The evaluation of American option prices under stochastic volatility and jump-diffusion dynamics

Carl Chiarella*, Boda Kang*, Gunter Meyer† and Andrew Ziogas‡

*School of Finance and Economics, UTS
†School of Mathematics, Georgia Institute of Technology, Atlanta
‡Integral Energy, Australia

Workshop on Mathematical Finance
Wolfgang Runggaldier’s 65th Birthday
Bressanone

16-20 July, 2007
1 Introduction

- Implied volatilities found using traded option prices vary with respect to option moneyness; smiles and skews are observed.

- Use alternative to the Black-Scholes asset return dynamics to capture the leptokurtosis found in financial time series data e.g. Merton’s (1976) jump-diffusion model, Heston’s (1993) stochastic volatility model, Scott’s (1997) \( SV + Jumps \) model, Lévy processes - Cont & Tankov (2004).

- Pricing European options under these alternative dynamics well-developed. American option prices are much harder to evaluate in these cases.
Aims of this paper are:

– to derive the integral equations for the price and early exercise boundary of an American call option under the combination of Heston’s (1993) square root and Merton’s (1976) jump diffusion processes;

– to extend the Fourier transform method as reformulated by Jamshidian (1992) to this; and

– to investigate numerical solution of the IPDE via the method of lines.
2 Presentation Overview

- Review of existing literature.
- Problem definition.
- The solution using the Fourier transform.
- Structure of the solution.
- Solving the IPDE directly via the method of lines.
- Solving the IPDE via componentwise splitting.
- Numerical results.
- Conclusion.
3 Literature Review: American Option

- **Kim (1990), Jacka (1991), Carr, Jarrow & Myneni (1992):** various derivation methods; decompose option price into European price plus an early premium term.

- **Jamshidian (1992):** transforms homogeneous FBVP to inhomogeneous unrestricted BVP.

- **Meyer and Van der Hoek (1997):** Method of Lines.

- **Various authors:** Finite difference methods, finite element methods etc.
4 Literature Review: Stochastic Volatility


5 Literature Review: Jump Diffusions

- **Merton (1976):** European call options under jump-diffusion.
- **Pham (1997):** American puts under jump-diffusion; behaviour of the price and free boundary using Ito calculus.
- **Gukhal (2001):** Kim’s method for American calls and puts under jump-diffusion.
- **Meyer (1998):** Method of lines.
6 Literature Review: SV + JD

- Cont and Tankov (2004): From the perspective of Levy processes.
7 SV + JD Dynamics

- SDE for $S$:

$$dS = (\mu - \lambda k)Sdt + \sqrt{v}SdZ_1 + (Y - 1)Sd\bar{q},$$

$$dv = \kappa_v(\theta - v)dt + \sigma \sqrt{v}dZ_2,$$

$$dZ_j \sim N(0, dt),$$

$$\mathbb{E}[dZ_1 dZ_2] = \rho dt,$$

$$k = \mathbb{E}_Q[(Y - 1)] = \int_0^\infty (Y - 1)G(Y)dY.$$
8 American Call: Free Boundary Value Problem

- IPDE for $C(S, v, \tau)$:

$$\frac{\partial C}{\partial \tau} = \frac{vS^2}{2} \frac{\partial^2 C}{\partial S^2} + \rho \sigma v S \frac{\partial^2 C}{\partial S \partial v} + \frac{\sigma^2 v}{2} \frac{\partial^2 C}{\partial v^2}$$

$$+ \left( r - q - \lambda \int_0^\infty (1 - \lambda_J(Y))(Y - 1)G(Y)dY \right) \frac{\partial C}{\partial S}$$

$$+ (\kappa_v[\theta - v] - \lambda_v v) \frac{\partial C}{\partial v} - rC$$

$$+ \lambda \int_0^\infty (1 - \lambda_J(Y))[C(SY, v, \tau) - C(S, v, \tau)]G(Y)dY,$$

- $\lambda_J(Y)$ - the MPR associated with a jump in $S$ with magnitude $Y$.

