A MARSDEN–WEINSTEIN REDUCTION THEOREM FOR PRESYMPLECTIC MANIFOLDS

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Abstract. In this paper we prove an analogue of the Marsden–Weinstein reduction theorem for presymplectic actions of Lie groups on presymplectic manifolds.

Introduction.

The well-known symplectic reduction procedure of Marsden and Weinstein is an important method of construction of new symplectic manifolds starting from old ones (with a Lie group of symmetries acting on them). On the other hand, the phase space corresponding to many physical systems, for instance those possessing gauge symmetries, is only a presymplectic manifold (where by this we mean that the closed 2-form may be degenerate). It is then natural to consider the problem of reduction in this more general context. In this paper we generalize the Marsden–Weinstein reduction procedure to the case of presymplectic actions of Lie groups on presymplectic manifolds.

This paper is organized as follows. In the first two sections we briefly recall some basic facts on symplectic and Poisson manifolds. In Section 3 we introduce the notion of presymplectic structure and study the basic properties of presymplectic manifolds. Finally, in Section 4, we study the actions of Lie groups on presymplectic manifolds, introduce the appropriate notion of moment map, and prove the reduction theorem.

1. Symplectic structures.

In this section we recall some basic definitions and results of symplectic geometry.

Definition 1.1. A symplectic structure on a C^∞ manifold M is a closed, non-degenerate, 2-form on M.

A symplectic structure ω on M determines an isomorphism

$$b : TM \to T^*M,$$

called the *Hamiltonian isomorphism*, defined by setting $b(X) = i_X\omega$, i.e.,

$$\langle b(X), Y\rangle = \omega(X, Y),$$

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for any two vector fields X and Y on M.

Using this isomorphism we may define the Hamiltonian vector field associated to a C^∞ function on M as follows:

Definition 1.2. The Hamiltonian vector field X_f of a function $f \in C^\infty(M)$ is given by $X_f = b^{-1}(df)$.

Finally we can define the *Poisson bracket* of two functions:

Definition 1.3. Let $f, g \in C^\infty(M)$. Their Poisson bracket is the function defined by setting

$$\{f, g\} = \omega(X_g, X_f) = X_f(g) = -X_g(f).$$

Proposition 1.4. *The Poisson bracket defines a structure of Lie algebra on $C^\infty(M)$. It also satisfies the following identity (called Leibnitz identity)*

$$\{f, gh\} = \{f, g\}h + g\{f, h\},$$

which may be restated by saying that the Poisson bracket of functions is a derivation in each of its entries.

Proof. The only non-trivial part of this proposition is the verification of the Jacobi identity:

$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0.$$

This identity is equivalent to the closure of the 2-form ω, as may be verified by a direct computation using local coordinates. □

Remark 1.5. From the Jacobi identity it follows that

$$[X_f, X_g] = X_{\{f,g\}},$$

where $[\cdot, \cdot]$ denotes the commutator of two vector fields. It follows that the map $C^\infty(M) \to \mathfrak{X}(M)$ sending a function f to its Hamiltonian vector field X_f is a homomorphism of Lie algebras.

Definition 1.6. A vector field X is called hamiltonian if the 1-form $b(X)$ is exact (if $b(X) = df$ it follows that $X = X_f$ is the hamiltonian vector field of the function f). A vector field X is called locally hamiltonian if the 1-form $b(X)$ is closed. This means that on a sufficiently small open neighborhood U of any point P there is a function $f_U \in C^\infty(U)$ such that $X|_U$ is the hamiltonian vector field of the function f_U.

The locally hamiltonian vector fields may be characterized by the following property:

Proposition 1.7. *A vector field X is locally hamiltonian if and only if $L_X\omega = 0$, i.e., if and only if the symplectic form ω is constant along the flow of X.*

Proof. From the identity

$$L_X\omega = i_X d\omega + d i_X \omega$$

and the fact that ω is a closed form, it follows that $di_X\omega = L_X\omega$, which proves the first statement. The equivalence of the first and second statements is well known (it follows from the fact that

$$\frac{d}{d\lambda}F_\lambda^*\omega = F_\lambda^* L_X\omega,$$

where F_λ denotes the local flow of X). □

We shall denote by $\mathfrak{X}_h(M)$ (resp. by $\mathfrak{X}_{lh}(M)$) the set of hamiltonian (resp. locally hamiltonian) vector fields on M.

Proposition 1.8. *The sets $\mathfrak{X}_h(M)$ and $\mathfrak{X}_{lh}(M)$ are Lie subalgebras of $\mathfrak{X}(M)$.*

Proof. We shall prove this proposition later, in the more general context of presymplectic manifolds. □

The Hamiltonian isomorphism $b : TM \to T^*M$ sets up a one-to-one correspondence between $\mathfrak{X}_h(M)$ and the set $B^1(M)$ of exact 1-forms on M (resp. between $\mathfrak{X}_{lh}(M)$ and the set $Z^1(M)$ of closed 1-forms). It follows that the quotient $\mathfrak{X}_{lh}(M)/\mathfrak{X}_h(M)$ is isomorphic to $Z^1(M)/B^1(M) = H^1_{DR}(M)$, the first De Rham cohomology group of M. This proves that the obstruction for a locally hamiltonian vector field to be globally hamiltonian is of a topological nature and lies in $H^1_{DR}(M)$.

