the data of

$$1+\epsilon\eta^1_{ij}+\epsilon'\eta^2_{ij}+\epsilon\epsilon'(\dot{D}^2_j\eta^1_{ij}-\eta^1_{ij}\dot{D}^2_i). \tag{5.4}$$

Finally, it is easy to see that the vector field $[D^1,D^2]$ is determined by gluing isomorphisms of the form

$$1+\epsilon\epsilon'([\dot{D}^1,\dot{D}^2]_j-[\dot{D}^1,\dot{D}^2]_i).$$

Now we are able to prove the following

**Theorem 5.1.** *Let $S$ be a Poisson surface and $s\in H^0(S,\omega_S^{-1})$ a Poisson structure on $S$. The antisymmetric contravariant 2-tensor $\theta=\theta_s\in H^0(\mathcal{M}^0,\wedge^2T\mathcal{M}^0)$ defines a Poisson structure on the moduli space $\mathcal{M}^0$ of $H$-stable vector bundles on $S$.*

*Proof*. We have to prove that $\tilde{d}\theta=0$. Let $\eta^1,\eta^2,\eta^3$ be three 1-forms on $\mathcal{M}^0$, i.e., three global sections of $R^1p_*(\mathcal{E}nd(\mathcal{E})\otimes q^*(\omega_S))$. To compute $(B(\eta^1))(\theta(\eta^2,\eta^3))=(B(\eta^1))(\langle s\eta^2,\eta^3\rangle)$, i.e., the derivative of the function $\langle s\eta^2,\eta^3\rangle$ along the vector field $B(\eta^1)$, we shall use first-order Taylor series expansions of $\eta^2$ and $\eta^3$, i.e., we shall compute $\langle s\eta^2_\epsilon,\eta^3_\epsilon\rangle$, where $\eta^2_\epsilon$ and $\eta^3_\epsilon$ are 'infinitesimal deformations' along the vector field $B(\eta^1)$ of $\eta^2$ and $\eta^3$ respectively.

Let us represent the cohomology classes $\eta^h$ by 1-cocycles $\eta^h=\{\eta^h_{ij}\}$, for $h=1,2,3$, so that $B(\eta^h)$ is represented by the 1-cocycle $\{s\eta^h_{ij}\}$. We know that there exist first-order differential operators $\dot{D}^h_i$ such that

$$s\eta^h_{ij} = \dot{D}^h_j - \dot{D}^h_i, \quad \text{for } h = 1, 2, 3. \tag{5.5}$$

If we apply what we have previously seen to the second-order differential operator $B(\eta^h)B(\eta^k)$, we find that the infinitesimal deformation of $B(\eta^k) = s\eta^k$ along the vector field $B(\eta^h)$ is given by

$$s\eta^k_\epsilon = \{s\eta^k_{ij} + \epsilon(\dot{D}^h_j(s\eta^k_{ij}) - s\eta^k_{ij}\dot{D}^h_i)\} = \{s\eta^k_{ij} + s\epsilon(\dot{D}^h_j\eta^k_{ij} - \eta^k_{ij}\dot{D}^h_i)\},$$

because of the $\mathcal{O}_S$-linearity of the differential operators.

Since the multiplication by $s$ is injective at the level of cocycles, it follows that

$$\eta^k_\epsilon = \{\eta^k_{ij} + \epsilon(\dot{D}^h_j\eta^k_{ij} - \eta^k_{ij}\dot{D}^h_i)\}, \tag{5.6}$$

for $h, k = 1, 2, 3$.

Then we have:

$$\begin{aligned}\langle s\eta^2_\epsilon, \eta^3_\epsilon\rangle &= \langle\{s\eta^2_{ij} + s\epsilon(\dot{D}^1_j\eta^2_{ij} - \eta^2_{ij}\dot{D}^1_i)\}, \{\eta^3_{ij} + \epsilon(\dot{D}^1_j\eta^3_{ij} - \eta^3_{ij}\dot{D}^1_i)\}\rangle \\ &= \{s\eta^2_{ij}\circ\eta^3_{jk}\} + \epsilon\{(s\eta^2_{ij}(\dot{D}^1_k\eta^3_{jk} - \eta^3_{jk}\dot{D}^1_j) + s(\dot{D}^1_j\eta^2_{ij} - \eta^2_{ij}\dot{D}^1_i)\eta^3_{jk})\},\end{aligned}$$

from which it follows that

$$(B(\eta^1))(\theta(\eta^2, \eta^3)) = \{s\eta^2_{ij}(\dot{D}^1_k\eta^3_{jk} - \eta^3_{jk}\dot{D}^1_j) + s(\dot{D}^1_j\eta^2_{ij} - \eta^2_{ij}\dot{D}^1_i)\eta^3_{jk}\}. \tag{5.7}$$

