Efficient valuation of exotic derivatives in Lévy models

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Volatility smile and surface

Volatility surfaces of foreign exchange and interest rate options

- Volatilities vary in strike (**smile**)
- Volatilities vary in time to maturity (**term structure**)
- Volatility clustering
The model

Let $\mathcal{B}_T = (\Omega, \mathcal{F}, \mathbb{F}, P)$ be a stochastic basis, where $\mathcal{F} = \mathcal{F}_T$ and $\mathbb{F} = (\mathcal{F}_t)_{0 \leq t \leq T}$. We model the price process of a financial asset as an exponential semimartingale

$$S_t = e^{H_t}, \quad 0 \leq t \leq T. \quad (1)$$

$H = (H_t)_{0 \leq t \leq T}$ is a semimartingale with canonical representation

$$H = H_0 + B + H^c + h(x) \ast (\mu^H - \nu) + (x - h(x)) \ast \mu^H. \quad (2)$$

For the processes $B, \ C = \langle H^c \rangle$, and the measure $\nu$ we use the notation

$$\mathbb{T}(H|P) = (B, C, \nu)$$

which is called the *triplet of predictable characteristics* of the semimartingale $H$. 

References
Alternative model description

\[ \mathcal{E}(X) = (\mathcal{E}(X)_t)_{0 \leq t \leq T} \quad \text{stochastic exponential} \]

\[ S_t = \mathcal{E}(\tilde{H})_t, \quad 0 \leq t \leq T \]
\[ dS_t = S_t d\tilde{H}_t \]

where
\[ \tilde{H}_t = H_t + \frac{1}{2} \langle H^c \rangle_t + \int_0^t \int_{\mathbb{R}} (e^x - 1 - x) \mu^H(ds, dx) \]

Note
\[ \mathcal{E}(\tilde{H})_t = \exp(\tilde{H}_t - \frac{1}{2} \langle \tilde{H}^c \rangle_t) \prod_{0 < s \leq t} (1 + \Delta \tilde{H}_s) \exp(-\Delta \tilde{H}_s) \]

Asset price positive only if \( \Delta \tilde{H} > -1. \)
Martingale modeling

Let $\mathcal{M}_{\text{loc}}(P)$ be the class of local martingales.

**Assumption (ES)**

The process $1_{\{x>1\}}e^x \ast \nu$ has bounded variation.

Then

$$S = e^H \in \mathcal{M}_{\text{loc}}(P) \iff B + \frac{C}{2} + (e^x - 1 - h(x)) \ast \nu = 0. \quad (3)$$

Throughout, we assume that $P$ is a (local) martingale measure for $S$. By the *Fundamental Theorem of Asset Pricing*, the value of an option on $S$ equals the discounted expected payoff under a martingale measure.

We assume zero interest rates.
Supremum and infimum processes

Let $X = (X_t)_{0 \leq t \leq T}$ be a stochastic process. We denote by

$$
\overline{X}_t = \sup_{0 \leq u \leq t} X_u \quad \text{and} \quad \underline{X}_t = \inf_{0 \leq u \leq t} X_u
$$

the supremum and infimum process of $X$ respectively. Since the exponential function is monotone and increasing

$$
\overline{S}_T = \sup_{0 \leq t \leq T} S_t = \sup_{0 \leq t \leq T} \left( e^{H_t} \right) = e^{\sup_{0 \leq t \leq T} H_t} = e^{\overline{H}_T}.
$$

(4)

Similarly

$$
\underline{S}_T = e^{\underline{H}_T}.
$$

(5)
Valuation formulae – payoff functional

We want to price an option with payoff $f(X_T)$, where $X_T = p(H_t, 0 \leq t \leq T)$ is an $\mathcal{F}_T$-measurable functional.

The functionals we consider are “European style”, and consist of two parts:

1. The *payoff function* is an arbitrary function $f : \mathbb{R} \rightarrow \mathbb{R}^+$; for example $f(x) = (e^x - K)^+$ or $f(x) = 1_{\{e^x > B\}}$, for $K, B \in \mathbb{R}^+$.

2. The *underlying process* can be the asset price or the supremum/infimum or an average of the asset price process (e.g. $X = H$ or $X = \overline{H}$).

- Exotic options
Valuation formulae – assumptions

Assumptions:

(R1) Assume that \( \int_{\mathbb{R}} e^{-Rx} f(x) \, dx < \infty \) for all \( R \in l_1 \subset \mathbb{R} \).

(R2) Assume that \( M_{X_T}(z) = E[e^{zX_T}] < \infty \), for all \( z \in l_2 \subset \mathbb{R} \).