2. Poisson structures.

One way to generalize the notion of a symplectic manifold is simply to forget about the 2-form ω and require instead the existence of a Poisson bracket on $C^\infty(M)$ with all the "good" properties, as in the symplectic case. More precisely:

Definition 2.1. A Poisson structure on a manifold M is a Lie algebra structure $\{\cdot,\cdot\}$ on the sheaf of smooth functions on M which is a derivation in each entry, i.e., satisfies $\{f, gh\} = \{f, g\}h + g\{f, h\}$.

It is easy to see that to give a Poisson structure on M is equivalent to giving a bivector field (i.e., an antisymmetric contravariant 2-tensor) $\theta \in H^0(M, \wedge^2 TM)$, defined by setting

$$\{f, g\} = \langle\theta, df \wedge dg\rangle, \tag{2.1}$$

with an additional requirement. In fact, given $\theta \in H^0(M, \wedge^2 TM)$, the bracket defined by (2.1) satisfies all the properties required to be a Poisson structure, except for the Jacobi identity

$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0.$$

We must then impose an additional condition on θ. We shall see later how to express this condition in a suitable way.

Definition 2.2. Let (M, θ) be a Poisson manifold. The Hamiltonian morphism is the homomorphism of vector bundles

$$B = B_\theta : T^*M \to TM$$

defined by setting $\langle \theta, \alpha \wedge \beta \rangle = \langle B_\theta(\alpha), \beta \rangle$, for 1-forms α and β on M.

The homomorphism B is called the Hamiltonian morphism by analogy with the symplectic case.

We can now define an operator $\tilde{d} : H^0(M, \wedge^2 TM) \to H^0(M, \wedge^3 TM)$ by setting

$$\begin{aligned} \tilde{d}\theta(\alpha, \beta, \gamma) = {} & B_\theta(\alpha)\theta(\beta, \gamma) - B_\theta(\beta)\theta(\alpha, \gamma) + B_\theta(\gamma)\theta(\alpha, \beta) \\ & - \langle [B_\theta(\alpha), B_\theta(\beta)], \gamma \rangle + \langle [B_\theta(\alpha), B_\theta(\gamma)], \beta \rangle - \langle [B_\theta(\beta), B_\theta(\gamma)], \alpha \rangle, \end{aligned} \tag{2.2}$$

for any three 1-forms α, β, γ on M, where $[\cdot, \cdot]$ denotes the usual commutator of vector fields. We have the following result, whose proof consists in a straightforward computation using local coordinates.

Proposition 2.3. *The bracket $\{\cdot, \cdot\}$ defined by an element $\theta \in H^0(M, \wedge^2 TM)$ as in (2.1) is a Poisson structure, i.e., satisfies the Jacobi identity, if and only if $\tilde{d}\theta = 0$.*

Remark 2.4. The element $\tilde{d}\theta \in H^0(M, \wedge^3 TM)$ coincides (up to a factor of 2) with the so-called Schouten bracket $[\theta, \theta]$ (see [V] for the definition).

Remark 2.5. When θ has maximal rank everywhere, i.e., when $B_\theta : T^*M \to TM$ is an isomorphism, to give θ is equivalent to giving its inverse 2-form $\omega \in \Omega^2_M$, which corresponds to the inverse isomorphism $b = B_\theta^{-1} : TM \to T^*M$. It is easy to check that, in this situation, the condition $\tilde{d}\theta = 0$ is equivalent to $d\omega = 0$, i.e., to the closure of the 2-form ω. In this case we say that the Poisson structure is symplectic. Note that a necessary condition for the existence of a symplectic structure on M is that the dimension of M be even.

Let now M be a Poisson manifold, and $\theta \in H^0(M, \wedge^2 TM)$ its Poisson structure. For any $x \in M$, we set $\mathcal{D}(x) = \mathrm{Im}(B_\theta(x)) \subseteq T_x M$. The collection $\mathcal{D} = (\mathcal{D}(x))_{x \in M}$ of subspaces of the tangent spaces of M is called the characteristic distribution of the Poisson manifold (M, θ). It is actually a distribution in a generalized sense, since the rank of the Poisson structure, i.e., the rank of the map B, will not be constant, in general.

It turns out that, for any Poisson manifold, the (generalized) characteristic distribution is completely integrable, and the Poisson structure of M determines a symplectic structure on the integral leaves of this distribution. These integral leaves are then called the symplectic leaves of the Poisson manifold (X, θ). For more details on the structure of Poisson manifolds, we refer to [V].

Let us come now to the definition of the Hamiltonian vector fields.

Definition 2.6. For any smooth function f on M, the Hamiltonian vector field of f is the vector field $X_f = B(df)$. Equivalently, it is the vector field (derivation) on M defined by $X_f(g) = \{f, g\}$, for any smooth function g.

Definition 2.7. A function f is called a Casimir function if $df \in \ker(B)$, i.e., if its Hamiltonian vector field is zero. Equivalently, f is a Casimir function if $\{f, g\} = 0$, for every function g. We shall denote by $\mathrm{Cas}(M)$ the set of Casimir functions on M.