Now, by using the decomposition (5.5), and by recalling that the antisymmetry of $\theta$ implies that $\langle s\eta^h, \eta^l\rangle = \{s\eta^h_{ij}\circ\eta^l_{jk}\} = -\{\eta^h_{ij}\circ s\eta^l_{jk}\} = -\langle\eta^h, s\eta^l\rangle$, we may decompose the various terms of (5.7) in different ways as follows:

$$\begin{aligned}s\eta^2_{ij}(\dot{D}^1_k\eta^3_{jk} - \eta^3_{jk}\dot{D}^1_j) &= \dot{D}^2_j\dot{D}^1_k\eta^3_{jk} - \dot{D}^2_j\eta^3_{jk}\dot{D}^1_j - \dot{D}^2_i\dot{D}^1_k\eta^3_{jk} + \dot{D}^2_i\eta^3_{jk}\dot{D}^1_j \\ &= -\eta^2_{ij}\dot{D}^1_k\dot{D}^3_k + \eta^2_{ij}\dot{D}^1_k\dot{D}^3_j + \eta^2_{ij}\dot{D}^3_k\dot{D}^1_j - \eta^2_{ij}\dot{D}^3_j\dot{D}^1_j,\end{aligned}$$

and

$$\begin{aligned}s(\dot{D}^1_j\eta^2_{ij} - \eta^2_{ij}\dot{D}^1_i)\eta^3_{jk} &= \dot{D}^1_j\dot{D}^2_j\eta^3_{jk} - \dot{D}^1_j\dot{D}^2_i\eta^3_{jk} - \dot{D}^2_j\dot{D}^1_i\eta^3_{jk} + \dot{D}^2_i\dot{D}^1_i\eta^3_{jk} \\ &= -\dot{D}^1_j\eta^2_{ij}\dot{D}^3_k + \eta^2_{ij}\dot{D}^1_i\dot{D}^3_k + \dot{D}^1_j\eta^2_{ij}\dot{D}^3_j - \eta^2_{ij}\dot{D}^1_i\dot{D}^3_j.\end{aligned}$$

The last expression we have to compute is the following:

$$\begin{aligned}\langle[B_s(\eta^1), B_s(\eta^2)], \eta^3\rangle &= \{\langle([\dot{D}^1, \dot{D}^2]_j - [\dot{D}^1, \dot{D}^2]_i), \eta^3\rangle\} \\ &= \{(\dot{D}^1_j\dot{D}^2_j - \dot{D}^2_j\dot{D}^1_j - \dot{D}^1_i\dot{D}^2_i + \dot{D}^2_i\dot{D}^1_i)\eta^3_{jk}\} \\ &= \{\dot{D}^1_j\dot{D}^2_j\eta^3_{jk} - \dot{D}^2_j\dot{D}^1_j\eta^3_{jk} - \dot{D}^1_i\dot{D}^2_i\eta^3_{jk} + \dot{D}^2_i\dot{D}^1_i\eta^3_{jk}\}.\end{aligned}$$

As for the last term, we may use the following decomposition:

$$\begin{aligned}\langle[B_s(\eta^2), B_s(\eta^3)], \eta^1\rangle &= \langle\eta^1, [B_s(\eta^2), B_s(\eta^3)]\rangle \\ &= \{\eta^1_{ij}([\dot{D}^2, \dot{D}^3]_k - [\dot{D}^2, \dot{D}^3]_j)\} \\ &= \{\eta^1_{ij}\dot{D}^2_k\dot{D}^3_k - \eta^1_{ij}\dot{D}^3_k\dot{D}^2_k - \eta^1_{ij}\dot{D}^2_j\dot{D}^3_j + \eta^1_{ij}\dot{D}^3_j\dot{D}^2_j\}.\end{aligned}$$

