CAPITAL GROWTH THEORY UNDER TRANSACTION COSTS: AN APPROACH BASED ON THE VON NEUMANN-GALE MODEL

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In the paper by Dempster, Evstigneev and Taksar (Annals of Finance, 2006), it has been shown that the von Neumann-Gale growth model provides a convenient and natural framework for the analysis of questions of asset pricing and hedging under transaction costs. The talk will review recent results pertaining to a different area of applications of the model in Finance. It will demonstrate how methods and concepts developed in the context of von Neumann-Gale dynamical systems can be used to build a complete and self-consistent theory of optimal financial growth under transaction costs.
The term "von Neumann-Gale model" refers to a special class of multivalued (set-valued) dynamical systems. The classical theory (von Neumann 1937, Gale 1956, and others) deals with deterministic models and aims basically at economic applications.

First attempts to build stochastic generalizations of the von Neumann-Gale model were undertaken in the pioneering work of Dynkin, Radner and their research groups in the 1970s.

The initial attack on the problem left many questions unanswered. Significant progress was achieved only recently.

The progress was motivated and the new methods were suggested by the applications of the model in Finance.
Von Neumann-Gale dynamical systems: the deterministic case.

Given:

a closed convex cone $A \subseteq \mathbb{R}^n$;

for each $t = 1, 2, \ldots$, a set-valued mapping 

$$a \mapsto G_t(a), \quad a \in A, \quad \emptyset \neq G_t(a) \subseteq A,$$

satisfying the following condition:

the graph of the mapping $G_t(\cdot)$,

$$Z_t = \{(a, b) \in A \times A : b \in G_t(a)\}$$

is a closed convex cone.

Most of the classical deterministic theory has been developed for

$$A = \mathbb{R}^n_+.$$

This will be assumed throughout the talk.
Multivalued dynamical system defined by $G_t(\cdot)$.

A path (trajectory) $\{b_0, b_1, b_2, \ldots\}$:

$$b_t \in G_t(b_{t-1}), \ t = 1, 2, \ldots$$

or, equivalently,

$$(b_{t-1}, b_t) \in Z_t.$$ 

In economics contexts, states of the system $b_t = (b_t^1, \ldots, b_t^n) \geq 0$ are typically interpreted as commodity vectors. The process of economic growth is regarded as an evolution of $b_t$ in time. Elements $(a, b) \in Z_t$ are feasible input-output pairs, or technological processes (for the time period $t-1, t$). The sets $Z_t$ are termed technology sets.

In financial applications, which will be considered in detail later, vectors $b_t$ represent portfolios of assets, and the sets $Z_t$ describe self-financing (solvency) constraints.
Example: von Neumann (1937) model.
The cone $Z_t$ is polyhedral: there is a finite set of basic technological processes

$$(a^{(1)}, b^{(1)}), ..., (a^{(m)}, b^{(m)})$$

and

$$(a, b) \in Z_t \iff (a, b) = \sum_{j=1}^{m} d^j (a^{(j)}, b^{(j)})$$

where

$$d^1 \geq 0, ..., d^m \geq 0$$

are intensities of operating the technological processes

$$(a^{(1)}, b^{(1)}), ..., (a^{(m)}, b^{(m)})$$

Gale (1956): general, not necessarily polyhedral, cones.
Stochastic von Neumann–Gale dynamical systems

Given: \((\Omega, F, P)\) probability space;

\(\ldots \subseteq F_{-1} \subseteq F_0 \subseteq F_1 \subseteq \ldots \subseteq F_t \subseteq \ldots \subseteq F\) filtration.