- Assume the market price of volatility risk is of the form

$\lambda(v) = \lambda_v \sqrt{v}$.
• **Solved in the region** $0 \leq \tau \leq T$, $0 < S \leq b(v, \tau)$.

• Subject to BCs

\[
C(S, v, 0) = \max(S - K, 0),
\]
\[
C(b(v, \tau), v, \tau) = b(v, \tau) - K,
\]
\[
\lim_{S \to b(v, \tau)} \frac{\partial C}{\partial S} = 1, \quad \lim_{S \to b(v, \tau)} \frac{\partial C}{\partial v} = 0.
\]
• In the volatility domain the boundary conditions are

\[
\frac{\partial C'(S, 0, \tau)}{\partial \tau} = \left( r - q - \lambda \int_0^\infty (1 - \lambda J(Y))(Y - 1)G(Y)dY \right) S \frac{\partial C(S, 0, \tau)}{\partial S} \\
+ \kappa_v \theta \frac{\partial C(S, 0, \tau)}{\partial v} - r C(S, 0, \tau) \\
+ \lambda \int_0^\infty (1 - \lambda J(Y))[C(SY, 0, \tau) - C(S, 0, \tau)]G(Y)dY,
\]

\[
\lim_{v \to \infty} C(S, v, \tau) = S.
\]
9 The Different Approaches to American Option Pricing

\[ M^C \text{Kean: - Homogeneous PDE} \]
\[ 0 < S < a(\tau) \]

\[ \text{Domain Charge} \]

\[ \text{Jamshidiam: - Inhomogeneous PDE} \]
\[ 0 < S < \infty \]

\[ \text{Kim: - Compound Option} \]

\[ C^{Am}(S,\tau) = F_M \left( \int_0^T G_M(a(\xi),a'(\xi),S,\xi) \, d\xi \right) \]
\[ a(\tau) = F^a_M \left( \int_0^T G^a_M(a(\xi),a'(\xi),a(\tau),\xi) \, d\xi \right) \]

\[ C^{Am}(S,\tau) = F_K \left( \int_0^T G_K(a(\xi),S,\xi) \, d\xi \right) \]
\[ a(\tau) = F^a_K \left( \int_0^T G^a_K(a(\xi),a(\tau),\xi) \, d\xi \right) \]
Cost incurred by the investor from downward jumps in $S$. 
13 General Form of the Solution

- The Jamshidiam approach gives the solution of the form

\[
C(S, v, \tau) = \Omega_C(S, v, \tau) + \int_0^\tau \Psi_C[b(\xi), \xi, \tau, S, v; C(\cdot, \xi)] d\xi,
\]

\[
b(v, \tau) = \Omega_a(b(\tau), v, \tau) + \int_0^\tau \Psi_a[b(\xi), \xi, \tau, b(\tau), v; C(\cdot, \xi)] d\xi,
\]
**14 Numerical Solution Using the Method of Lines**

- The method of lines has several strengths when dealing with American options:
  - The **price**, **free boundary**, **delta** and **gamma** are all found as part of the computation.
  - The method discretises the IPDE in an intuitive manner, and is readily adapted to be **second order accurate in time**.
- The key idea behind the method of lines is to replace an IPDE with an equivalent system of one-dimensional integro-differential equations (IDEs).
- The **system of IDEs** is developed by discretising the time derivative and the derivative terms involving the volatility, $v$.
- We must provide a means of dealing with the integral term.
• **The IPDE** to be solved is

\[
\frac{\partial C}{\partial \tau} = \frac{v S^2}{2} \frac{\partial^2 C}{\partial S^2} + \rho \sigma v S \frac{\partial^2 C}{\partial S \partial v} + \frac{\sigma^2 v}{2} \frac{\partial^2 C}{\partial v^2} 
\]

\[
+ (r - q - \lambda^* k^*) S \frac{\partial C}{\partial S} + (\alpha - \beta v) \frac{\partial C}{\partial v} 
\]

\[
- (r + \lambda^*) C + \lambda^* \int_0^\infty C(SY, v, \tau) G^*(Y) dY,
\]

where \( \alpha \equiv \kappa_v \theta \) and \( \beta \equiv \kappa_v + \lambda_v \).

