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DIFFERENTIAL FORMS ON MODULI SPACES OF PRINCIPAL BUNDLES

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Abstract. Let X be a smooth projective variety, G a connected reductive algebraic group, and let $\mathcal{M}_G$ be a moduli space of stable principal G-bundles over X. By defining a suitable local version of the Atiyah class of a family of principal bundles and applying it to a (locally defined) universal family of principal G-bundles over $\mathcal{M}_G$, we are able to construct, in a natural way, closed differential forms on the moduli space $\mathcal{M}_G$. We remark that no assumption about the smoothness of the moduli spaces is made.

1. INTRODUCTION

Moduli spaces of vector bundles or, more generally, of principal bundles, over a variety X are very interesting geometrical objects which, in general, inherit lots of structure from the variety X itself.

When X is a non-singular projective variety defined over an algebraically closed field k of characteristic 0 and G is a connected reductive algebraic group over k, moduli spaces of (semi)stable principal G-bundles over X are known to exist and to be quasi-projective schemes (usually singular). They were first constructed by Ramanathan (cf. [11], [12] and [13]), when $\dim X = 1$, and subsequently the construction was extended, by various authors, to the case of higher dimensional varieties. For a modern construction of moduli spaces of stable principal G-bundles (and of their compactifications) we refer to [5].

In this paper we shall consider a moduli space $\mathcal{M}_G$ of stable principal G-bundles over a smooth projective variety X of dimension n, defined over an algebraically closed field k of characteristic 0, and we shall describe a natural procedure which leads to the construction of closed differential forms on $\mathcal{M}_G$ starting with some cohomology classes on X.

More precisely, for any $i \leq j$ and any invariant polynomial F, of homogeneous degree $k = n - i$, on the Lie algebra $\mathfrak{g}$, for the adjoint action of G, we shall define a

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map
$$f^F\colon H^i(X,\Omega_X^j)\to H^0(\mathcal{M}_G,\Omega_{\mathcal{M}_G}^{j-i}),$$
and prove that the differential forms on $\mathcal{M}_G$ obtained as images of cohomology classes in $H^i(X,\Omega_X^j)$ are closed.

This result generalizes, to the case of principal G-bundles, a construction of differential forms on moduli spaces of stable vector bundles carried out in [4] by using a different approach to the problem.

Let us briefly describe now the organization of the paper. In Section 2, after recalling the definition of the Atiyah class of a principal G-bundle, we introduce the notion of the *local Atiyah class* of a family of principal G-bundles over X and prove that equivalent families of principal G-bundles have the same local Atiyah class. This shows that, in a relative setting, the local Atiyah class behaves much better than the usual Atiyah class.

In Section 3 we first prove that it is possible to define the local Atiyah class of a universal family of principal G-bundles on a moduli space $\mathcal{M}_G$ of stable principal G-bundles over X even if, in general, such a universal family does not exist! In fact, universal families usually exist only locally on $\mathcal{M}_G$ and, as we shall see, this will be enough to allow us to glue together the local Atiyah classes of the locally defined universal families in order to construct a local Atiyah class globally defined over $\mathcal{M}_G$.

Finally by evaluating on this local Atiyah class a homogeneous invariant polynomial F on $\mathfrak{g}$, we obtain closed differential forms on the product $X\times\mathcal{M}_G$. By using these differential forms, we are finally able to define a natural map
$$f^F\colon H^i(X,\Omega_X^j)\to H^{k+i-n}(\mathcal{M}_G,\Omega_{\mathcal{M}_G}^{k+j-n}),$$
where k is the degree of F. The closedness of the elements in the image of f^F will then follow easily from the closedness of the differential forms constructed from the local Atiyah class via the invariant polynomial F.

2. LOCAL ATIYAH CLASSES

In this section we shall define the *local Atiyah class* of a family of principal G-bundles on a smooth projective variety.

Let X be a smooth n-dimensional projective variety over an algebraically closed field k of characteristic zero, let G be a connected reductive algebraic group over k, and let us denote by $\mathfrak{g}$ its Lie algebra. For a principal G-bundle P over X we denote by $\mathrm{ad}(P)$ its adjoint bundle (the vector bundle over X associated to the adjoint representation of G). To any such P we can associate a cohomology class $a(P)\in H^1(X,\mathrm{ad}(P)\otimes\Omega_X^1)$, called the Atiyah class of P (introduced by Atiyah in [1]). We also recall that, for any homogeneous invariant polynomial F on $\mathfrak{g}$, by evaluating F on $a(P)$ we obtain a cohomology class $F(a(P))\in H^k(X,\Omega_X^k)$, where k is the degree of F. All these cohomology classes are represented by closed differential form, and they generate the characteristic cohomology ring of P.