Definition 2.8. We define the set of hamiltonian vector fields $\mathfrak{X}_h(M)$ to be the image, through the Hamiltonian morphism B, of the set $B^1(M)$ of exact 1-forms;

$$\mathfrak{X}_h(M) = B(B^1(M)).$$

Similarly, we define the set of locally hamiltonian vector fields $\mathfrak{X}_{lh}(M)$ to be the image, through the Hamiltonian morphism B, of the set $Z^1(M)$ of closed 1-forms;

$$\mathfrak{X}_{lh}(M) = B(Z^1(M)).$$

As in the symplectic case, we have:

Proposition 2.9. *The sets $\mathfrak{X}_h(M)$ and $\mathfrak{X}_{lh}(M)$ are Lie subalgebras of $\mathfrak{X}(M)$.*

Proof. Let $X_f, X_g \in \mathfrak{X}_h(M)$ be two Hamiltonian vector fields. Recall that we have $X_f(h) = \{f, h\}$, for any smooth function h. Hence:

$$\begin{aligned}
[X_f, X_g](h) &= X_f X_g(h) - X_g X_f(h) \\
&= \{f, \{g, h\}\} - \{g, \{f, h\}\} \\
&= \{f, \{g, h\}\} + \{g, \{h, f\}\} \\
&= \{\{f, g\}, h\} = X_{\{f,g\}}(h).
\end{aligned}$$

It follows that $[X_f, X_g] = X_{\{f,g\}} \in \mathfrak{X}_h(M)$.

To prove that $\mathfrak{X}_{lh}(M)$ is a Lie subalgebra of $\mathfrak{X}(M)$ we just use a local version of the preceeding computation. □

Remark 2.10. We note that the fact that $\mathfrak{X}_h(M)$ and $\mathfrak{X}_{lh}(M)$ are Lie subalgebras of $\mathfrak{X}(M)$ is a direct consequence of the Jacobi identity satisfied by the Poisson bracket, which, in turn, is equivalent to the vanishing of $\tilde{d\theta}$.

We have natural isomorphisms

$$\mathfrak{X}_{lh}(M) \cong Z^1(M)/\ker(B) \cap Z^1(M), \qquad \mathfrak{X}_h(M) \cong B^1(M)/\ker(B) \cap B^1(M).$$

By using the following isomorphisms

$$H^1_{DR}(M) = \frac{Z^1(M)}{B^1(M)} \cong \frac{Z^1(M)/\ker(B) \cap B^1(M)}{B^1(M)/\ker(B) \cap B^1(M)}$$

and

$$\frac{\mathfrak{X}_{lh}(M)}{\mathfrak{X}_h(M)} \cong \frac{Z^1(M)/\ker(B) \cap Z^1(M)}{B^1(M)/\ker(B) \cap B^1(M)},$$

we deduce that the Hamiltonian morphism B induces a surjective map

$$H^1_{DR}(M) \to \frac{\mathfrak{X}_{lh}(M)}{\mathfrak{X}_h(M)},$$

whose kernel consists of all cohomology classes which may be represented by a closed 1-form lying in the kernel of the map B.

It follows that the obstruction for a locally hamiltonian vector field to be globally hamiltonian lies in a quotient of $H^1_{DR}(M)$. This obstruction now is not purely topological, since it may well happen that $H^1_{DR}(M) \neq 0$ but the quotient obstruction group is zero (this happens, for instance, in the case of the zero Poisson structure, i.e., $\theta = 0$).

3. Presymplectic structures.

We have seen how the notion of a Poisson manifold may be considered a generalization of the notion of a symplectic manifold. There is also another way to generalize the notion of a symplectic manifold, which is, in some sense, "dual" to the notion of a Poisson manifold. This time we shall insist on the existence of a closed 2-form ω but we shall release the requirements on the existence of a Poisson bracket. This leads to the notion of a presymplectic manifold. Precisely, we have:

Definition 3.1. A presymplectic structure on a manifold M is a closed 2-form ω on M.

Remark 3.2. In the literature it is usually required that ω have constant rank on M (which is supposed to be connected). In this paper we shall not impose such a restriction on ω.

As in the symplectic case, a presymplectic structure ω on M determines a homomorphism

$$b : TM \to T^*M,$$

called the *Hamiltonian morphism*, defined by setting $b(X) = i_X\omega$, i.e.,

$$\langle b(X), Y\rangle = \omega(X, Y),$$

for any two vector fields X and Y on M. This morphism will be an isomorphism if and only if the presymplectic structure is actually a symplectic structure.

We shall denote by $\mathcal{K}$ the kernel of b; it is a subsheaf of the tangent bundle TM.

