PRICING FINANCIAL CONTRACTS ON INFLATION

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Stylized facts

- Inflation-indexed bonds have been issued since the 80's, but it is only in the very last years that these bonds, and inflation-indexed derivatives in general, have become quite popular.

- Inflation is defined as the percentage increment of a reference index, the Consumer Price Index (CPI), which is a basket of goods and services.

- Denoting by $I(t)$ the CPI's value at time t, the inflation rate over the time interval $[t, T]$ is therefore:

$$i(t, T) := \frac{I(T)}{I(t)} - 1.$$

- In theory, but also in practice, inflation can become negative.

Stylized facts (cont'd)

Historical plots of CPI's

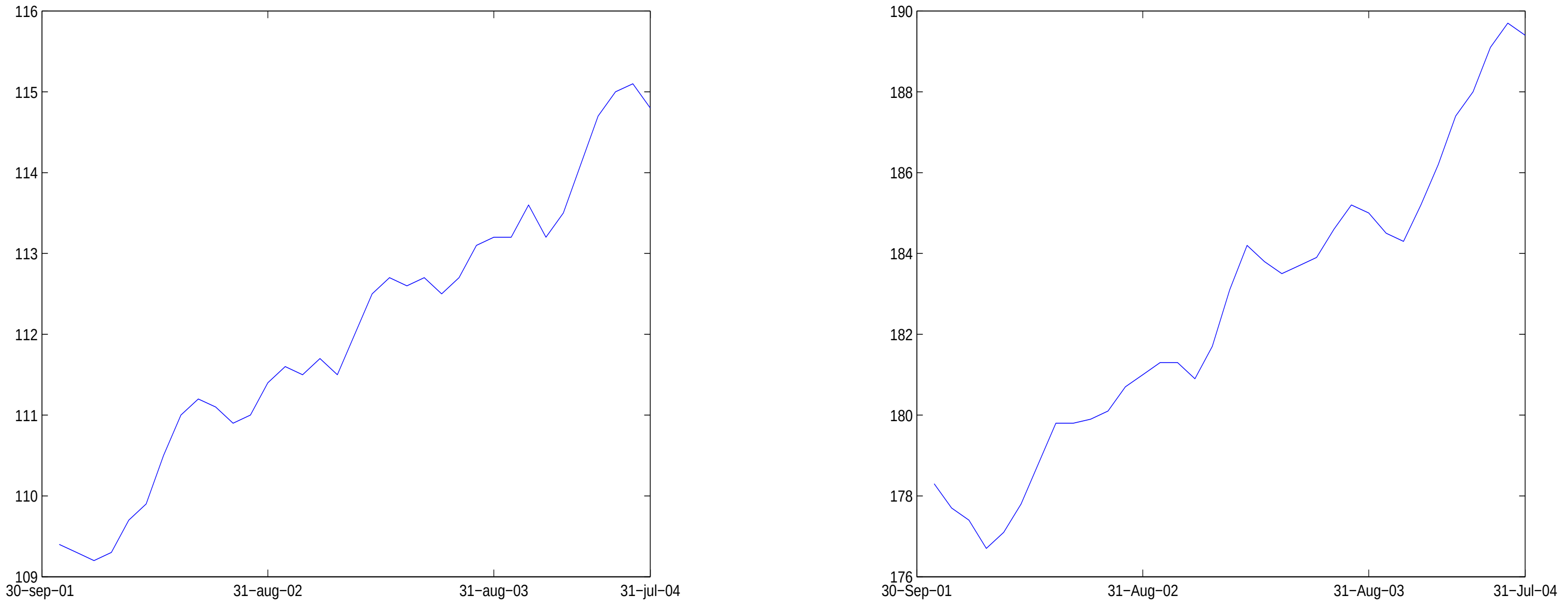


Figure 1: Left: EUR CPI Unrevised Ex-Tobacco. Right: USD CPI Urban Consumers NSA. Monthly closing values from 30-Sep-01 to 21-Jul-04.