Now let Y be a locally noetherian scheme over k and let $\mathcal{P}$ be a family of stable principal G-bundles over X parametrized by Y (i.e., $\mathcal{P}$ is a principal G-bundle over $X \times Y$, such that for every closed point $y \in Y$ the principal G-bundle $\mathcal{P}|_{X\times\{y\}}$ is stable).

Any such family $\mathcal{P}$ defines a morphism

$$\rho_{\mathcal{P}} \colon Y \to \mathcal{M}_G,$$

where $\mathcal{M}_G$ is a suitable moduli space of stable principal G-bundles over X. Two such families $\mathcal{P}$ and $\mathcal{Q}$ of stable principal G-bundles over X are said to be *equivalent* if $\rho_{\mathcal{P}} = \rho_{\mathcal{Q}}$.

By considering the usual Atiyah class of the principal G-bundle $\mathcal{P}$ over $X \times Y$, we obtain a cohomology class

$$a(\mathcal{P}) \in H^1(X \times Y, \operatorname{ad}(\mathcal{P}) \otimes \Omega^1_{X\times Y}).$$

However it may happen that two equivalent families $\mathcal{P}$ and $\mathcal{Q}$ of stable principal G-bundles as above have different Atiyah classes. This is a clear indication that, in a relative situation, the usual Atiyah class is not the "right" object to consider. We shall now define a local version of it.

Definition 2.1. The local Atiyah class of a family $\mathcal{P}$ of principal G-bundles over X parametrized by Y, denoted by $\tilde{a}(\mathcal{P})$, is the image of $a(\mathcal{P})$ under the natural map

$$H^1(X \times Y, \operatorname{ad}(\mathcal{P}) \otimes \Omega^1_{X\times Y}) \to H^0(Y, R^1 q_*(\operatorname{ad}(\mathcal{P}) \otimes \Omega^1_{X\times Y})),$$

where $q \colon X \times Y \to Y$ is the canonical projection.

As we shall see, the local Atiyah class, being a global section of a sheaf, behaves much better than its classical global analogue. In fact, if $\mathcal{P}$ and $\mathcal{Q}$ are two equivalent families of stable principal G-bundles over X, we have $\tilde{a}(\mathcal{P}) = \tilde{a}(\mathcal{Q})$.

Lemma 2.1. *Let $\mathcal{P}$ and $\mathcal{Q}$ be two equivalent families of stable principal G-bundles over X parametrized by Y. Then, for any $i \geq 0$, there is a natural isomorphism of sheaves over Y*

$$R^i q_* \operatorname{ad}(\mathcal{P}) \cong R^i q_* \operatorname{ad}(\mathcal{Q}).$$

Moreover, the local Atiyah classes $\tilde{a}(\mathcal{P})$ and $\tilde{a}(\mathcal{Q})$ are identified under the natural isomorphism $R^1 q_* \operatorname{ad}(\mathcal{P}) \cong R^1 q_* \operatorname{ad}(\mathcal{Q})$.

Proof. Let us recall that $R^i q_* \operatorname{ad}(\mathcal{P})$ is the sheaf over Y associated to the presheaf $U \mapsto H^i(X \times U, \operatorname{ad}(\mathcal{P}))$ and that, for any closed point $y \in Y$, the stalk of $R^i q_* \operatorname{ad}(\mathcal{P})$ over y is isomorphic to $H^i(X \times \{y\}, \operatorname{ad}(\mathcal{P}|_{X\times\{y\}})$. Since the two families $\mathcal{P}$ and $\mathcal{Q}$ are equivalent, we have $\mathcal{P}|_{X\times\{y\}} \cong \mathcal{Q}|_{X\times\{y\}}$, for any $y \in Y$. These isomorphisms actually define an isomorphism of sheaves $R^i q_* \operatorname{ad}(\mathcal{P}) \cong R^i q_* \operatorname{ad}(\mathcal{Q})$.

Let us remark that the local Atiyah class $\tilde{a}(\mathcal{P})$ is the global section of the sheaf $R^1 q_* \operatorname{ad}(\mathcal{P})$ determined by the section $a(\mathcal{P})$ of the corresponding presheaf. Since, for any closed point $y \in Y$, the germ of $\tilde{a}(\mathcal{P})$ in y is the usual Atiyah class of the principal G-bundle $\mathcal{P}|_{X\times\{y\}}$, and since we have an isomorphism $\mathcal{P}|_{X\times\{y\}} \cong \mathcal{Q}|_{X\times\{y\}}$, it follows that $\tilde{a}(\mathcal{P}) = \tilde{a}(\mathcal{Q})$ as claimed. □

If F is a homogeneous invariant polynomial on $\mathfrak{g}$, by evaluating it on the local Atiyah class of $\mathcal{P}$ we obtain a global section $F(\tilde{a}(\mathcal{P}))$ of the sheaf $R^k q_*(\Omega^k_{X\times Y})$ over Y, where k is the degree of F. Note that, as in the case of the usual Atiyah class, the section $F(\tilde{a}(\mathcal{P}))$ is represented by a closed differential form.