(R3) Assume that \( l_1 \cap l_2 \neq \emptyset \).

Valuation formulae based on Fourier transforms; similar to Raible (2000), but no need for Lebesgue density.

Consider the Fourier transform of the payoff function like Borovkov and Novikov (2002); also Hubalek et al. (2006) and Černý (2007), for hedging.

Carr and Madan (1999) and Raible (2000) transform the option price.
Valuation formulae

Theorem 1

Assume that (R1)–(R3) are in force. Then, the price $V_f(X)$ of an option on $S = (S_t)_{0 \leq t \leq T}$ with payoff $f(X_T)$ is given by

$$V_f(X) = \frac{1}{2\pi} \int_{\mathbb{R}} \varphi_{X_T}(-u - iR)\mathcal{F}_f(u + iR)du,$$

where $\varphi_{X_T}$ denotes the extended characteristic function of $X_T$ and $\mathcal{F}_f$ denotes the Fourier transform of $f$.

Proof

Introduce the *dampened payoff function* $g(x) = e^{-Rx} f(x), R \in \mathbb{I}_1$. Then

$$V_f(X) = E[f(X_T)] = E[e^{RX_T} g(X_T)] = \int_{\mathbb{R}} e^{Rx} g(x) P_{X_T}(dx).$$

cont. next page
Proof (cont.)

Under assumption (R1), $g$ has a Fourier transform $\tilde{g}$; inverting it, we get a representation as

$$g(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-ixu} \tilde{g}(u) du. \quad (8)$$

Returning to the valuation problem (7) we get

$$V_f(X) = \int_{\mathbb{R}} e^{Rx} \left( \frac{1}{2\pi} \int_{\mathbb{R}} e^{-ixu} \tilde{g}(u) du \right) P_{X_T}(dx)$$

$$= \frac{1}{2\pi} \int_{\mathbb{R}} \left( \int_{\mathbb{R}} e^{i(-u-iR)x} P_{X_T}(dx) \right) \tilde{g}(u) du$$

$$= \frac{1}{2\pi} \int_{\mathbb{R}} \varphi_{X_T}(-u - iR) \tilde{f}_f(u + iR) du. \quad (9)$$
Valuation formulae II – options

Valuation formulae for options that depend on two functionals of the driving process.

Examples: barrier, slide-in or corridor and two-asset correlation option

\[
(S_T - K)^+ 1_{\{S_T > B\}};
\]

\[
(S_T - K)^+ \sum_{i=1}^{N} 1_{\{L < S_{T_i} < H\}};
\]

\[
(S_T^1 - K)^+ 1_{\{S_T^2 > B\}}.
\]
Valuation formulae II

**Theorem 2**

The price $V_{f,g}(X, Y)$ of an option on $S = (S_t)_{0 \leq t \leq T}$ with payoff function $f(X_T)g(Y_T)$ is given by

$$V_{f,g}(X, Y) = \frac{1}{4\pi^2} \int_{\mathbb{R}} \int_{\mathbb{R}} \varphi_{X_T,Y_T}(-u - iR_1, -v - iR_2)$$

$$\times \mathcal{F}_g(v + iR_2)\mathcal{F}_f(u + iR_1)dvdu,$$

where $\varphi_{X_T,Y_T}$ denotes the extended characteristic function of the random vector $(X_T, Y_T)$.

**Proof.**

Assumptions and proof are similar to Theorem 1.
Examples of payoff functions

Example (Call and put option)

**Call payoff** \( f(x) = (e^x - K)^+ \), \( K \in \mathbb{R}_+ \),

\[
\tilde{f}(u + iR) = \frac{K^{1+iu-R}}{(iu - R)(1 + iu - R)}, \quad R \in I_1 = (1, \infty). \tag{11}
\]

**Similarly, if** \( f(x) = (K - e^x)^+ \), \( K \in \mathbb{R}_+ \),

\[
\tilde{f}(u + iR) = \frac{K^{1+iu-R}}{(iu - R)(1 + iu - R)}, \quad R \in I_1 = (-\infty, 0). \tag{12}
\]
Example (Digital option)

Call payoff $1_{\{e^x > B\}}$, $B \in \mathbb{R}^+$. 

\[ \zeta_f(u + iR) = -B^{iu-R} \frac{1}{iu - R}, \quad R \in I_1 = (0, \infty). \]  

(13)

Similarly, for the payoff $f(x) = 1_{\{e^x < B\}}$, $B \in \mathbb{R}^+$,

\[ \zeta_f(u + iR) = B^{iu-R} \frac{1}{iu - R}, \quad R \in I_1 = (-\infty, 0). \]  

(14)

Example (Double digital option)