Now, by collecting all the pieces, we find

$$\begin{aligned}
\tilde{d}\theta(\eta^1,\eta^2,\eta^3) &= B_s(\eta^1)(\theta(\eta^2,\eta^3)) - B_s(\eta^2)(\theta(\eta^1,\eta^3)) + B_s(\eta^3)(\theta(\eta^1,\eta^2)) \\
&\qquad - \langle [B_s(\eta^1), B_s(\eta^2)], \eta^3\rangle + \langle [B_s(\eta^1), B_s(\eta^3)], \eta^2\rangle \\
&\qquad - \langle [B_s(\eta^2), B_s(\eta^3)], \eta^1\rangle \\
&= \{ s\eta^2_{ij}(\dot{D}^1_k \eta^3_{jk} - \eta^3_{jk}\dot{D}^1_j) - \dot{D}^3_j \dot{D}^1_i \eta^2_{jk} - \dot{D}^1_j \dot{D}^3_i \eta^2_{jk} \\
&\qquad + \dot{D}^1_j \dot{D}^3_j \eta^2_{jk} + \dot{D}^3_i \dot{D}^1_i \eta^2_{jk} \} \\
&= \{ s\eta^2_{ij}(\dot{D}^1_k \eta^3_{jk} - \eta^3_{jk}\dot{D}^1_j) \\
&\qquad + (\dot{D}^1_j \dot{D}^3_j - \dot{D}^1_j \dot{D}^3_i - \dot{D}^3_j \dot{D}^1_i + \dot{D}^3_i \dot{D}^1_i)\eta^2_{jk} \} \\
&= \{ s\eta^2_{ij}(\dot{D}^1_k \eta^3_{jk} - \eta^3_{jk}\dot{D}^1_j) + (\dot{D}^1_j (s\eta^3_{ij}) - (s\eta^3_{ij})\dot{D}^1_i)\eta^2_{jk} \} \\
&= \{ s\eta^2_{ij}(\dot{D}^1_k \eta^3_{jk} - \eta^3_{jk}\dot{D}^1_j) + s(\dot{D}^1_j \eta^3_{ij} - \eta^3_{ij}\dot{D}^1_i)\eta^2_{jk} \} \\
&= \{ s\eta^2_{ij} \circ \xi^3_{jk} + s\xi^3_{ij} \circ \eta^2_{jk} \} \\
&= 0,
\end{aligned}$$

where, for the infinitesimal deformation $\eta^3_\epsilon$ of $\eta^3$ along $B_s(\eta^1)$, we have set $\eta^3_{\epsilon\, ij} = \eta^3_{ij} + \epsilon\xi^3_{ij}$. Note that the last equality follows from the antisymmetry of $\theta$. $\square$

*Remark 5.2.* We note here that a perfectly analogous result holds for the moduli spaces of $H$-stable sheaves with fixed determinant; the only difference being that, in this case, the tangent (resp. cotangent) space to the moduli variety at a point $E$ is given by $\operatorname{Ext}^1_0(E,E)$ (resp. $\operatorname{Ext}^1_0(E, E\otimes\omega_S)$), where the subscript 0 means that we are considering only trace-free classes.

## 6. The rank of $\theta$

Let us turn now to the computation of the rank of the Poisson structure $\theta$, i.e., of the dimension of the symplectic leaves of the Poisson variety $\mathcal{M}^0$.

Let $S$ be a Poisson surface and let us fix a Poisson structure $s \in H^0(S, \omega_S^{-1})$ on $S$. If $s$ is a symplectic structure, i.e., if $S$ is an abelian or a $K3$ surface, then $\theta$ actually defines a symplectic structure on $\mathcal{M}^0$, hence it has everywhere maximal rank (equal to the dimension of $\mathcal{M}^0$).

If $S$ is not symplectic, let us denote by $D$ the divisor of $s$ and suppose, for simplicity, that $D$ is an irreducible nonsingular curve (note that from the adjunction formula it follows that the canonical line bundle of $D$ is trivial, hence $D$ is an elliptic curve). For any vector bundle $E$ on $S$, we have an exact sequence

$$0 \to \mathcal{E}nd(E)\otimes\omega_S \xrightarrow{s} \mathcal{E}nd(E) \to \mathcal{E}nd(E|_D) \to 0. \tag{6.1}$$