• The domain for the problem is \( 0 \leq \tau \leq T, \ 0 \leq S \leq b(v, \tau) \) and \( 0 \leq v < \infty \).
• We discretise according to $\tau_n = n\Delta\tau$ and $v_m = m\Delta v$.

• $C(S, v_m, \tau_n) = C^m_m(S)$,

$$V(S, v_m, \tau_n) \equiv \frac{\partial C(S, v_m, \tau_n)}{\partial S} = V^m_m(S).$$

• We use the **standard central difference scheme**

$$\frac{\partial^2 C}{\partial v^2} = \frac{C^m_{m+1} - 2C^m_m + C^m_{m-1}}{(\Delta v)^2},$$

$$\frac{\partial^2 C}{\partial S \partial v} = \frac{V^m_{m+1} - V^m_{m-1}}{2\Delta v}.$$

• We use an **upwinding finite difference scheme** for the first order derivative term

$$\frac{\partial C}{\partial v} = \begin{cases} \frac{C^m_{m+1} - C^m_m}{\Delta v} & \text{if } v \leq \frac{\alpha}{\beta}, \\ \frac{C^m_m - C^m_{m-1}}{\Delta v} & \text{if } v > \frac{\alpha}{\beta}. \end{cases}$$
• **Integral term at each grid point** estimated via num. integration.

• We assume that the jump sizes are log-normally distributed.

• Applying the Hermite Gauss-quadrature scheme, we have

\[
W_m^n = \frac{1}{\sqrt{\pi}} \sum_{j=0}^{J} w_j^H C_m^n \left( S \exp \left\{ (\gamma - \delta^2/2) + \sqrt{2}\delta X_j^H \right\} \right),
\]

• We interpolate for the required values of \( C_m^n \) using cubic splines fitted in \( S \) along each line in \( v \).

• **A second order approximation for the time derivative**,\[
\frac{\partial C}{\partial \tau} = \frac{3}{2} \frac{C_m^n - C_m^{n-1}}{\Delta \tau} + \frac{1}{2} \frac{C_m^{n-1} - C_m^{n-2}}{\Delta \tau}.
\]
After taking the boundary conditions into consideration, at each time step \( n \) we must **solve a system of \( M - 1 \) second order IDEs**.

This is done using a **two stage iterative scheme**.

First we view the IDEs as ODEs by using \( C_{m-1}^n \) as an initial approximation for \( C_m^n \) in the integral term \( W_m^n \).

We then solve the ODEs for increasing values of \( v \), using the latest available estimates for \( C_{m+1}^n, C_{m-1}^n, V_{m+1}^n \) and \( V_{m-1}^n \).

We iterate until the price profile converges to a desired level of accuracy.

We **update the estimate of the integral term** \( W_m^n \) using the current price profile estimate, and repeat the process until convergence is obtained for both levels of iteration.
• The generic first order form of the ODE

\[
\Delta \frac{dC^n_m}{dS} = V^n_m,
\]

\[
\Gamma \frac{dV^n_m}{dS} = A_m(S)C^n_m + B_m(S)V^n_m + P^n_m(S),
\]

where \( P^n_j(S) \) is also a function of \( C^n_{m+1}, C^n_{m-1}, V^n_{m+1}, V^n_{m-1}, C^{n-1}_m, C^{n-2}_m \) and \( W^n_m \).