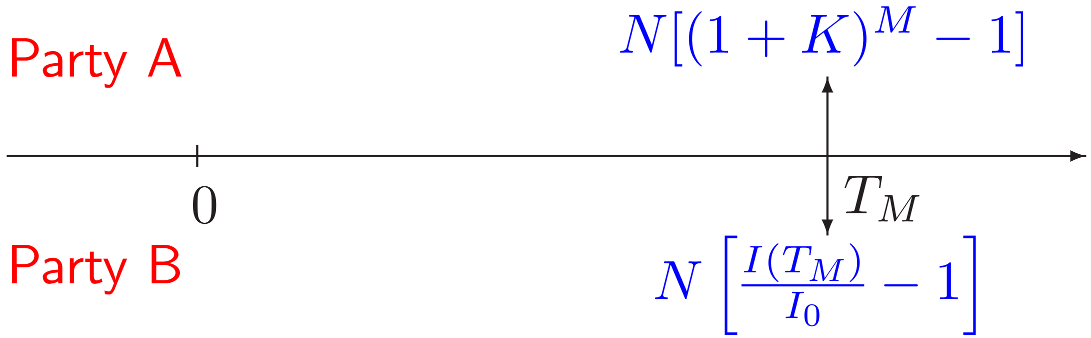
Stylized facts (cont'd)

- Banks are used to issue inflation-linked bonds, where a zero-strike floor is offered in conjunction with the "pure" bond.
- To grant positive coupons, the inflation rate is typically floored at zero.
- Accordingly, floors with low strikes are the most actively traded options on inflation rates.
- Other extremely popular derivatives are inflation-indexed swaps.
- Two are the main inflation-indexed swaps traded in the market:
  - the zero coupon (ZC) swap;
  - the year-on-year (YY) swap.

The related literature

- Inflation-indexed derivatives require a specific model to be valued.
- Main references: Barone and Castagna (1997), van Bezooyen et al. (1997), Hughston (1998), Kazziha (1999), Cairns (2000), Jamshidian (2002), Jarrow and Yildirim (2003), Korn and Kruse (2003), Belgrade et al. (2004), Mercurio (2005), Kruse and Nögel (2006) and Mercurio and Moreni (2006).
- Inflation derivatives are priced with a foreign-currency analogy (the pricing is equivalent to that of a cross-currency interest-rate derivative).
- In a short rate approach, one models the evolution of the instantaneous nominal and real rates and of the CPI (interpreted as the "exchange rate" between the nominal and real economies).
- Recent approaches are based on market models, where one models forward CPI indices and nominal rates.