For each \(t = 1, 2, \ldots\), let

\[ (\omega, a) \mapsto G_t(\omega, a) \subseteq \mathbb{R}_+^n \]

be a set-valued mapping assigning to each \(\omega \in \Omega\) and \(a \in \mathbb{R}_+^n\) a set \(G_t(\omega, a) \subseteq \mathbb{R}_+^n\) so that

(i) for each \(\omega\), the graph

\[ Z_t(\omega) := \{(a, b) : b \in G_t(\omega, a)\} \]

of the mapping \(G_t(\omega, \cdot)\) is a closed convex cone (transition cone);

(ii) the set-valued mapping \((\omega, a) \mapsto G_t(\omega, a)\) is \(F_t \times B(\mathbb{R}_+^n)\)-measurable.

Random (multivalued) dynamical system:

Paths (trajectories)

\[ \{y_0(\omega), y_1(\omega), \ldots\} \]

\[ y_t(\omega) \in G_t(\omega, y_{t-1}(\omega)) \text{ (a.s.), } t = 1, 2, \ldots, \]

or, equivalently,

\[ (y_{t-1}(\omega), y_t(\omega)) \in Z_t(\omega) \text{ (a.s.)} \]

and

\[ y_t \in L_+^\infty(\Omega, F_t, P, \mathbb{R}_+^n). \]
Dynamic securities market model.

Given \((Ω, F, P), \{F_t\}\). There are \(n\) assets traded on the market at dates \(t = 0, 1, \ldots\). A (contingent) portfolio of assets held by an investor at date \(t\) is a vector

\[
y_t(ω) = (y^1_t(ω), \ldots, y^n_t(ω))
\]

where \(y^i_t(ω)\) is the amount of money invested in asset \(i\) (the value of asset \(i\) in the portfolio in terms of the current market prices). It is supposed that \(y_t(ω)\) is \(F_t\)-measurable. In the applications we will deal with (capital growth theory), the classical model excludes short selling, and so the vectors \(y_t\) are supposed to be non-negative.

A trading strategy is a sequence of contingent portfolios \(y_0, y_1, y_2, \ldots\).

We will focus on the analysis of self-financing strategies. They are defined as follows. In the model, we are given sets \(Z_t(ω) ⊆ R^n_+ \times R^n_+\) (defining the self-financing constraints), and a strategy \(y_0, y_1, y_2, \ldots\) is called self-financing if

\[
(y_{t-1}(ω), y_t(ω)) \in Z_t(ω) \text{ (a.s.) for all } t.
\]

It is assumed that \(Z_t(ω)\) is a closed convex cone depending \(F_t\)-measurably on \(ω\). This assumption means that the model takes into account proportional transaction costs.

The cones \(Z_t(ω)\) define a stochastic von Neumann-Gale model. Trading strategies are paths in this model.
Examples

No transaction costs. Let $S_t(\omega) = (S^1_t(\omega), ..., S^n_t(\omega))$ be the vector of asset prices at time $t$ ($F_t$-measurable). Define

$$Z_t(\omega) := \{(a, b) \in R^n_+ \times R^n_+: \sum_{i=1}^n b^i \leq \sum_{i=1}^n \frac{S^i_t(\omega)}{S^i_{t-1}(\omega)} a^i\}.$$ 

A portfolio $a$ can be rebalanced to a portfolio $b$ (without transaction costs) if and only if $(a, b) \in Z_t(\omega)$.

Proportional transaction costs: single currency. Let $Z_t(\omega)$ be the set of those $(a, b) \in R^n_+ \times R^n_+$ for which

$$\sum_{i=1}^n (1 + \lambda^+_{t,i}) (b^i - \frac{S^i_t}{S^i_{t-1}} a^i)_+ \leq \sum_{i=1}^n (1 - \lambda^-_{t,i}) (\frac{S^i_t}{S^i_{t-1}} a^i - b^i)_+,$$

where

$$a_+ := \max\{a, 0\}.$$ 

The transaction cost rates for buying and selling are given by the numbers $\lambda^+_{t,i}(\omega) \geq 0$ and $1 > \lambda^-_{t,i}(\omega) \geq 0$, respectively. A portfolio $a$ can be rebalanced to a portfolio $b$ (with transaction costs) if and only if $(a, b) \in Z_t(\omega)$. The above inequality expresses the fact that purchases of assets are made only at the expense of sales of other assets.
Proportional transaction costs: several currencies.