• We solve above system using the Riccati transform.
• **The Riccati transformation**

\[ C^n_m(S) = R_m(S)V^n_m(S) + W^n_m(S). \]

• Where \( R \) and \( W \) are solutions to the initial value problems

\[
\begin{align*}
\frac{dR_m}{dS} &= 1 - B_m(S)R_m - A_m(S)(R_m)^2, \quad R_m(0) = 0, \\
\frac{dW^n_m}{dS} &= -A_m(S)R_m(S)W^n_m - R_m(S)P^n_m(S), \quad W^n_m(0) = 0,
\end{align*}
\]

and \( V^n_m \) is the solution to

\[
\begin{align*}
\frac{dV^n_m}{dS} &= A_m(S)(R_m(S)V^n_m + W^n_m(S)) + B_m(S)V^n_m + P^n_m(S), \\
V^n_m(b^n_m) &= 1,
\end{align*}
\]
• We solve for increasing values of $S$.

• We then **find the value $S^*$ such that**

\[ S^* - K = R_m(S^*) + W^m(S^*), \]

• Thus $S^*$ is the value of the free boundary at grid point $(v_m, \tau_n)$. 
Boundary condition \( v = v_M \)

Initial Condition (Payoff)

Boundary condition \( v = 0 \)
Stencil for $C_m^n = f(C_{m-1}^n, C_m^n, C_{m+1}^n, C_m^{n-1}, C_m^{n-2})$. 
Solving for the free boundary point along a \((v_m, \tau_n)\) line.
15 Numerical Solutions using Componentwise Splitting Method

- A non-standard finite difference method on nonuniform grids.
- In $S$–direction: $(S - h_l, v, \tau) \leftarrow (S, v, \tau) \rightarrow (S + h_r, v, \tau)$. A uniform grid in $v$ direction with a step size $h$.
- We approximate the derivatives in the $S$–direction:

$$\frac{\partial C}{\partial S} \approx \frac{1}{h_l + h_r} \left( \frac{h_l}{h_r} C(S + h_r, v, \tau) - \left( \frac{h_l}{h_r} - \frac{h_r}{h_l} \right) C - \frac{h_r}{h_l} C(S - h_l, v, \tau) \right),$$

(20)

$$\frac{\partial^2 C}{\partial S^2} \approx \frac{2}{h_l + h_r} \left( \frac{1}{h_l} C(S - h_l, v, \tau) - \left( \frac{1}{h_l} + \frac{1}{h_r} \right) C + \frac{1}{h_r} C(S + h_r, v, \tau) \right).$$

(21)
• Discretize the underlying IPDE without jumps by a seven-point finite difference stencil:
\[
\frac{\partial C}{\partial \tau} + AC = 0, \tag{22}
\]
where \( A \) is a block tridiagonal matrix and \( C \) is a vector.

• Use the CN method to discretize the semi-discrete (22):
\[
\left( I + \frac{1}{2} \Delta \tau A \right) C^{(k+1)} = \left( I - \frac{1}{2} \Delta \tau A \right) C^{(k)}, \quad k = 0, \ldots, N - 1, \tag{23}
\]
where \( N \) is the number of time steps and \( I \) is the identity matrix. The initial value \( C^{(0)} \) is given by the payoff function.
• Solve a sequence of linear complementarity problems (LCPs)

\[
\begin{cases}
BC^{(k+1)} \geq DC^{(k)}, \quad C^{(k+1)} \geq c, \\
\left(BC^{(k+1)} - DC^{(k)}\right)^T \left(C^{(k+1)} - c\right) = 0,
\end{cases}
\]

for \( k = 0, \ldots, N - 1 \).

• We implement the componentwise splitting methods for LCPs based on the decomposition of the matrix \( A \):

\[ A = A_S + A_{Sv} + A_v. \]  

• The matrices \( A_S, A_{Sv}, A_v \) contain the couplings of the finite difference stencil in the \( S \)–direction, in the \( Sv \)–direction, and in the \( v \)–direction, respectively.