Zero-coupon inflation-indexed swaps



Party A

$N[(1+K)^M - 1]$

0

T_M

Party B

$N\left[\frac{I(T_M)}{I_0} - 1\right]$

In a ZCIIS, at time $T_M = M$ years, Party B pays Party A the fixed amount

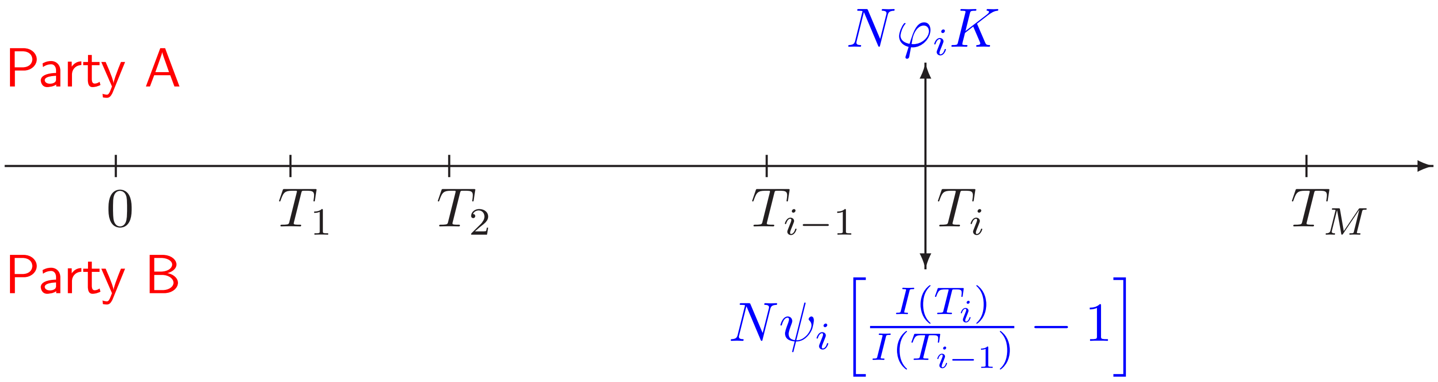
$$N[(1+K)^M - 1],$$

where K and N are, respectively, the contract fixed rate and the contract nominal value.

Party A pays Party B, at the final time T_M, the floating amount

$$N\left[\frac{I(T_M)}{I_0} - 1\right].$$

Year-on-year inflation-indexed swaps



In a YYIIS, at each time T_i, Party B pays Party A the fixed amount

$$N\varphi_i K,$$

while Party A pays Party B the (floating) amount

$$N\psi_i \left[\frac{I(T_i)}{I(T_{i-1})} - 1\right],$$

where φ_i and ψ_i are, respectively, the fixed- and floating-leg year fractions for the interval $[T_{i-1}, T_i]$, $T_0 := 0$ and N is again the swap nominal value.

ZCIIS and YYIIS rates

Both ZC and YY swaps are quoted, in the market, in terms of the corresponding fixed rate K.

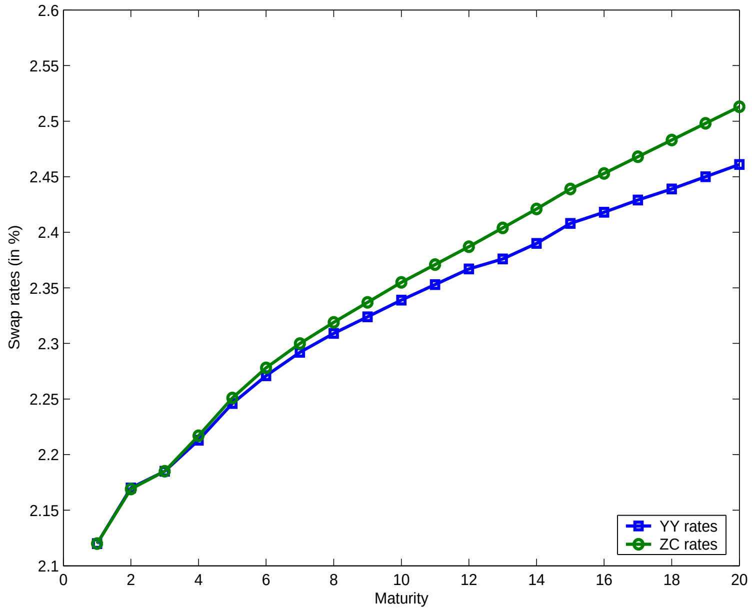


Figure 2: Euro inflation swap rates as of October 7, 2004. The reference CPI is the Euro-zone ex-tobacco index.

Inflation-indexed caplets

An Inflation-Indexed Caplet (IIC) is a call option on the inflation rate implied by the CPI index.

Analogously, an Inflation-Indexed Floorlet (IIF) is a put option on the same inflation rate.

In formulas, at time T_i, the IICF payoff is

$$N\psi_i \left[\omega\left(\frac{I(T_i)}{I(T_{i-1})} - 1 - \kappa\right)\right]^+,$$

where κ is the IICF strike, ψ_i is the contract year fraction for the interval $[T_{i-1}, T_i]$, N is the contract nominal value, and $\omega = 1$ for a caplet and $\omega = -1$ for a floorlet.