The payoff of a double digital call option is $1_{\{B < e^x < B\}}$, $B, \overline{B} \in \mathbb{R}^+$. 

\[ \zeta_f(u + iR) = \frac{1}{iu - R} \left( B^{iu-R} - \overline{B}^{iu-R} \right), \quad R \in I_1 = \mathbb{R} \setminus \{0\}. \]  

(15)
Example (Asset-or-nothing digital)

Call payoff  \( f(x) = e^x 1_{\{e^x > B\}} \)

\[ \tilde{\mathbb{F}} f(u + iR) = -\frac{B^{1+iu-R}}{1 + iu - R}, \quad R \in I_1 = (1, \infty) \]

Put payoff  \( f(x) = e^x 1_{\{e^x < B\}} \)

\[ \tilde{\mathbb{F}} f(u + iR) = \frac{B^{1+iu-R}}{1 + iu - R}, \quad R \in I_1 = (-\infty, 1) \]

Example (Self-quanto option)

Call payoff  \( f(x) = e^x (e^x - K)^+ \)

\[ \tilde{\mathbb{F}} f(u + iR) = \frac{K^{2+iu-R}}{(1 + iu - R)(2 + iu - R)}, \quad R \in I_1 = (2, \infty) \]
Lévy processes

Let $L = (L_t)_{0 \leq t \leq T}$ be a Lévy process with triplet of local characteristics $(b, c, \lambda)$, i.e. $B_t(\omega) = bt$, $C_t(\omega) = ct$, $\nu(\omega; dt, dx) = dt\lambda(dx)$, $\lambda$ Lévy measure.

**Assumption (EM)**

*There exists a constant $M > 1$ such that*

$$\int_{\{|x| > 1\}} e^{ux} \lambda(dx) < \infty, \quad \forall u \in [-M, M].$$

Using (EM) and Theorems 25.3 and 25.17 in Sato (1999), we get that

$$E[e^{uL_t}] < \infty, \quad E[e^{-uL_t}] < \infty \quad \text{and} \quad E[e^{uL_t}] < \infty$$

for all $u \in [-M, M]$. 
On the characteristic function of the supremum I

Lemma 3

Let \( L = (L_t)_{0 \leq t \leq T} \) be a Lévy process that satisfies assumption \((EM)\). Then, the moment generating function of \( \bar{L}_t \) is defined for all \( u \in (-\infty, M] \) and \( t \in [0, T] \).

Lemma 4

Let \( L = (L_t)_{0 \leq t \leq T} \) be a Lévy process that satisfies assumption \((EM)\). Then, the characteristic function \( \varphi_{\bar{L}_t} \) of \( \bar{L}_t \) is holomorphic in the half plane \( \{ z \in \mathbb{C} : -M < \ImaginaryPart z < \infty \} \) and can be represented as a Fourier integral in the complex domain

\[
\varphi_{\bar{L}_t}(z) = E\left[ e^{iz\bar{L}_t} \right] = \int_{\mathbb{R}} e^{izx} P_{\bar{L}_t}(dx).
\]
Theorem 5 (Wiener–Hopf factorization)

Let $L$ be a Lévy process. The Laplace transform of $L$ at an independent and exponentially distributed time $\theta$ can be identified from the Wiener–Hopf factorization of $L$ via

$$E[e^{-\beta L_\theta}] = \frac{\kappa(q, 0)}{\kappa(q, \beta)}$$

(16)

where $\kappa(\alpha, \beta)$, $\alpha \geq 0$, $\beta \geq 0$, is given by

$$\kappa(\alpha, \beta) = k \exp \left( \int_0^\infty \int_0^\infty \left( e^{-t} - e^{-\alpha t - \beta x} \right) \frac{1}{t} P_{L_t}(dx) \, dt \right).$$

(17)

Moreover, $\kappa$ can be analytically extended to $\alpha, \beta \in \mathbb{C}$ with $\Re \alpha \geq 0$ and $\Re \beta \geq -M$.

Proof.

Linking fixed and exponential times

**Lemma 6**

Let $L = (L_t)_{0 \leq t \leq T}$ be a Lévy process that satisfies assumption (EM) and consider $\beta \in \mathbb{C}$ with $\Re \beta \in [-M, \infty)$. The Laplace transforms of $L_t$, $t \in [0, T]$ and $L_\theta$, $\theta \sim \text{Exp}(q)$, are related via

$$E[e^{-\beta L_\theta}] = q \int_0^\infty e^{-qt} E[e^{-\beta L_t}] \, dt. \quad (18)$$

Moreover, the Laplace transform of $L_\theta$ is finite for $\beta \in \mathbb{C}$ with $\Re \beta \in [-M, \infty)$.