• They can be made tridiagonal after performing different re-orderings of unknowns for each of those matrices.
We implemented a second-order accurate splitting method by performing a Strang symmetrization for a basic “three steps” splitting method which uses the Crank-Nicolson method.

Perform first a half time step with $A_S$ and then with $A_v$, a full time step with $A_{Sv}$, and finally a half time step with $A_v$ and then with $A_S$. The notations are as follow:

\[
B_{S/2} = I + \frac{1}{4} \Delta \tau A_S, \quad B_{v/2} = I + \frac{1}{4} \Delta \tau A_v, \\
B_{Sv} = I + \frac{1}{2} \Delta \tau A_{Sv}, \quad D_{Sv} = I - \frac{1}{2} \Delta \tau A_{Sv}, \quad (26) \\
D_{S/2} = I - \frac{1}{4} \Delta \tau A_S, \quad D_{v/2} = I - \frac{1}{4} \Delta \tau A_v.
\]
• We approximate the original LCP (24) by five LCPs as follows:

\[
\begin{align*}
B_{S/2}C^{(k+1/5)} & \geq D_{S/2}C^{(k)}, \quad C^{(k+1/5)} \geq c, \\
\left( B_{S/2}C^{(k+1/5)} - D_{S/2}C^{(k)} \right)^T \left( C^{(k+1/5)} - c \right) & = 0, \\
B_{v/2}C^{(k+2/5)} & \geq D_{v/2}C^{(k+1/5)}, \quad C^{(k+2/5)} \geq c, \\
\left( B_{v/2}C^{(k+2/5)} - D_{v/2}C^{(k+1/5)} \right)^T \left( C^{(k+2/5)} - c \right) & = 0, \\
B_{Sv}C^{(k+3/5)} & \geq D_{Sv}C^{(k+2/5)}, \quad C^{(k+3/5)} \geq c, \\
\left( B_{Sv}C^{(k+3/5)} - D_{Sv}C^{(k+2/5)} \right)^T \left( C^{(k+3/5)} - c \right) & = 0, \\
B_{v/2}C^{(k+4/5)} & \geq D_{v/2}C^{(k+3/5)}, \quad C^{(k+4/5)} \geq c, \\
\left( B_{v/2}C^{(k+4/5)} - D_{v/2}C^{(k+3/5)} \right)^T \left( C^{(k+4/5)} - c \right) & = 0, \\
B_{S/2}C^{(k+1)} & \geq D_{S/2}C^{(k+4/5)}, \quad C^{(k+1)} \geq c, \\
\left( B_{S/2}C^{(k+1)} - D_{S/2}C^{(k+4/5)} \right)^T \left( C^{(k+1)} - c \right) & = 0,
\end{align*}
\]

for \( k = 0, \ldots, N - 1 \).

• Finally, we add the jump integral term \( I(S, v, \tau) \) to the explicit side of
Eqs (27) to (31) and evaluate them in a similar way as they are in MOL:

1. \( I(S, v, \tau) \) is evaluated at payoff or from the most recent prices;
2. 1/6 of \( I(S, v, \tau) \) is added to the \( S \)-direction;
3. 1/6 of \( I(S, v, \tau) \) is added to the \( v \)-direction;
4. 1/3 of \( I(S, v, \tau) \) is added to the \( Sv \)-direction;
5. 1/6 of \( I(S, v, \tau) \) is added to the \( v \)-direction;
6. 1/6 of \( I(S, v, \tau) \) is added to the \( S \)-direction;
7. Evaluate \( I(S, v, \tau) \) with a cubic spline interpolation based on the updated prices;
8. If the average error in the grid is:
   - bigger than the tolerance then return to 2 and start to calculate the new price based on the new integral,
   - less than the tolerance then start next time step.
## 16 Numerical Results