We set $K := 1 + \kappa$.

Inflation-indexed caplets (cont'd)

Standard no-arbitrage pricing theory implies that the value at time $t \le T_{i-1}$ of the IICF at time T_i is

$$\textbf{IICplt}(t, T_{i-1}, T_i, \psi_i, K, N, \omega)$$

$$= N\psi_i P_n(t, T_i) E_n^{T_i} \left\{ \left[\omega \left(\frac{I(T_i)}{I(T_{i-1})} - K \right) \right]^+ \Big| \mathcal{F}_t \right\}$$

$$= N\psi_i P_n(t, T_i) E_n^{T_i} \left\{ \left[\omega \left(\frac{\mathcal{I}_i(T_i)}{\mathcal{I}_{i-1}(T_{i-1})} - K \right) \right]^+ \Big| \mathcal{F}_t \right\},$$

where we define the T_i-forward CPI by

$$\mathcal{I}_i(t) := I(t) \frac{P_r(t, T_i)}{P_n(t, T_i)}.$$

A market model with stochastic volatility

We assume that, under a reference measure $\mathbb{Q}$:

- Nominal rates F_i are lognormally distributed with constant volatilities;
- Forward CPI's $\mathcal{I}_i$ follow Heston-like dynamics with a common volatility process $V(t)$:

$$\begin{aligned} dF_i(t)/F_i(t) &= (\ldots)\, dt + \sigma_i^F \, dZ_i^{\mathbb{Q},F} \\ d\mathcal{I}_i(t)/\mathcal{I}_i(t) &= (\ldots)\, dt + \sigma_i^I \sqrt{V(t)}\, dZ_i^{\mathbb{Q},I} \\ dV(t) &= \alpha(\theta - V(t))\, dt + \epsilon\sqrt{V(t)}\, dW^{\mathbb{Q}}, \quad V(0) = V_0, \end{aligned}$$

where σ_i^I, σ_i^F, α, θ, ϵ and V_0 are positive constants, and $2\alpha\theta > \epsilon$ to ensure positiveness of V.

We allow for correlations between Brownian motions $Z_i^{\mathbb{Q},F}, Z_i^{\mathbb{Q},I}, W^{\mathbb{Q}}$.

A market model with stochastic volatility (cont'd)

We take $\mathbb{Q} = \mathbb{Q}^0$, where $\mathbb{Q}^0$ is the spot LIBOR measure corresponding to the numeraire

$$B_d(t) = P(t, \beta(t)) \prod_{l=1}^{\beta(t)} [1 + \tau_l F_l(t)], \quad \beta(t) = T_j \text{ if } T_{j-1} < t \leq T_j.$$

By definition of B_d and the change-of-measure technique, we have, under $\mathbb{Q}^0$,

$$dF_i(t)/F_i(t) = \sigma_i^F \left[-\sum_{l=\beta(t)+1}^{i} \sigma_l^F \rho_{i,l}^F \frac{\tau_l F_l(t)}{1 + \tau_l F_l(t)} dt + dZ_i^{0,F}(t) \right]$$

$$d\mathcal{I}_i(t)/\mathcal{I}_i(t) = \sqrt{V(t)}\, \sigma_i^I \left[-\sum_{l=\beta(t)+1}^{i} \sigma_l^F \rho_{l,i}^{F,I} \frac{\tau_l F_l(t)}{1 + \tau_l F_l(t)} dt + dZ_i^{0,I}(t) \right]$$

$$dZ_i^{0,F} dZ_l^{0,F}(t) = \rho_{i,l}^F dt, \quad dZ_i^{0,I} dZ_l^{0,F}(t) = \rho_{l,i}^{F,I} dt$$