There are $n$ currencies.

A matrix

$$
\mu_t^{ij}(\omega) \text{ with } \mu_t^{ij} > 0 \text{ and } \mu_t^{ii} = 1
$$

is given, specifying the exchange rates of the currencies $i = 1, 2, ..., n$ (including transaction costs).

A portfolio $a \geq 0$ of currencies can be exchanged to a portfolio $b$ at date $t$ if and only if there exists a nonnegative matrix $(d_t^{ij})$ (exchange matrix) such that

$$
a^i \geq \sum_{j=1}^{n} d_t^{ji}, \quad 0 \leq b^i \leq \sum_{j=1}^{n} \mu_t^{ij}(\omega)d_t^{ij}.
$$

Here, $d_t^{ij}$ ($i \neq j$) stands for the amount of currency $j$ exchanged into currency $i$. The amount $d_t^{ii}$ of currency $i$ is left unexchanged. The second inequality says that, at time $t$, the $i$th position of the portfolio cannot be greater than the sum $\sum_{j=1}^{n} \mu_t^{ij}d_t^{ij}$ obtained as a result of the exchange.

A version of the models considered by Kabanov, Stricker and others.
Stationary models. This is an important class of models in which the random cone-valued process $Z_t(\omega)$ is stationary. Formally, in a stationary model we are given in addition to the above data a measure preserving one-to-one transformation $T$ of the probability space $(\Omega, F, P)$ (time shift) such that

(a) the filtration $\cdots \subseteq F_{-1} \subseteq F_0 \subseteq F_1 \subseteq \cdots \subseteq F_t \subseteq \cdots$ is invariant with respect to the time shift

$$T^{-1}F_t = F_{t+1},$$

(b) for each $t$,

$$Z_t(T\omega) = Z_{t+1}(\omega).$$

The last condition means that the probabilistic structure of the transition cones is invariant with respect to the time shift (stationarity of the cone-valued process $Z_t$).

Examples. In the above examples, the transition cones form stationary processes if the vector-valued processes of asset returns

$$R_t(\omega) = \left( \frac{S_t^1(\omega)}{S_{t-1}^1(\omega)}, \ldots, \frac{S_t^n(\omega)}{S_{t-1}^n(\omega)} \right),$$

or the matrix-valued processes

$$M_t(\omega) = (\mu_t^{ij}(\omega))_{i,j=1}^n$$

of the currencies’ exchange rates are stationary.
How to invest in order to maximize the asymptotic growth rate of wealth?


How to define asymptotic optimality? In the definition below, we follow essentially Algoet and Cover (1988).

For a vector \( b = (b^1, ..., b^n) \), put \( |b| = |b^1| + ... + |b^n| \). If \( b \geq 0 \), then \( |b| = b^1 + ... + b^n \), and if \( b \geq 0 \) represents a portfolio, then \( |b| \) is the value of this portfolio—the total amount of money invested in all its assets.

**Definition.** Let \( y_0, y_1, ... \) be an investment strategy. It is called asymptotically optimal if for any other investment strategy \( y'_0, y'_1, ... \) there exists a supermartingale \( \xi_t \) such that

\[
\frac{|y'_t|}{|y_t|} \leq \xi_t, \ t = 0, 1, ... \ (a.s.)
\]

Note that the above property remains valid if \(| \cdot |\) is replaced by any (possibly random) function \( \psi(\cdot) \), where \( c|a| \leq \psi(a) \leq C|a| \), where \( 0 < c < C \) are non-random constants.
Implications of asymptotic optimality. The strength of the above definition, which might seem not immediately intuitive, is illustrated by the following implications of asymptotic optimality. As long as

$$\frac{|y'_t|}{|y_t|} \leq \xi_t, \quad t = 0, 1, \ldots \quad \text{(a.s.),}$$

where $\xi_t$ is a supermartingale, the following properties hold.