3. DIFFERENTIAL FORMS ON MODULI SPACES

In this section we shall apply the preceding results in order to construct closed differential forms on moduli spaces of principal G-bundles over a smooth projective variety.

From now on we shall take as Y a moduli space $\mathcal{M}_G$ of stable principal G-bundles over X. We recall that, in general, there does not exist a universal family of principal G-bundles on $\mathcal{M}_G$, however universal families do exist locally on $\mathcal{M}_G$, for the complex analytic or étale topology.

Let us consider a suitable open covering $\mathcal{U} = \{U_i\}_{i\in I}$ of $\mathcal{M}_G$ and let $\mathcal{P}_i$ be a universal family over $X \times U_i$. By Lemma 2.1, the sheaves $R^1 q_* \operatorname{ad}(\mathcal{P}_i)$ and $R^1 q_* \operatorname{ad}(\mathcal{P}_j)$ coincide on $U_i \cap U_j$, hence we can glue the family of sheaves $\{R^1 q_* \operatorname{ad}(\mathcal{P}_i)\}_{i\in I}$ in order to obtain a sheaf defined over $\mathcal{M}_G$ which, by abuse of notation, we shall denote by $R^1 q_* \operatorname{ad}(\mathcal{P})$, even if there is no universal family $\mathcal{P}$ over $\mathcal{M}_G$.

Again by recalling Lemma 2.1, we see that the local Atiyah classes $\tilde{a}(\mathcal{P}_i)$ and $\tilde{a}(\mathcal{P}_j)$ agree on the intersection $U_i \cap U_j$. It follows that the family of local Atiyah classes $\{\tilde{a}(\mathcal{P}_i)\}_{i\in I}$ define a global section of the sheaf $R^1 q_* \operatorname{ad}(\mathcal{P})$. By abuse of notation we shall denote this section by $\tilde{a}(\mathcal{P})$ and call it the local Atiyah class of $\mathcal{P}$.

Let now F be a homogeneous invariant polynomial of degree k on $\mathfrak{g}$. By evaluating F on $\tilde{a}(\mathcal{P})$ we obtain a global section of the sheaf $R^k q_*(\Omega^k_{X\times\mathcal{M}_G})$, that we shall denote by $\tilde{\gamma}^F(\mathcal{P})$.

Let us remark that, for any open subset $U \subseteq \mathcal{M}_G$, we have $\Omega^1_{X\times U} = p^*\Omega^1_X \oplus q^*\Omega^1_U$, where $p : X \times \mathcal{M}_G \to X$ and $q : X \times \mathcal{M}_G \to \mathcal{M}_G$ are the canonical projections. Since X is a smooth variety, it follows that there is a Künneth decomposition

$$H^k(X \times U, \Omega^k_{X\times U}) = \bigoplus_{i,j=0}^{k} H^i(X, \Omega^j_X) \otimes H^{k-i}(U, \Omega^{k-j}_U),$$

for every $k \geq 0$.

Since $R^k q_*(\Omega^k_{X\times\mathcal{M}_G})$ is the sheaf over $\mathcal{M}_G$ associated to the presheaf

$$U \mapsto H^k(X \times U, \Omega^k_{X\times U}),$$

we obtain a similar Künneth decomposition of sheaves

$$R^k q_*(\Omega^k_{X\times\mathcal{M}_G}) = \bigoplus_{i,j=0}^{k} H^i(X,\Omega^j_X)\otimes\mathcal{H}^{k-i}(\mathcal{M}_G,\Omega^{k-j}_{\mathcal{M}_G}),$$

where $\mathcal{H}^{k-i}(\mathcal{M}_G,\Omega^{k-j}_{\mathcal{M}_G})$ is the sheaf associated to the presheaf

$$U\mapsto H^{k-i}(U,\Omega^{k-j}_U).$$

Definition 3.1. Given a homogeneous invariant polynomial F of degree k on $\mathfrak{g}$, we shall write

$$\tilde{\gamma}^F(\mathcal{P}) = \sum_{i,j}\tilde{\gamma}^F_{i,j}(\mathcal{P}),$$

where $\tilde{\gamma}^F_{i,j}(\mathcal{P})$ is a global section of the sheaf $H^i(X,\Omega^j_X)\otimes\mathcal{H}^{k-i}(\mathcal{M}_G,\Omega^{k-j}_{\mathcal{M}_G})$.