The pricing of caplets

The price at time $t \le T_{j-1}$ of the j-th caplet, is, under the measure Q^{T_j},

$$\begin{aligned}\textbf{IICplt}_j(t,K) &= P(t,T_j)E_t^{T_j}\left(\frac{\mathcal{I}_j(T_j)}{\mathcal{I}_{j-1}(T_{j-1})} - K\right)^+ \\ &= P(t,T_j)\int_{-\infty}^{+\infty}(e^s - e^k)^+ q_t^j(s)\,ds\end{aligned}$$

where $k = \ln(K)$ and $q_t^j(s)ds = Q^{T_j}\left\{\ln\left[\mathcal{I}_j(T_j)/\mathcal{I}_{j-1}(T_{j-1})\right] \in [s, s+ds]|\mathcal{F}_t\right\}.$

Remark. Instead of having a payoff depending on a single asset $S(t)$, as it is for standard or cliquet options (paying off $[S(T_j)/S(T_{j-1}) - K]^+$ in T_j), here the payoff depends on the ratio between two different assets at two different times.

The pricing of caplets (cont'd)

Following Carr and Madan (1999), we rewrite the caplet price in term of its (renormalized) Fourier transform:

$$\begin{aligned}\textbf{IICplt}_j(t, \mathrm{e}^k) &= P(t, T_j)\frac{\mathrm{e}^{-\eta k}}{2\pi}\int_{-\infty}^{+\infty} \mathrm{e}^{-isk}\psi_t^j(\eta, s)ds \\ &= P(t, T_j)\frac{\mathrm{e}^{-\eta k}}{\pi}\,\mathrm{Re}\int_{0}^{+\infty} \mathrm{e}^{-isk}\psi_t^j(\eta, s)ds \\ \psi_t^j(\eta, u) &= \frac{\phi_t^j(u-(\eta+1)i)}{(\eta+iu)(\eta+1+iu)}\end{aligned}$$

where the only unknown is the conditional characteristic function $\phi_t^j(\cdot)$ of $\ln\left(\mathcal{I}_j(T_j)/\mathcal{I}_{j-1}(T_{j-1})\right)$, and where $\eta \in \mathbb{R}^+$ is used to ensure L^2-integrability when $k \to -\infty$.

Derivation of the characteristic function

Our objective is now to find an explicit formula for ϕ_t^j.

Setting $Y_j(t) := \ln \mathcal{I}_j(T_j)$, we recall that, by definition of characteristic function and the Markov property:

$$\phi_t^j(u) = E_t^{T_j}\left[\mathrm{e}^{iu \ln \frac{\mathcal{I}_j(T_j)}{\mathcal{I}_{j-1}(T_{j-1})}}\right] = H\left(V(t), Y_j(t), Y_{j-1}(t), F_1(t), \ldots, F_j(t)\right).$$

Applying the Feynman-Kač theorem, H can then be found by solving a related PDE.

Remark. In the general case, due to the unpleasant presence of drift terms of type $\sqrt{V(t)}F_l(t)/(1+\tau_l F_l(t))$, there are no a priori reasons for the PDE to be explicitly solvable. In the following, we thus investigate a particular case allowing for an explicit solution.

Derivation of the characteristic function (cont'd)

We assume that, for each $i, l = 1, \ldots, M$:

$$\rho_{i,l}^{F,I} = \rho_i^{F,V} = 0$$

We allow, however, for non-zero correlations $\rho_{j,l}^{I} = dZ_j^I dZ_l^I / dt$ (between different forward CPI's) and $\rho_i^{I,V} = dZ_i^I dW/dt$ (between forward CPI's and the volatility).