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>SV Parameter</th>
<th>Value</th>
<th>JD Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T$</td>
<td>0.50</td>
<td>$\theta$</td>
<td>0.04</td>
<td>$\lambda^*$</td>
<td>5.00</td>
</tr>
<tr>
<td>$r$</td>
<td>0.03</td>
<td>$\kappa_v$</td>
<td>2.00</td>
<td>$\gamma$</td>
<td>0.00</td>
</tr>
<tr>
<td>$q$</td>
<td>0.05</td>
<td>$\sigma$</td>
<td>0.40</td>
<td>$\delta$</td>
<td>0.10</td>
</tr>
<tr>
<td>$K$</td>
<td>100</td>
<td>$\lambda_v$</td>
<td>0.00</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\rho$</td>
<td></td>
<td></td>
<td></td>
<td>$\pm 0.50$</td>
<td></td>
</tr>
</tbody>
</table>

Table 1: Parameter values used for the American call option. The stochastic volatility (SV) parameters correspond to the Heston model. The jump-diffusion (JD) parameters correspond to the Merton model with log-normal jump sizes.
Table 2: Sample convergence pattern for the method of lines iterative procedures. Parameter values are as given in Table 1.
Figure 2: Early exercise surface for a 6-month American call option, generated using the method of lines. Parameter values are as listed in Table 1.
Table 3: Parameters used to match the time-averaged variance for the GBM, JD, SV and SVJD models for a 6-month option. The equivalent Black-Scholes volatilities, $\sqrt{v_{\text{GBM}}}$, are 29.7860% for $\rho = 0.50$, and 30.2760% for $\rho = -0.50$. The value of $v$ in the SVJD model is 4%.

<table>
<thead>
<tr>
<th>Model</th>
<th>Parameter</th>
<th>Value: $\rho = 0.50$</th>
<th>Value: $\rho = -0.50$</th>
</tr>
</thead>
<tbody>
<tr>
<td>GBM</td>
<td>$v_{\text{BSM}}$</td>
<td>8.8721%</td>
<td>9.1664%</td>
</tr>
<tr>
<td>JD</td>
<td>$v_{\text{JD}}$</td>
<td>3.8596%</td>
<td>4.1539%</td>
</tr>
<tr>
<td>SV</td>
<td>$\theta_{\text{SV}}$</td>
<td>9.0000%</td>
<td>8.5250%</td>
</tr>
<tr>
<td></td>
<td>$v_{\text{SV}}$</td>
<td>10%</td>
<td>9%</td>
</tr>
</tbody>
</table>
Figure 3: Exploring the effect of jump-diffusion and stochastic volatility on the early exercise boundary for an American call option. The correlation is $\rho = 0.50$; all other parameter values are as listed in Tables 1 and 3.
\( \rho = 0.50, \nu = 0.04 \)