Lemma 3.3. *$\mathcal{K}$ is a sheaf of Lie subalgebras of TM.*

Proof. Let $X, Y \in \Gamma(U, \mathcal{K})$ and $Z \in \Gamma(U, TM)$ be vector fields defined on an open set U. We have $i_X\omega = i_Y\omega = 0$, and the closedness of ω implies that

$$\begin{aligned} 0 = d\omega(X, Y, Z) &= X\omega(Y, Z) - Y\omega(X, Z) + Z\omega(X, Y) \\ &\qquad - \omega([X, Y], Z) + \omega([X, Z], Y) - \omega([Y, Z], X) \\ &= -\omega([X, Y], Z) \end{aligned}$$

for any vector field Z. This shows that $[X, Y] \in \Gamma(U, \mathcal{K})$. □

This lemma shows that $\mathcal{K}$ defines an involutive distribution on M (this is not a distribution in the classical sense, since we do not assume the rank of ω to be constant on M; it is sometimes called a general distribution, see [V, Chapter 2]).

In the regular case, i.e., when the rank of ω is constant, this distribution is completely integrable, by the classical Frobenius theorem. This is also true in the general case, as we shall prove later. Before doing that, we need to introduce the notions of Hamiltonian and locally Hamiltonian vector fields.

Definition 3.4. A vector field X on M is called a Hamiltonian vector field of a function $f \in C^\infty(M)$ if $b(X) = df$.

Note that, in the presymplectic case, it is not true that to any function $f \in C^\infty(M)$ there corresponds a Hamiltonian vector field. When this happens, i.e., when there is a Hamiltonian vector field corresponding to a function f, this vector field is defined only up to the addition of vector fields in $\mathcal{K}$. It is instructive to compare this situation with the case of Poisson manifolds, where it is the function f corresponding to a Hamiltonian vector field to be determined only up to the addition of a Casimir function.

We can now define the sets of Hamiltonian and locally Hamiltonian vector fields as follows:

Definition 3.5. The set $\mathfrak{X}_h(M)$ of Hamiltonian vector fields on M is the inverse image, through b, of the set of exact 1-forms:

$$\mathfrak{X}_h(M) = b^{-1}(B^1(M)).$$

Analogously, the set $\mathfrak{X}_{lh}(M)$ of locally Hamiltonian vector fields is the inverse image, through b, of the set of closed 1-forms:

$$\mathfrak{X}_{lh}(M) = b^{-1}(Z^1(M)).$$

We have then an isomorphism, induced by the map b,

$$\frac{\mathfrak{X}_{lh}(M)}{\mathfrak{X}_h(M)} \cong \frac{Z^1(M) \cap \mathrm{Im}(b)}{B^1(M) \cap \mathrm{Im}(b)}.$$

From this we deduce the existence of an injective morphism

$$\bar{b} : \frac{\mathfrak{X}_{lh}(M)}{\mathfrak{X}_h(M)} \to \frac{Z^1(M)}{B^1(M)} = H^1_{DR}(M),$$

induced by the Hamiltonian morphism b.

This proves that the obstruction for a locally hamiltonian vector field to be globally hamiltonian lies in a (generally proper) subset of $H^1_{DR}(M)$ (hence it is not of a purely topological nature).

Let us come now to the definition of a Poisson bracket. Obviously, in the case of a presymplectic manifold, it is not possible to define the Poisson bracket of any two smooth functions. It is however possible to define a Poisson bracket if we restrict to a suitable class of functions.

Definition 3.6. Let $C^\infty_h(M)$ be the subset of $C^\infty(M)$ consisting of Hamiltonian functions, i.e., functions having a Hamiltonian vector field.

We can define the Poisson bracket of two Hamiltonian functions as follows:

Definition 3.7. Let $f, g \in C^\infty_h(M)$. Their Poisson bracket is the function defined by

$$\{f, g\} = \omega(X_g, X_f),$$

where X_f and X_g are any two Hamiltonian vector fields corresponding to the functions f and g respectively.

Remark 3.8. It is easy to prove that the Poisson bracket of two Hamiltonian functions is well defined. In fact, if $X'_f = X_f + Y$ and $X'_g = X_g + Z$, with $Y, Z \in \mathcal{K}$, are two different Hamiltonian vector fields for f and g, we have

$$\omega(X'_g, X'_f) = \omega(X_g, X_f).$$

On the other hand, if we try to define the Poisson bracket of a Hamiltonian function $f \in C^\infty_h(M)$ with a generic function $g \in C^\infty(M)$ by setting (as in the symplectic case)

$$\{f, g\} = X_f(g),$$

we can easily see that this will not work, because the result will depend on the choice of the Hamiltonian vector field of the function f. A consequence of this fact is that the map

$$g \mapsto \{f, g\},$$

does not define a derivation on $C^\infty(M)$, but only on the subset $C^\infty_h(M)$. It follows that it is not possible to identify a Hamiltonian vector field of a Hamiltonian function f with the map $g \mapsto \{f, g\}$ (which, by the way, is obvious, otherwise the Hamiltonian vector field of a Hamiltonian function would be uniquely determined). However, it is true that the restriction of any Hamiltonian vector field X_f of f to the subset $C^\infty_h(M)$ of $C^\infty(M)$ coincides with the derivation $g \mapsto \{f, g\}$.