Setting $X_j(t) := Y_j(t) - Y_{j-1}(t)$, we then have, under Q^{T_j},

$$dY_j(t) = -\frac{1}{2} V(t) (\sigma_j^I)^2 \, dt + \sqrt{V(t)} \, \sigma_j^I(t) \, dZ_j^I(t)$$

$$dX_j(t) = \frac{V(t)}{2} ((\sigma_{j-1}^I)^2 - (\sigma_j^I)^2) \, dt + \sqrt{V(t)} (\sigma_j^I \, dZ_j^I(t) - \sigma_{j-1}^I \, dZ_{j-1}^I(t))$$

$$dV(t) = [\alpha\theta - \alpha V(t)] \, dt + \epsilon \sqrt{V(t)} \, dW(t)$$

Derivation of the characteristic function (cont'd)

To make ϕ_t^j explicit, we write

$$\phi_t^j(u) = E_t^{T_j}\left[\mathrm{e}^{iu\left(Y_j(T_j)-Y_{j-1}(T_{j-1})\right)}\right] = E_t^{T_j}\left[\mathrm{e}^{-iuY_{j-1}(T_{j-1})}E_{T_{j-1}}^{T_j}\left(\mathrm{e}^{iuY_j(T_j)}\right)\right]$$

Noting that $E_{T_{j-1}}^{T_j}\left(\mathrm{e}^{iuY_j(T_j)}\right)$ is the characteristic function of $\ln \mathcal{I}_j(T_j)$ conditional on $\mathcal{F}_{T_{j-1}}$, solving a Heston-like PDE, we have that

$$E_{T_{j-1}}^{T_j}\left[\mathrm{e}^{iuY_j(T_j)}\right] = \exp\left\{A_Y(\bar{\tau}_j, u) + B_Y(\bar{\tau}_j, u)V(T_{j-1}) + iuY_j(T_{j-1})\right\}$$

where $\bar{\tau}_j := T_j - T_{j-1}$ and A_Y and B_Y are deterministic complex functions.

Consequently,

$$\phi_t^j(u) = \mathrm{e}^{A_Y(\bar{\tau}_j,u)}E_t^{T_j}\left[\mathrm{e}^{iuX_j(T_{j-1})+B_Y(\bar{\tau}_j,u)V(T_{j-1})}\right]$$

Derivation of the characteristic function (cont'd)

The last conditional expectation is nothing but the characteristic function of the couple $(X_j(T_{j-1}), V(T_{j-1}))$ evaluated at point $(u, -iB_Y(\bar{\tau}_j, u))$.

By again solving a PDE of Heston's type with suitable boundary conditions, we obtain

$$\phi_t^j(u) = \exp\left\{A_Y(\bar{\tau}_j, u) + A_X(T_{j-1} - t, u) + B_X(T_{j-1} - t, u)V(t) + iuX_j(t)\right\}$$

where A_X and B_X are other deterministic complex functions.

The II caplet price is finally calculated by numerical integration:

$$\mathbf{IICplt}_j(t, \mathrm{e}^k) = P(t, T_j)\frac{\mathrm{e}^{-\eta k}}{\pi}\,\mathrm{Re}\int_0^{+\infty} \mathrm{e}^{-isk}\frac{\phi_t^j(s - (\eta + 1)i)}{(\eta + is)(\eta + 1 + is)}\,ds$$

Derivation of the characteristic function (cont'd)

The coefficients A_Y and B_Y:

$$B_Y(s,u) = \frac{\gamma - b}{2a}\left[\frac{1 - \mathrm{e}^{\gamma s}}{1 - \frac{b-\gamma}{b+\gamma}\mathrm{e}^{\gamma s}}\right]$$

$$A_Y(s,u) = \frac{\alpha\theta(\gamma - b)}{2a}s - \frac{\alpha\theta}{a}\ln\left[\frac{1 - \frac{b-\gamma}{b+\gamma}\mathrm{e}^{\gamma s}}{1 - \frac{b-\gamma}{b+\gamma}}\right]$$

where

$$a := \epsilon^2/2, \quad c := -iu(\sigma_j^I)^2/2 - (\sigma_j^I)^2 u^2/2,$$

$$b := iu\sigma_j^I \epsilon \rho_j^{I,V} - \alpha, \quad \gamma := \sqrt{b^2 - 4ac}.$$