<table>
<thead>
<tr>
<th>Method ((N, M, S_{\text{pts}}))</th>
<th>80</th>
<th>90</th>
<th>100</th>
<th>110</th>
<th>120</th>
<th>RMSRD (%)</th>
<th>Runtime (sec)</th>
</tr>
</thead>
<tbody>
<tr>
<td>MOL (50, 100, 1138)</td>
<td>1.4844</td>
<td>3.7123</td>
<td>7.6982</td>
<td>13.6686</td>
<td>21.3645</td>
<td>0.0505</td>
<td>485</td>
</tr>
<tr>
<td>MOL (200, 100, 1138)</td>
<td>1.4847</td>
<td>3.7130</td>
<td>7.6993</td>
<td>13.6697</td>
<td>21.3654</td>
<td>0.0381</td>
<td>1162</td>
</tr>
<tr>
<td>MOL (200, 250, 2995)</td>
<td>1.4848</td>
<td>3.7146</td>
<td>7.7018</td>
<td>13.6715</td>
<td>21.3657</td>
<td>0.0148</td>
<td>12120</td>
</tr>
<tr>
<td>CS (2.5) (200, 100, 294)</td>
<td>1.4841</td>
<td>3.7070</td>
<td>7.6806</td>
<td>13.6387</td>
<td>21.3357</td>
<td>0.2113</td>
<td>100</td>
</tr>
<tr>
<td>CS (2.5) (300, 100, 294)</td>
<td>1.4747</td>
<td>3.6853</td>
<td>7.6442</td>
<td>13.5972</td>
<td>21.3029</td>
<td>0.6470</td>
<td>118</td>
</tr>
<tr>
<td>CS (2.5) (300, 200, 549)</td>
<td>1.4770</td>
<td>3.7027</td>
<td>7.6868</td>
<td>13.6563</td>
<td>21.3537</td>
<td>0.2990</td>
<td>345</td>
</tr>
<tr>
<td>CS (2.5) (1000, 1000, 2764)</td>
<td>1.4825</td>
<td>3.7120</td>
<td>7.6996</td>
<td>13.6690</td>
<td>21.3628</td>
<td>0.0823</td>
<td>25985</td>
</tr>
<tr>
<td>PSOR (200, 100, 200)</td>
<td>1.4960</td>
<td>3.7415</td>
<td>7.7507</td>
<td>13.7300</td>
<td>21.4103</td>
<td>0.5779</td>
<td>470</td>
</tr>
<tr>
<td>PSOR (500, 500, 1000)</td>
<td>1.4861</td>
<td>3.7181</td>
<td>7.7086</td>
<td>13.6793</td>
<td>21.3707</td>
<td>0.0668</td>
<td>31269</td>
</tr>
<tr>
<td>PSOR (1000, 2000, 4000)</td>
<td>1.4847</td>
<td>3.7152</td>
<td>7.7037</td>
<td>13.6732</td>
<td>21.3660</td>
<td>−</td>
<td>1650931</td>
</tr>
</tbody>
</table>

Table 4: American call prices computed using method of lines (MOL), componentwise splitting (CS) and Crank-Nicolson with PSOR (PSOR). Parameter values are given in Table 1, with \( \rho = 0.50 \) and \( \nu = 0.04 \). For CS, the first number in brackets for the CS method indicates the ratio between the grid step sizes at \( S_{\text{max}} \) and \( K \) imposed on the the non-uniform grid in \( S \).
$\rho = -0.50, v = 0.04$

<table>
<thead>
<tr>
<th>Method ($N, M, S_{pts}$)</th>
<th>80</th>
<th>90</th>
<th>100</th>
<th>110</th>
<th>120</th>
<th>RMSRD (%)</th>
<th>Runtime (sec)</th>
</tr>
</thead>
<tbody>
<tr>
<td>MOL (50, 100, 1138)</td>
<td>1.1369</td>
<td>3.3512</td>
<td>7.5922</td>
<td>13.8786</td>
<td>21.7156</td>
<td>0.0595</td>
<td>485</td>
</tr>
<tr>
<td>MOL (200, 100, 1138)</td>
<td>1.1370</td>
<td>3.3518</td>
<td>7.5932</td>
<td>13.8798</td>
<td>21.7168</td>
<td>0.0514</td>
<td>1159</td>
</tr>
<tr>
<td>MOL (200, 250, 2995)</td>
<td>1.1363</td>
<td>3.3530</td>
<td>7.5959</td>
<td>13.8827</td>
<td>21.7191</td>
<td>0.0119</td>
<td>12122</td>
</tr>
<tr>
<td>CS (2.5) (200, 100, 294)</td>
<td>1.1368</td>
<td>3.3526</td>
<td>7.5950</td>
<td>13.8807</td>
<td>21.7162</td>
<td>0.0347</td>
<td>98</td>
</tr>
<tr>
<td>CS (2.5) (300, 100, 294)</td>
<td>1.1233</td>
<td>3.3199</td>
<td>7.5440</td>
<td>13.8309</td>
<td>21.6834</td>
<td>0.7803</td>
<td>117</td>
</tr>
<tr>
<td>CS (2.5) (300, 200, 549)</td>
<td>1.1298</td>
<td>3.3433</td>
<td>7.5855</td>
<td>13.8734</td>
<td>21.7120</td>
<td>0.3055</td>
<td>323</td>
</tr>
<tr>
<td>CS (2.5) (1000, 1000, 2764)</td>
<td>1.1336</td>
<td>3.3501</td>
<td>7.5940</td>
<td>13.8808</td>
<td>21.7174</td>
<td>0.1215</td>
<td>25707</td>
</tr>
<tr>
<td>PSOR (200, 100, 200)</td>
<td>1.1651</td>
<td>3.4050</td>
<td>7.6510</td>
<td>13.9196</td>
<td>21.7358</td>
<td>1.3621</td>
<td>490</td>
</tr>
<tr>
<td>PSOR (500, 500, 1000)</td>
<td>1.1394</td>
<td>3.3594</td>
<td>7.6035</td>
<td>13.8875</td>
<td>21.7210</td>
<td>0.1436</td>
<td>32979</td>
</tr>
<tr>
<td>PSOR (1000, 2000, 4000)</td>
<td>1.1363</td>
<td>3.3541</td>
<td>7.5981</td>
<td>13.8839</td>
<td>21.7192</td>
<td>—</td>
<td>1756066</td>
</tr>
</tbody>
</table>