Proposition 3.9. *The Poisson bracket of Hamiltonian functions defines a structure of Lie algebra on* $C^\infty_h(M)$*, and it satisfies the Leibnitz identity*

$$\{f, gh\} = \{f, g\}h + g\{f, h\}.$$

Proof. The only non-trivial part of this proposition is the verification of the Jacobi identity:

$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0.$$

This identity is equivalent to the closure of the 2-form ω, as may be verified by a direct computation using local coordinates. □

Remark 3.10. If we denote by D_f the derivation on $C_h^\infty(M)$ defined by

$$D_f(g) = \{f, g\},$$

it follows from the Jacobi identity that

$$[D_f, D_g] = D_{\{f,g\}}.$$

As in the case of a symplectic manifold, the locally Hamiltonian vector fields may be characterized by the following property, whose proof is exactly the same as in the symplectic case.

Proposition 3.11. *A vector field X on a presymplectic manifold (M, ω) is locally Hamiltonian if and only if $L_X\omega = 0$, i.e., if and only if the presymplectic form ω is constant along the flow of X.*

We can now prove the assertion about the complete integrability of the (general) distribution defined by $\mathcal{K}$.

Proposition 3.12. *The distribution defined by $\mathcal{K}$ is completely integrable.*

Proof. Since all vector fields in $\mathcal{K}$ are Hamiltonian, we have $L_X\omega = 0$ for any $X \in \mathcal{K}$, by Proposition 3.11. This means that $\exp(tX)$ is a presymplectic automorphism. It follows that the rank of ω is constant along the integral curves of the vector fields $X \in \mathcal{K}$. This, together with the involutivity of $\mathcal{K}$, is precisely the condition that ensures that the distribution defined by $\mathcal{K}$ is completely integrable (cf. [V, Theorem 2.6] or [V, Theorem 2.9″]). □

Remark 3.13. Since the distribution defined by $\mathcal{K}$ is completely integrable, it determines a foliation of M by integral submanifolds. When the quotient $\bar{M} = M/\sim$ of M by this foliation exists as a smooth manifold, i.e., when the space of leaves has the structure of a smooth manifold, it is easy to prove that the presymplectic structure ω on M determines, in a natural way, a symplectic structure $\bar{\omega}$ on $\bar{M}$.

Let us go back to the study of the Hamiltonian and locally Hamiltonian vector fields. To prove the next proposition we shall need the following lemma:

Lemma 3.14. *The differential of a 2-form may be expressed, in terms of Lie derivatives, by the following formula:*

$$d\omega(X, Y, Z) = L_X\omega(Y, Z) - L_Y\omega(X, Z) + L_Z\omega(X, Y) + \omega([X, Y], Z) - \omega([X, Z], Y) + \omega([Y, Z], X). \tag{3.1}$$

Proof. Let us recall the following properties of the Lie derivative:

$$L_X f = X(f), \qquad L_X Y = [X, Y],$$

and

$$L_X(\omega(Y, Z)) = L_X\omega(Y, Z) + \omega([X, Y], Z) + \omega(Y, [X, Z]).$$

Then we have:

$$L_X\omega(Y,Z) = X(\omega(Y,Z)) - \omega([X,Y],Z) + \omega([X,Z],Y).$$

Now, by an easy computation, we find:

$$\begin{aligned}
&L_X\omega(Y,Z) - L_Y\omega(X,Z) + L_Z\omega(X,Y) \\
&\qquad + \omega([X,Y],Z) - \omega([X,Z],Y) + \omega([Y,Z],X) \\
&= X(\omega(Y,Z)) - Y(\omega(X,Z)) + Z(\omega(X,Y)) \\
&\qquad - \omega([X,Y],Z) + \omega([X,Z],Y) - \omega([Y,Z],X) \\
&= d\omega(X,Y,Z).
\end{aligned}$$

□

Proposition 3.15. *The sets* $\mathfrak{X}_h(M)$ *and* $\mathfrak{X}_{lh}(M)$ *are Lie subalgebras of* $\mathfrak{X}(M)$.

Proof. We shall prove the assertion for $\mathfrak{X}_h(M)$. The same reasoning, done locally, will prove the assertion also for $\mathfrak{X}_{lh}(M)$.

Let $X = X_f$ and $Y = Y_g$ be two Hamiltonian vector fields. Let $Z \in \mathfrak{X}(M)$ be any vector field. Since ω is closed, we have:

$$\begin{aligned}
0 = d\omega(X,Y,Z) &= L_X\omega(Y,Z) - L_Y\omega(X,Z) + L_Z\omega(X,Y) \\
&\qquad + \omega([X,Y],Z) - \omega([X,Z],Y) + \omega([Y,Z],X) \\
&= L_Z\omega(X,Y) + \omega([X,Y],Z) - \omega([X,Z],Y) + \omega([Y,Z],X) \\
&= Z(\omega(X,Y)) - \omega([Z,X],Y) + \omega([Z,Y],X) \\
&\qquad + \omega([X,Y],Z) - \omega([X,Z],Y) + \omega([Y,Z],X) \\
&= Z(\omega(X,Y)) + \omega([X,Y],Z).
\end{aligned}$$

It follows that

$$Z(\{f,g\}) = Z(\omega(Y,X)) = \omega([X,Y],Z) = \langle Z, b([X,Y])\rangle,$$

for any vector field Z, hence

$$d\{f,g\} = b([X,Y]),$$

which means that $\{f,g\}$ is a Hamiltonian function and

$$X_{\{f,g\}} = [X_f, Y_g] \mod \mathcal{K}.$$

From this equation it follows that $\mathfrak{X}_h(M)$ is closed for the Lie bracket, hence is a Lie subalgebra of $\mathfrak{X}(M)$. □