(a) With probability one

$$\sup_t \frac{|y'_t|}{|y_t|} < \infty,$$

i.e. for no strategy wealth can grow asymptotically faster than for $y_0, y_1, \ldots \quad \text{(a.s.).}$

(b) The strategy $y_0, y_1, \ldots$ a.s. maximizes the exponential growth rate of wealth

$$\limsup_{t \to \infty} \frac{1}{t} \ln |y_t|.$$

(c) We have

$$\sup_t E \frac{|y'_t|}{|y_t|} < \infty \quad \text{and} \quad \sup_t E \ln \frac{|y'_t|}{|y_t|} < \infty.$$
This work aims at obtaining results on asymptotic optimality in models with transaction costs.

**The main results:**

- existence of asymptotically optimal strategies in general (non-stationary) models;
- existence of asymptotically optimal strategies of a special structure (so-called balanced strategies) in stationary models.

From now on we will use the terms ”trading strategy” in the dynamic securities market model and ”path” in the associated stochastic von Neumann-Gale model interchangeably.

**Relevant studies.**


The idea of our approach. Our main tool for analysing the questions of asymptotic optimality is the concept of a rapid path in the stochastic von Neumann-Gale model.

**Rapid paths.** A path \( \{x_0, x_1, \ldots\} \) (finite or infinite) is called rapid if there is a sequence of random vectors \( \{p_0, p_1, \ldots\}, p_t \in L^1(\Omega, F_t, P, \mathbb{R}^n) \), such that

\[
p_t x_t = 1 \text{ (a.s.)} \quad (1)
\]

and any of the four equivalent conditions holds:

for all \((x, y) \in Z_t \) (a.s.),

\[
E \ln \frac{p_t y}{p_{t-1} x} \leq E \ln \frac{p_t x_t}{p_{t-1} x_{t-1}} = 0, \quad (2)
\]

\[
E \frac{p_t y}{p_{t-1} x} \leq E \frac{p_t x_t}{p_{t-1} x_{t-1}} = 1, \quad (3)
\]

\[
Ep_t y \leq Ep_{t-1} x, \quad (4)
\]

\[
E(p_t y | F_{t-1}) \leq p_{t-1} x. \quad (5)
\]

In the financial applications, the notion of a rapid path is a counterpart of the notion of a numeraire portfolio (Long 1990). Note that for any path \( \{y_0, y_1, \ldots\} \), the sequence \( p_0 y_0, p_1 y_1, p_2 y_2, \ldots \) is a supermartingale. The proofs of the main results regarding asymptotic optimality are based on the following fact. Under quite general assumptions,

an infinite rapid path is asymptotically optimal.
Existence results for rapid paths (implying the existence results for asymptotically optimal paths) are based on the following assumptions:

**Assumptions.** The transition cones $Z_t(\omega)$ satisfy:

(A1) The set $Z_t(\omega)$ contains with every $(a, b)$ all $(a', b')$ such that $a' \geq a$ and $0 \leq b' \leq b$.

(A2) There exists a constant $M$ such that $|b| \leq M|a|$ for all $(a, b) \in Z_t(\omega)$.

(A3) There exists a constant $\gamma > 0$ such that $(e, \gamma e) \in Z_t(\omega)$, where $e = (1, 1, ..., 1)$.

(A4) There exists an integer $l \geq 1$ such that for every $t \geq 0$ and $i = 1, ..., n$ there is a path $y_{t,i}, ..., y_{t+l,i}$ satisfying

\[ y_{t,i} = e_i, ..., y_{t+l,i} \geq \gamma e, \]

where $e_i = (0, 0, ..., 1, ..., 0)$ (all the coordinates are 0 except the $i$th which is 1).
Existence results for non-stationary models.