Derivation of the characteristic function (cont'd)

The coefficients A_X and B_X:

$$B_X(\tau,u) = B_Y(\tau_j,u) + \frac{\bar\gamma - \bar b - 2\bar a B_Y(\tau_j,u)}{2\bar a}\left[\frac{1-\mathrm{e}^{\bar\gamma\tau}}{1-\frac{2\bar a B_Y(\tau_j,u)+\bar b-\bar\gamma}{2\bar a B_Y(\tau_j,u)+\bar b+\bar\gamma}\mathrm{e}^{\bar\gamma\tau}}\right]$$

$$A_X(\tau,u) = \frac{\alpha\theta(\bar\gamma-\bar b)}{2\bar a}\tau - \frac{\alpha\theta}{\bar a}\ln\left[\frac{1-\frac{2\bar a B_Y(\tau_j,u)+\bar b-\bar\gamma}{2\bar a B_Y(\tau_j,u)+\bar b+\bar\gamma}\mathrm{e}^{\bar\gamma\tau}}{1-\frac{2\bar a B_Y(\tau_j,u)+\bar b-\bar\gamma}{2\bar a B_Y(\tau_j,u)+\bar b+\bar\gamma}}\right]$$

where

$$\bar a := \epsilon^2/2, \qquad \bar b := iu\epsilon(\sigma^I_j\rho^{I,V}_j - \sigma^I_{j-1}\rho^{I,V}_{j-1}) - \alpha$$

$$\bar c := iu\big((\sigma^I_{j-1})^2 - (\sigma^I_j)^2\big)/2 - \big((\sigma^I_{j-1})^2 + (\sigma^I_j)^2 - 2\sigma^I_j\sigma^I_{j-1}\rho^I_{j,j-1}\big)u^2/2$$

$$\bar\gamma := \sqrt{\bar b^2 - 4\bar a\bar c}$$

Calibration to a matrix of II caps/floors

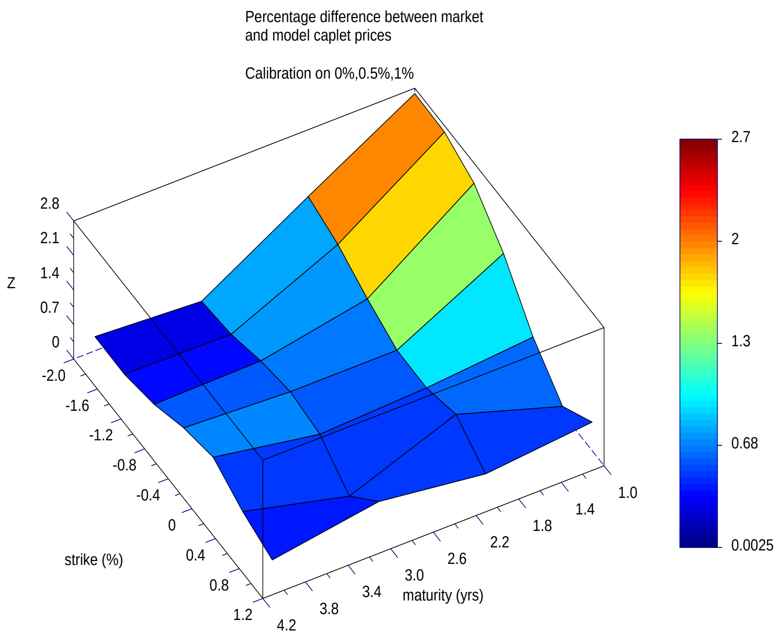


Figure 3: Absolute percentage differences between calibrated prices and market prices (market quotes as of October 7, 2004).