Remark 3.1. For any homogeneous invariant polynomial F, the corresponding section $\tilde{\gamma}^F(\mathcal{P})$ of the sheaf $R^k q_*(\Omega^k_{X\times\mathcal{M}_G})$ is represented by a d-closed differential form. It follows that all its components $\tilde{\gamma}^F_{i,j}(\mathcal{P})$ are also d-closed.

We can now prove the following result:

Theorem 3.1. *Let $n=\dim X$. For any $i,j=1,\dots,n$ and any homogeneous invariant polynomial F of degree k on $\mathfrak{g}$, with $k\geq\max\{n-i,n-j\}$, there is a natural map*

$$f^F\colon H^i(X,\Omega^j_X)\to H^{k+i-n}(\mathcal{M}_G,\Omega^{k+j-n}_{\mathcal{M}_G}).$$

Moreover, for any $\sigma\in H^i(X,\Omega^j_X)$, the cohomology class $f^F(\sigma)$ is d-closed.

Proof. To define the map f^F we first consider the isomorphism

$$H^i(X,\Omega^j_X)\xrightarrow{\sim}H^{n-i}(X,\Omega^{n-j}_X)^*$$

given by Serre duality; then we compose it with the map

$$H^{n-i}(X,\Omega^{n-j}_X)^*\to H^{k+i-n}(\mathcal{M}_G,\Omega^{k+j-n}_{\mathcal{M}_G})$$

defined by multiplication by the section $\tilde{\gamma}^F_{n-i,n-j}(\mathcal{P})$ of the sheaf

$$H^{n-i}(X,\Omega^{n-j}_X)\otimes\mathcal{H}^{k+i-n}(\mathcal{M}_G,\Omega^{k+j-n}_{\mathcal{M}_G}).$$

It only remains to prove that, for any $\sigma\in H^i(X,\Omega^j_X)$, the cohomology class $f^F(\sigma)$ is d-closed. This follows easily from the closedness of the section $\tilde{\gamma}^F_{n-i,n-j}(\mathcal{P})$. In fact, if we write $\tilde{\gamma}^F_{n-i,n-j}(\mathcal{P})=\sum_\ell\alpha_\ell\otimes\beta_\ell$, for some $\alpha_\ell\in H^{n-i}(X,\Omega^{n-j}_X)$ and some sections β_ℓ of $\mathcal{H}^{k+i-n}(\mathcal{M}_G,\Omega^{k+j-n}_{\mathcal{M}_G})$, we have:

$$0=d\tilde{\gamma}^F_{n-i,n-j}(\mathcal{P})=\sum_\ell\big(d_X\alpha_\ell\otimes\beta_\ell+\alpha_\ell\otimes d_{\mathcal{M}_G}\beta_\ell\big).$$

Since X is a non-singular projective variety, we have $d_X\alpha_\ell = 0$, hence

$$\sum_\ell \alpha_\ell \otimes d_{\mathcal{M}_G}\beta_\ell = 0.$$

By recalling the definition of f^F, we can write

$$f^F(\sigma) = \sum_\ell \langle \sigma, \alpha_\ell \rangle \, \beta_\ell,$$

where $\langle \cdot, \cdot \rangle$ is the Serre duality pairing. It follows that

$$d(f^F(\sigma)) = \sum_\ell \langle \sigma, \alpha_\ell \rangle \, d\beta_\ell = 0.$$

□

As a special case of this theorem, namely for an invariant polynomial F of homogeneous degree $k = n - i$, we obtain a natural map

$$f^F \colon H^i(X, \Omega_X^j) \to H^0(\mathcal{M}_G, \Omega_{\mathcal{M}_G}^{j-i}),$$

for any $i \leq j$. It follows that, if there exists such an invariant polynomial, we can construct closed holomorphic p-forms on $\mathcal{M}_G$ by starting with elements in $H^i(X, \Omega_X^{i+p})$, for any $i \geq 0$. As an example, if we take G to be $\mathrm{GL}(n)$ (resp. $\mathrm{SL}(n)$), the methods developed in this paper provide a way to construct closed differential forms on moduli spaces of stable vector bundles (resp. stable vector bundles with fixed determinant) over X. A different (and more explicit) construction of such differential forms was given in [4], under the additional assumption of smoothness of the moduli spaces of stable vector bundles.

Finally we remark that for $p = 2$ we obtain a natural construction of presymplectic structures on moduli spaces of principal G-bundles over X. It turns out that in some cases the corresponding 2-form is actually non-degenerate, hence they define holomorphic symplectic structures on the corresponding moduli spaces.

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