Proposition 3.16. *The kernel* $\mathcal{K}$ *of the Hamiltonian morphism is a Lie ideal of the Lie algebras* $\mathfrak{X}_h(M)$ *and* $\mathfrak{X}_{lh}(M)$ *(but not of* $\mathfrak{X}(M)$*).*

Proof. We shall prove the assertion for $\mathfrak{X}_{lh}(M)$. Let $X \in \mathcal{K}$, $Y \in \mathfrak{X}_{lh}(M)$ and $Z \in \mathfrak{X}(M)$ be three vector fields. From the closure of ω it follows that

$$\begin{aligned} 0 = d\omega(X,Y,Z) &= L_X\omega(Y,Z) - L_Y\omega(X,Z) + L_Z\omega(X,Y) \\ &\quad + \omega([X,Y],Z) - \omega([X,Z],Y) + \omega([Y,Z],X) \\ &= L_Z\omega(X,Y) + \omega([X,Y],Z) - \omega([X,Z],Y) \\ &= -\omega([Z,X],Y) + \omega([X,Y],Z) - \omega([X,Z],Y) \\ &= \omega([X,Y],Z), \end{aligned}$$

for any Z, hence $[X,Y] \in \mathcal{K}$. □

4. Actions of Lie groups on presymplectic manifolds.

The usual notion of a symplectic action of a Lie group on a symplectic manifold admits a straightforward generalization to the presymplectic case.

Let (M,ω) be a presymplectic manifold and let G be a (connected) Lie group acting on M. We shall denote by $\Phi : G \times M \to M$ the action of G, by $\Phi_g : M \to M$ the map $\Phi_g(x) = \Phi(g,x)$ and by $\Phi_x : G \to M$ the map $\Phi_x(g) = \Phi(g,x)$, for any $g \in G$ and $x \in M$.

Definition 4.1. The action Φ of G on M is called a presymplectic action if $\Phi_g^*\omega = \omega$, for any $g \in G$.

Let $\mathfrak{g}$ be the Lie algebra of G. For any $\xi \in \mathfrak{g}$ we shall denote by ξ_M the fundamental vector field on M corresponding to ξ. The set of all fundamental vector fields will be denoted by $\mathfrak{g}_M$. The action Φ of G on M is a presymplectic action if and only if the 2-form ω is constant along the flow of ξ_M, i.e., if and only if $L_{\xi_M}\omega = 0$, for any $\xi \in \mathfrak{g}$. By Proposition 3.11 this is equivalent to saying that the vector field ξ_M is locally Hamiltonian, i.e., that the 1-form $b(\xi_M)$ is closed, for any $\xi \in \mathfrak{g}$.

We can now state the following definition:

Definition 4.2. The action Φ of G on M is called a strongly presymplectic, or a Hamiltonian action if, for any $\xi \in \mathfrak{g}$, the fundamental vector field ξ_M is a Hamiltonian vector field, i.e., if the 1-form $b(\xi_M)$ is exact.

4.1. Momentum mapping.

Let (M,ω) be a presymplectic manifold and let G be a Lie group acting presymplectically on M. Let $\mathfrak{g}$ be the Lie algebra of G.

Definition 4.3. A map $J : M \to \mathfrak{g}^*$ is a moment map (or momentum map) for the action of G on M if, for any $\xi \in \mathfrak{g}$, $b(\xi_M) = dJ_\xi$, where ξ_M is the fundamental vector field on M associated to ξ, and $J_\xi \in C^\infty(M)$ is the function defined by setting $J_\xi(x) = \langle J(x), \xi\rangle$, for any $x \in M$.

Note that a necessary condition for the existence of a moment map is that the 1-form $b(\xi_M)$ be an exact 1-form, for any $\xi \in \mathfrak{g}$, i.e., that the action of G on M

be strongly presymplectic. Actually the converse is also true (as in the symplectic case):

Proposition 4.4. *A moment map $J : M \to \mathfrak{g}^*$ exists if and only if the action of G on M is strongly presymplectic.*

Proof. We have already seen that this condition is necessary. Let us assume now that the action of G on M is strongly presymplectic. Let $\xi_1, \dots, \xi_k$ be a basis of $\mathfrak{g}$ and let $J_{\xi_i} \in C^\infty(M)$, be functions such that $b(\xi_{iM}) = dJ_{\xi_i}$, for $i = 1, \dots, k$. For any $\xi = \sum_{i=1}^k \lambda_i \xi_i \in \mathfrak{g}$ we define $J_\xi \in C^\infty(M)$ by setting $J_\xi = \sum_{i=1}^k \lambda_i J_{\xi_i}$. Then the map $J : M \to \mathfrak{g}^*$ defined by $\langle J(x), \xi\rangle = J_\xi(x)$ is a moment map for the action of G on M. □

Remark 4.5. It is clear from the definition that, if J_1 and J_2 are two moment maps for the same action of G on M, and M is connected, then there exists $\mu \in \mathfrak{g}^*$ such that $J_1(x) - J_2(x) = \mu$, for any $x \in M$. On the other hand, if J is a moment map for the action of G on M then $J_\mu(x) = J(x) + \mu$ is another moment map for the same action of G on M, for any $\mu \in \mathfrak{g}^*$.