**Proof.**

An application of Fubini’s theorem yields

$$E[e^{-\beta L_\theta}] = E\left[\int_0^\infty qe^{-qt} e^{-\beta L_t} \, dt\right] = q \int_0^\infty e^{-qt} E[e^{-\beta L_t}] \, dt.$$
On the characteristic function of the supremum II

Theorem 7

Let \( L = (L_t)_{0 \leq t \leq T} \) be a Lévy process. The Laplace transform of \( \bar{L}_t \) at a fixed time \( t, t \in [0, T] \), is given by

\[
E[e^{-\beta \bar{L}_t}] = \frac{1}{2\pi} \int_{\mathbb{R}} \frac{e^{t(Y + iv)}}{Y + iv} \frac{\kappa(Y + iv, 0)}{\kappa(Y + iv, \beta)} dv,
\]

for \( Y > 0 \). Moreover, the Laplace transform can be extended to the complex plane for \( \beta \in \mathbb{C} \) with \( \Re \beta \in [-M, \infty) \).

Proof.

Combining eqs. (16) and (18) we get

\[
q \int_0^\infty e^{-qt} E[e^{-\beta \bar{L}_t}] dt = \frac{\kappa(q, 0)}{\kappa(q, \beta)}.
\]

Applying Doetsch (1950), we invert the Laplace transform and the claim follows.
Non-path-dependent options

European option on an asset with price process $S_t = e^{H_t}$

Examples: call, put, digitals, asset-or-nothing, double digitals, self-quanto options

$X_t \equiv H_T$, i.e. we need $\varphi_{H_T}$

Generalized hyperbolic model (GH model):

$$\varphi_{H_1}(u) = e^{iu\mu} \left( \frac{\alpha^2 - \beta^2}{\alpha^2 - (\beta + iu)^2} \right)^{\lambda/2} \frac{K_{\lambda}(\delta \sqrt{\alpha^2 - (\beta + iu)^2})}{K_{\lambda}(\delta \sqrt{\alpha^2 - \beta^2})}$$

$$l_2 = (-\alpha - \beta, \alpha - \beta)$$

$$\varphi_{H_T}(u) = (\varphi_{H_1}(u))^T$$

similar: NIG, CGMY, Meixner
Non-path-dependent options II

Stochastic volatility Lévy models: Carr, Geman, Madan, Yor (2003)

Stochastic clock \[ Y_t = \int_0^t y_s ds \quad (y_s > 0) \]
e.g. CIR process
\[ dy_t = K(\eta - y_t)dt + \lambda y_t^{1/2} dW_t \]

Define for a pure jump Lévy process \( X = (X_t)_{t \geq 0} \)
\[ H_t = X_{Y_t} \quad (0 \leq t \leq T) \]
Then
\[ \varphi_{H_t}(u) = \frac{\varphi_{Y_t}(-i\varphi_{X_t}(u))}{(\varphi_{Y_t}(-iu\varphi_{X_t}(-i)))^{iu}} \]
Lookback options

Fixed strike lookback option: $(\bar{S}_T - K)^+$.

Combining Theorem 1 and Theorem 7, we get

$$C_T(\bar{S}; K) = \frac{1}{2\pi} \int_{\mathbb{R}} \varphi_{\mathcal{L}_T}(-u - iR) \frac{K^{1+iu-R}}{(iu - R)(1 + iu - R)} \, du$$  \hspace{1cm} (21)

where

$$\varphi_{\mathcal{L}_T}(-u - iR) = \frac{1}{2\pi} \int_{\mathbb{R}} \frac{e^{T(Y + iv)}}{Y + iv} \frac{\kappa(Y + iv, 0)}{\kappa(Y + iv, iu - R)} \, dv.$$  \hspace{1cm} (22)

• The floating strike lookback option, $(\bar{S}_T - S_T)^+$, is treated by a duality formula.
Floating strike lookback options (1)

Payoff of a put: \( \left( \beta \sup_{0 \leq t \leq T} S_t - S_T \right)^+ \) for a \( 0 < \beta \leq 1 \)

Assume \( H' = (H'_t)_{0 \leq t \leq T} \) satisfies
\[
\text{Law} \left( H'_T - \inf_{t \leq T} H'_t \big| P' \right) = \text{Law} \left( \sup_{t \leq T} H'_t \big| P' \right)
\]
(holds for Lévy processes), then
\[
\mathbb{P}_T(\beta \sup S; S) = \beta C'_T \left( \sup S'; \frac{1}{\beta} \right)
\]

Value of a *floating strike* lookback put
→ value of a *fixed strike* lookback call
Floating strike lookback options (2)