**Lemma 6.1.** *If $E$ is $H$-semistable, then $H^0(S, \mathcal{E}nd(E)\otimes\omega_S) = 0$.*

*Proof*. Note that an $H$-semistable sheaf is also $\mu$-$H$-semistable, where a coherent sheaf $E$ is said to be $\mu$-$H$-semistable (resp. $\mu$-$H$-stable) if it torsion-free and for any proper coherent torsion-free subsheaf $F$ of $E$, we have

$$\mu(F) = \frac{(c_1(F)\cdot H)}{\operatorname{rk}(F)} \le \frac{(c_1(E)\cdot H)}{\operatorname{rk}(E)} = \mu(E) \qquad (\text{resp. } \mu(F) < \mu(E)).$$

Now we recall that, in the proof of Proposition 3.3, we have shown that $(K_S \cdot H) < 0$, hence, if $\phi \in \mathrm{Hom}(E, E \otimes \omega_S)$ is not zero, we have $\mu(E) \leq \mu(\phi(E)) \leq \mu(E \otimes \omega_S) = \mu(E) + (K_S \cdot H) < \mu(E)$, which is a contradiction. $\square$

Now, by considering the long exact cohomology sequence of (6.1), we get

$$0 \to \mathbb{C} \to H^0(D, \mathcal{E}nd(E|_D)) \to H^1(S, \mathcal{E}nd(E) \otimes \omega_S) \stackrel{B(E)}{\to} H^1(S, \mathcal{E}nd(E)) \to \cdots.$$

We have thus proved the following

**Proposition 6.2.** *The kernel of the hamiltonian morphism $B(E)$ is given by*

$$\mathrm{Ker}\, B(E) = H^0(D, \mathcal{E}nd(E|_D))/\mathbb{C}.$$

*Hence*

$$\mathrm{rk}(B(E)) = \dim \mathcal{M}^0 - \dim H^0(D, \mathcal{E}nd(E|_D)) + 1.$$

This shows how the rank of the Poisson structure $\theta$ at a point $E \in \mathcal{M}^0$ is determined by the restriction $E|_D$ of $E$ to the curve $D$. On the open subset of $\mathcal{M}^0$ consisting of vector bundles $E$ that restrict to a simple bundle on $D$, the rank of $\theta$ is equal to the dimension of $\mathcal{M}^0$, i.e., $\theta$ is nondegenerate, or, in other terms, it induces a symplectic structure.

*Remark 6.3.* If $\dim \mathcal{M}^0 = 2$, then $\mathcal{M}^0$ is, like $S$, a Poisson surface, i.e., an abelian or a $K3$ surface in the symplectic case, or a ruled surface with an effective anticanonical divisor in the general case.

*Remark 6.4.* By recalling that the dimension of $\mathcal{M}^0 = \mathcal{M}^0(r, c_1, c_2)$ is given by $(1-r)c_1^2 + 2rc_2 - r^2\chi(\mathcal{O}_S) + 1$ (if it is non-empty), and that the dimension of a symplectic variety is even, we see that, if $r$ and $c_1$ are both even, then $\dim \mathcal{M}^0$ is odd, hence the symplectic leaves of $\mathcal{M}^0$ have dimension strictly less then the dimension of $\mathcal{M}^0$. It follows that there are no $H$-stable vector bundles $E \in \mathcal{M}^0$ such that $E|_D$ is simple. The same conclusion holds, for example, if $r$ is odd and $\chi(\mathcal{O}_S)$ is even.

*Remark 6.5.* If $S = \mathbb{P}^2$, much more is known on the structure of the moduli space $\mathcal{M}$. If we denote by $\mathcal{M}^0_\mu$ the moduli space of $\mu$-$H$-stable locally free sheaves on $\mathbb{P}^2$, then we know that $\mathcal{M}^0_\mu$ is irreducible and that, under some technical conditions on the rank and Chern classes, it is everywhere dense in $\overline{\mathcal{M}}$ (see [8] for details). It follows that, in this case, $\theta$ defines a Poisson structure on all of $\mathcal{M}$.

*Remark 6.6.* Let us denote now by $\mathcal{M}_D$ the moduli space of semistable vector bundles on the curve $D$ and by $\mathcal{M}_S$ the open subset of $\mathcal{M}^0$ such that

$$E \in \mathcal{M}_S \Rightarrow E|_D \in \mathcal{M}_D.$$

Let us denote by $\rho : \mathcal{M}_S \to \mathcal{M}_D$ the restriction map.