Finite rapid paths: Evstigneev and Flåm (1998) – obtained by the maximization of logarithmic functionals:

$$E \ln |x_N| \to \max$$

over all paths

$$x_0, ..., x_N$$

with fixed $$x_0$$ and $$N$$.

Infinite rapid paths: Bahsoun, Evstigneev and Taksar (2007) – obtained by passing to the limit from finite rapid paths with the help of the Fatou lemma in several dimensions (a conditional version of it).

Lemma. Let $$G$$ be a sub-$$\sigma$$-algebra of $$F$$. If $$w_N(\omega)$$, $$\omega \in \Omega, N = 1, 2, ..., are random vectors with values in $$R^m_+$$ and the conditional expectations $$E[w_N(\omega)|G]$$ are finite and converge a.s. to a random vector $$z(\omega)$$, then exists a sequence of integer-valued random variables

$$1 < N_1(\omega) < N_2(\omega) < ...$$

and a random vector $$w(\omega)$$ such that

$$\lim w_{N_k(\omega)}(\omega) = w(\omega) \text{ (a.s.) and}$$

$$E[w(\omega)|G] \leq z(\omega) \text{ (a.s.)}.$$ 

This result is a conditional version of the multidimensional Fatou lemma (Schmeidler 1970).
Existence results for rapid paths in stationary models.

Recall that in stationary models the random cones $Z_t(\omega)$ form a stationary process:

$$Z_t(\omega) = Z(T^t\omega).$$

In such models an important role is played by the notion of a balanced path.

**Balanced paths** are paths of the form

$$y_t = \lambda_1 \lambda_2 \ldots \lambda_t \tilde{y}_t,$$

where

$$\lambda_t(\omega) = \lambda(T^{t-1}\omega), \quad \tilde{y}_t(\omega) = y(T^t\omega).$$

The scalar function $\lambda(\omega) > 0$ is supposed to be $F_1$-measurable and bounded. The vector function $y(\omega) \geq 0$ is $F_0$-measurable and satisfies $|y(\omega)| = 1$.

A balanced path grows with stationary proportions (given by the stationary vector process $\tilde{y}_t(\omega)$) and at a stationary rate, growth rates at times $t = 1, 2, \ldots$ being $\lambda_1, \lambda_2, \ldots$. 
**Von Neumann path**: balanced path maximizing 
\[ E \log \lambda_t \]
among all balanced paths
\[ y_t = \lambda_1 \lambda_2 ... \lambda_t \bar{y}_t. \]
By virtue of stationarity, the functional \( E \log \lambda_t \) does not depend on \( t \).

To find a von Neumann path, we have to solve the following variational problem

\[
\text{maximize } E \ln \lambda
\]

over all
\[ \lambda \in L_+^\infty(\Omega, F_1, P, R^1), \quad y \in L_+^\infty(\Omega, F_0, P, R^n) \]
satisfying
\[
(y(\omega), \lambda(\omega)y(T\omega)) \in Z_1(\omega) \text{ (a.s.)},
\]
\[ |y(\omega)| = 1 \text{ (a.s.)}. \]
The main result on rapid paths for stationary models.

**Theorem.** Under assumptions (A1) - (A4), a von Neumann path exists and it is rapid.

By virtue of the theorem, the von Neumann path – which is the best among all balanced ones – turns out to be asymptotically optimal **in the class of all** (not necessarily balanced) paths.

This result gives the positive answer to a problem posed by Eugene Dynkin in the early 1970s.

The strategy of the proof is classical and goes back to Dvoretzky, Wald and Wolfowitz (1950). It is based on the idea of ”elimination of randomization”. We first construct an appropriate extension of the underlying probability space, using an auxiliary stochastic process serving as an additional source of randomness. Then, using some subtle properties of convexity, we eliminate randomization and establish the existence of a von Neumann path in the original system.