Payoff of a call: \( \left( S_T - \alpha \inf_{0 \leq t \leq T} S_t \right)^+ \) for an \( \alpha \geq 1 \)

Assume \( H' = (H'_t)_{0 \leq t \leq T} \) satisfies the reflection principle

\[
\text{Law} \left( \sup_{t \leq T} H'_t - H'_T \mid P' \right) = \text{Law}( - \inf_{t \leq T} H'_t \mid P')
\]

(holds for Lévy processes), then

\[
C_T(S; \alpha \inf S) = \alpha P'_T \left( \frac{1}{\alpha} ; \inf S' \right)
\]

Value of a floating strike lookback call
→ value of a fixed strike lookback put
## Proof

\[ C_T(S; \alpha \inf S) = E[(S_T - \alpha \inf_{t \leq T} S_t)^+] = E[S_T \left(1 - \frac{\alpha \inf_{t \leq T} S_t}{S_T}\right)^+] \]

\[ = E' \left[\left(1 - \alpha e^{\inf_{t \leq T} H_t - H_T}\right)^+\right] \]

\[ = E' \left[\left(1 - \alpha e^{H'_T - \sup_{t \leq T} H'_t}\right)^+\right] \]

The process \( H' = (H'_t)_{0 \leq t \leq T} \) satisfies the reflection principle:

\[ \text{Law} \left(\sup_{t \leq T} H'_t - H'_T \mid P'\right) = \text{Law} \left(- \inf_{t \leq T} H'_t \mid P'\right) \]

\[ C_T(S; \alpha \inf S) = \alpha E' \left[\left(\frac{1}{\alpha} - e^{\inf_{t \leq T} H'_t}\right)^+\right] \]

\[ = \alpha E' \left[\left(\frac{1}{\alpha} - \inf_{t \leq T} S'_t\right)^+\right] = \alpha \mathbb{P}'_T \left(\frac{1}{\alpha}; \inf S'\right) \]
One-touch options

One-touch call option: \( 1_{\{S_T > B\}} \).

Combining Theorem 1, Theorem 7 and the example for digital options, we get

\[
\mathbb{D}C_T(S; B) = \frac{1}{4\pi^2} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{e^{T(Y+iv)}}{Y + iv} \frac{\kappa(Y + iv, 0)}{\kappa(Y + iv, iu - R)} \frac{B^{iu-R}}{R - iu} dv du. \tag{23}
\]

Similarly for the one-touch put option: \( 1_{\{S_T \leq B\}} \).

\[
\mathbb{D}P_T(S; B) = \frac{1}{4\pi^2} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{e^{T(Y+iv)}}{Y + iv} \frac{\widehat{\kappa}(Y + iv, 0)}{\widehat{\kappa}(Y + iv, iu - R)} \frac{B^{iu-R}}{iu - R} dv du. \tag{24}
\]
Equity default swap (EDS)

- Fixed premium exchanged for payment at “default”
- default: drop of stock price by 30% or 50% of $S_0 \rightarrow$ first passage time
- fixed leg pays premium $\mathcal{K}$ at times $T_1, \ldots, T_N$, if $T_i \leq \tau_B$
- if $\tau_B \leq T$: protection payment, paid at time $\tau_B$
- premium of the EDS chosen such that initial value equals 0; hence

$$\mathcal{K} = \frac{E \left[ e^{-r\tau_B} 1_{\{\tau_B \leq T\}} \right]}{\sum_{i=1}^{N} E \left[ e^{-rT_i} 1_{\{\tau_B > T_i\}} \right]}.$$  \hspace{1cm} (25)

- Calculations similar to touch options, since $1_{\{\tau_B \leq T\}} = 1_{\{S_T \leq B\}}$. 
Options on two assets

Two-asset correlation options:
Payoff of a correlation call: \((S_T^1 - K)^+ 1_{\{S_T^2 > B\}}\)

Measurement asset \(S^2\) in the money \(\rightarrow\) call on a payment asset \(S^1\)

Asset price processes \(S^i_t = \exp(L^i_t)\) \(i = 1, 2\)
where \(L = (L^1, L^2)\) is a time-inhomogeneous \(\text{Lévy}\) process

\[
\text{TAC}_{T}(S^1, S^2; K, B) = \frac{1}{4\pi^2} \int_{\mathbb{R}} \int_{\mathbb{R}} \varphi_{LT}(-u - iR_1, -v - iR_2) \\
\times \frac{K^{1+iu-R_1}}{(iu - R_1)(1 + iu - R_1)} \frac{B^{iu-R_2}}{R_2 - iu} dv du
\]
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