Let us assume now that there exists a moment map $J : M \to \mathfrak{g}^*$ for the action of G on M. Since the functions J_ξ, for any $\xi \in \mathfrak{g}$, are Hamiltonian functions, the Poisson bracket of J_ξ and J_η is well defined for any $\xi, \eta \in \mathfrak{g}$. However it is not true, in general, that

$$\{J_\xi, J_\eta\} = J_{[\xi,\eta]}.$$

Related to this problem we have the following result, whose proof is the same as in the symplectic case (see [AM]):

Proposition 4.6. *The following two statements are equivalent:*

(1) $\{J_\xi, J_\eta\} = J_{[\xi,\eta]}$, *for every* $\xi, \eta \in \mathfrak{g}$,
(2) *for any* $g \in G$ *the following diagram is commutative*

$$\begin{tikzcd} M \arrow[r, "J"] \arrow[d, "\Phi_g"'] & \mathfrak{g}^* \arrow[d, "\mathrm{Ad}^*_g"] \\ M \arrow[r, "J"'] & \mathfrak{g}^*, \end{tikzcd}$$

i.e., the moment map J is Ad**-equivariant.*

We can now give the following definition:

Definition 4.7. The action Φ of G on M is called a Poissonian, or a strongly Hamiltonian action if there exists an Ad*-equivariant moment map J.

4.2. **Presymplectic reduction.** We shall now describe how the usual symplectic reduction procedure of Marsden–Weinstein can be extended to the case of presymplectic manifolds.

Let (M, ω) be a presymplectic manifold and G be a Lie group acting on M. We shall assume that the action is strongly Hamiltonian, so that there exists an Ad^*-equivariant moment map $J : M \to \mathfrak{g}^*$. Under these assumptions, it follows that, for any $\mu \in \mathfrak{g}^*$, the isotropy group

$$G_\mu = \{g \in G \mid \mathrm{Ad}_g^* \mu = \mu\}$$

acts on $J^{-1}(\mu)$ (it is actually the maximal subgroup of G leaving $J^{-1}(\mu)$ invariant), hence we can form the quotient $\overline{M}_\mu = J^{-1}(\mu)/G_\mu$. This is usually called the *reduced phase space*. Of course we are especially interested in the case when $\overline{M}_\mu$ is actually a "space", i.e., for instance, a smooth manifold. The usual way to ensure this is to assume that μ is a regular value of J and that the action of G_μ on $J^{-1}(\mu)$ is free and proper. In particular, from now on, we shall assume that both $J^{-1}(\mu)$ and $\overline{M}_\mu = J^{-1}(\mu)/G_\mu$ are smooth manifolds.

Let us denote by i the natural injection $i : J^{-1}(\mu) \hookrightarrow M$. The presymplectic structure ω on M determines a natural presymplectic structure $\omega_\mu = i^*\omega$ on $J^{-1}(\mu)$. In the following proposition we shall describe the kernel of ω_μ.

Proposition 4.8. *For any $x \in J^{-1}(\mu)$, we have $\ker \omega_\mu(x) = \mathfrak{g}_{\mu M_x} + \ker \omega(x)$, where*

$$\mathfrak{g}_{\mu M_x} = \{v \in T_x J^{-1}(\mu) \mid v = \xi_M(x), \ \textit{for some } \xi \in \mathfrak{g}_\mu\}.$$

To prove this result we need the following lemma:

Lemma 4.9. (1) $\mathfrak{g}_{\mu M} = \mathfrak{g}_M \cap \mathfrak{X}(J^{-1}(\mu))$;
(2) *For any $x \in J^{-1}(\mu)$, we have $T_x J^{-1}(\mu) = \mathfrak{g}_{M_x}^{\perp_\omega}$;*
(3) *For any $x \in J^{-1}(\mu)$, we have $\ker \omega(x) \subset T_x J^{-1}(\mu)$;*
(4) $\ker \omega_\mu(x) = T_x J^{-1}(\mu) \cap T_x J^{-1}(\mu)^{\perp_\omega}$.

Proof. (1) Let us recall that $\mathfrak{g}_\mu$ is the Lie algebra of G_μ. Since G_μ acts on $J^{-1}(\mu)$, we have $\mathfrak{g}_{\mu M} \subset \mathfrak{X}(J^{-1}(\mu))$. But, by recalling that G_μ is the maximal subgroup of G leaving $J^{-1}(\mu)$ invariant, it follows that

$$\mathfrak{g}_{\mu M} = \mathfrak{g}_M \cap \mathfrak{X}(J^{-1}(\mu)).$$

(2) Since μ is a regular value of J, we have $T_x J^{-1}(\mu) = \ker(T_x J)$. Hence $v \in T_x J^{-1}(\mu)$ if and only if, for every $\xi \in \mathfrak{g}$,

$$0 = \langle T_x J(v), \xi \rangle = \langle v, dJ_\xi(x) \rangle = \omega(\xi_M(x), v).$$

This means precisely that $v \in \mathfrak{g}_{M_x}^{\perp_\omega}$.