By recalling the Poisson structure of $\mathcal{M}_S$, for every $E \in \mathcal{M}_S$ we get the following commutative diagram

$$\begin{array}{ccc} T_E\mathcal{M}_S & \xrightarrow{T_E\rho} & T_{E|_D}\mathcal{M}_D \\ \uparrow{\scriptstyle B(E)} & & \uparrow{\scriptstyle \tilde{B}(E)} \\ T^*_E\mathcal{M}_S & \xleftarrow{T^*_E\rho} & T^*_{E|_D}\mathcal{M}_D, \end{array}$$

where $\tilde{B}(E) = T_E\rho \circ B(E) \circ T^*_E\rho$.

Since the canonical bundle of the curve $D$ is trivial, the preceding diagram is canonically identified with

$$\begin{array}{ccc} H^1(S, \mathcal{E}nd(E)) & \longrightarrow & H^1(D, \mathcal{E}nd(E|_D)) \\ \uparrow{\scriptstyle B(E)} & & \uparrow{\scriptstyle \tilde{B}(E)} \\ H^1(S, \mathcal{E}nd(E) \otimes \omega_S) & \longleftarrow & H^0(D, \mathcal{E}nd(E|_D)). \end{array}$$

Now, the long exact cohomology sequence of (6.1), gives

$$H^0(D, \mathcal{E}nd(E|_D)) \xrightarrow{\delta} H^1(S, \mathcal{E}nd(E) \otimes \omega_S) \xrightarrow{B(E)} H^1(S, \mathcal{E}nd(E)) \xrightarrow{\alpha} H^1(D, \mathcal{E}nd(E|_D)),$$

and it is easy to see that $\tilde{B}(E)$ is precisely the composition $\alpha \circ B(E) \circ \delta$, hence it is zero.

This proves that the Poisson structure of $\mathcal{M}_S$ induces on the image $\rho(\mathcal{M}_S) \subset \mathcal{M}_D$ a Poisson structure that is identically zero.

We would like to end with some remarks on the relations between the moduli space of stable vector bundles on $\mathbb{P}^2$ and the moduli space of anti self-dual connections on $S^4$.

Let $P$ be a principal $\mathrm{SU}(r)$-bundle on $S^4 = \mathbb{R}^4 \cup \{\infty\}$, and $k$ be minus its Pontryagin index. Let us denote by $\widetilde{M}(\mathrm{SU}(r), k)$ the moduli space parametrizing pairs consisting of an anti self-dual $\mathrm{SU}(r)$-connection on $P$ and an isomorphism $P_\infty \cong \mathrm{SU}(r)$. This is a manifold of (real) dimension $4rk$, if $k$ is sufficiently large. The usual moduli space of anti self-dual $\mathrm{SU}(r)$-connections on $P$ is then given by $M(\mathrm{SU}(r), k) = \widetilde{M}(\mathrm{SU}(r), k)/\,\mathrm{SU}(r)$.

Now let us fix a line $D \subset \mathbb{P}^2$ (so that we get isomorphisms $\mathbb{P}^2 \setminus D \cong \mathbb{C}^2 \cong \mathbb{R}^4$) and denote by $\widetilde{\mathcal{M}}(r, 0, k)$ the moduli space of pairs consisting of a rank $r$ holomorphic vector bundle $E$ on $\mathbb{P}^2$ with $c_1(E) = 0$ and $c_2(E) = k$ and a trivialization of $E|_D$.

Donaldson proved in [6] that there is a natural isomorphism

$$\widetilde{M}(\mathrm{SU}(r), k) \cong \widetilde{\mathcal{M}}(r, 0, k).$$

If $\mathcal{M}^0(r, 0, k)$ denotes the moduli space of stable rank $r$ vector bundles on $\mathbb{P}^2$ with Chern classes $c_1 = 0$ and $c_2 = k$, it is known that there are Zariski open subsets $U \subset \widetilde{\mathcal{M}}(r, 0, k)$ and $V \subset \mathcal{M}^0(r, 0, k)$ such that $U$ fibers over $V$. As a consequence we find that an open subset of the moduli space $M(\mathrm{SU}(r), k)$ fibers over $V$ with fiber $\mathrm{SL}(r, \mathbb{C})/\,\mathrm{SU}(r)$. Hence, after factoring out by this group, we get a complex structure and a (complex) Poisson structure on the real moduli space of anti self-dual connections.

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