Table 5: American call prices computed using method of lines (MOL), componentwise splitting (CS) and Crank-Nicolson with PSOR (PSOR). Parameter values are given in Table 1, with $\rho = -0.50$ and $v = 0.04$. RMSRD is calculated in the following way: $\sqrt{\frac{1}{\text{No. of prices}} \sum_{\text{price}} \left( \frac{\text{price} - \text{true price}}{\text{true price}} \right)^2}$. It is important to use RMSRD to measure the errors from all price, delta and gamma since they have different magnitude scale.
Efficiency Plots

- We worked out a number of efficiency plots in the next a couple of slides to compare the calculation efficiency with MOL, CS and PSOR.

- We take the solution from PSOR with a big grid consisting of 1,000 time steps, 2,000 volatility steps and 4,000 share steps which is shown in Tables 4 and 5 as a true solution for the price.

- We take the delta and gamma from MOL with a big grid consisting of 500 time steps, 1,000 volatility steps and 11,380 share steps as a true solution for delta and gamma.

- The root mean-square relative differences (RMSRDs) for each method in relation to this true solution in the corresponding cases with share prices range from 80 to 120 and $\rho = \pm 0.5$. 
Figure 4: Runtime efficiency of American call price with MOL, CS and PSOR.
Figure 5: Runtime efficiency of American call delta with MOL, CS and PSOR.
Figure 6: Runtime efficiency of American call gamma with MOL, CS and PSOR.
Figure 7: Overall runtime efficiency of American call price, delta and gamma with MOL, CS and PSOR.
Comments on the efficiency plots

Advantages of Method of Lines:

- Method of Lines (MOL) has more advantages.

- American call \textbf{delta} seems to have a faster convergence rate than the American call prices for MOL and CS.

- For a fixed grid of share prices and volatilities, the prices of MOL will “converge locally” with a relatively small time steps, say, usually 300 – 400 time steps and the accuracy could be up to almost 4 decimal places.
Disadvantages of Method of Lines:

- The stability region of MOL is smaller than CS and PSOR.
- Higher order of the tolerance, e.g. $10^{-8}$, within the SV iteration loop is necessary to update the prices and the jump integrals.
- For some parameters, the iteration within each time step no longer converge to the required tolerance, e.g. $10^{-8}$. 
17 Final Remarks

- Formulated American option under SV + JD dynamics.
- Representation of the solution using Jamshidian’s approach.
- Numerical solution by method of lines and component-wise splitting.
- Method of lines is the best in calculating all of American call price, delta and gamma.