(3) By (2) we know that $T_x J^{-1}(\mu) = \mathfrak{g}_{M_x}^{\perp_\omega}$, and it is obvious that $\ker \omega(x) \subset \mathfrak{g}_{M_x}^{\perp_\omega}$.

(4) By definition, we have that $v \in \ker \omega_\mu(x)$ if and only if $v \in T_x J^{-1}(\mu)$ and $\omega_\mu(v, w) = 0$ for any $w \in T_x J^{-1}(\mu)$, which means precisely that $v \in T_x J^{-1}(\mu)^{\perp_\omega}$. $\square$

We can now prove the preceeding proposition:

Proof. By the results of the lemma we have:

$$\begin{aligned}\ker\omega_\mu(x) &= T_xJ^{-1}(\mu)\cap T_xJ^{-1}(\mu)^{\perp_\omega}\\ &= T_xJ^{-1}(\mu)\cap({\mathfrak{g}_M}_x^{\perp_\omega})^{\perp_\omega}\\ &= T_xJ^{-1}(\mu)\cap({\mathfrak{g}_M}_x+\ker\omega(x))\\ &= {\mathfrak{g}_{\mu M}}_x+\ker\omega(x),\end{aligned}$$

where the equality

$$({\mathfrak{g}_M}_x^{\perp_\omega})^{\perp_\omega} = {\mathfrak{g}_M}_x+\ker\omega(x)$$

is left as a simple exercise in linear algebra. □

Now we shall consider the quotient $\overline{M}_\mu = J^{-1}(\mu)/G_\mu$. Let us denote by $\pi : J^{-1}(\mu) \to \overline{M}_\mu$ the canonical projection. For any $x \in J^{-1}(\mu)$, we have $T_{\pi(x)}\overline{M}_\mu = T_xJ^{-1}(\mu)/{\mathfrak{g}_{\mu M}}_x$. From the inclusion ${\mathfrak{g}_{\mu M}}_x \subset \ker\omega_\mu(x)$, it follows that we can define a 2-form $\overline{\omega}_\mu$ on $\overline{M}_\mu$ by setting

$$\overline{\omega}_\mu(\bar{v},\bar{w}) = \omega_\mu(v,w),$$

where $v,w \in T_xJ^{-1}(\mu)$ are any two vectors such that $T_x\pi(v) = \bar{v}$ and $T_x\pi(w) = \bar{w}$. It is evident that $\overline{\omega}_\mu$ is a 2-form on $\overline{M}_\mu$ satisfying $\pi^*\overline{\omega}_\mu = \omega_\mu = i^*\omega$. It is a closed 2-form because we have $\pi^*(d\overline{\omega}_\mu) = d(\pi^*\overline{\omega}_\mu) = d\omega_\mu = 0$, and from this it follows that $d\overline{\omega}_\mu = 0$ (because π and $T\pi$ are surjections).

From what we have seen before it is now clear that the kernel of $\overline{\omega}_\mu$ is the projection of the kernel of ω_μ, i.e., for any $x \in J^{-1}(\mu)$ we have

$$\ker\overline{\omega}_\mu(\pi(x)) = \ker\omega_\mu(x)/{\mathfrak{g}_{\mu M}}_x.$$

Finally, by recalling the preceeding description of $\ker\omega_\mu(x)$, we deduce that

$$\ker\overline{\omega}_\mu(\pi(x)) = \ker\omega(x)/{\mathfrak{g}_{\mu M}}_x.$$

We have thus proved the following result, which is the analogue in the presymplectic case of the usual symplectic reduction theorem:

Theorem 4.10. *Let (M,ω) be a presymplectic manifold. Let G be a Lie group acting on M. Let us assume that the action of G on M is strongly Hamiltonian and let us denote by $J : M \to \mathfrak{g}^*$ an* Ad**-equivariant moment map. Let $\mu \in \mathfrak{g}^*$ be a regular value of J and let us denote by $i : J^{-1}(\mu) \hookrightarrow M$ the natural inclusion. Let G_μ denote the isotropy subgroup for the coadjoint action of G on $\mathfrak{g}^*$. Finally let us assume that the quotient space $\overline{M}_\mu = J^{-1}(\mu)/G_\mu$ is a smooth manifold (e.g., assume that the action of G_μ on $J^{-1}(\mu)$ is free and proper), and let us denote by $\pi : J^{-1}(\mu) \to \overline{M}_\mu$ the canonical projection. Then there exists a unique presymplectic structure $\overline{\omega}_\mu$ on $\overline{M}_\mu$ such that $\pi^*\overline{\omega}_\mu = i^*\omega$. Moreover, for any $x \in J^{-1}(\mu)$, we have*

$$\ker\overline{\omega}_\mu(\pi(x)) = \ker\omega(x)/{\mathfrak{g}_{\mu M}}_x.$$

In particular, the reduced presymplectic structure $\overline{\omega}_\mu$ *is a symplectic structure if and only if we have* $\ker \omega(x) = \mathfrak{g}_{\mu M_x}$*, for any* $x \in J^{-1}(\mu)$*.*

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