Introduction to the theory of L_p-spaces

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Chapter 1

Preliminaries

The book is dedicated to the study of L_p-spaces where $p > 0$, consisting of functions f for which the function $|f|^p$ is Lebesgue integrable, and some other related function spaces, which are normed or more general spaces consisting of real- or complex-valued functions of n real variables ($n \in \mathbb{N}$).*

In general, a *function space* is a set of functions with a common domain. It will always be assumed that this space is a linear (vector) space with respect to point-wise addition and multiplication by a scalar.

We start with recalling some notions and facts in linear functional analysis, which will be required in the sequel. Then we shall speak about spaces of continuous, uniformly continuous, l times continuously differentiable and infinitely many times continuously differentiable functions. In the last, main, section of this chapter a brief exposition of the theory of the Lebesgue integration will be given, with emphasis on the properties related to passing to the limit, containing all tools which will be used in the book.

1.1 Normed, semi-normed, quasi-normed spaces

A linear (vector) space with respect to multiplication by complex numbers is called a *normed space* if for all $x \in X$ a real number † $\|x\| \equiv \|x\|_X$, the *norm* of x, is defined, and the following properties are satisfied:

1) for all $x \in X \qquad \|x\| \geqslant 0,$

2) if $\|x\| = 0$ then $x = \theta$, where θ is the null element of X,

3) for all $x \in X$ and for all $a \in \mathbb{C} \qquad \|ax\| = |a| \cdot \|x\|,$

4) (the *triangle inequality*) for all $x, y \in X$

$$\|x + y\| \leqslant \|x\| + \|y\| .$$

* $\mathbb{N}$ is the set of all natural numbers, $\mathbb{R} \equiv \mathbb{R}^1$ the set of all real numbers, $\mathbb{R}^n$ ($n \in \mathbb{N}$) the set of all $x = (x_1, \dots, x_n)$ where $x_1, \dots, x_n \in \mathbb{R}$, $\mathbb{C}$ is the set of all complex numbers. It will always be assumed that $n \in \mathbb{N}$. Moreover, the letter n will be only used to denote the dimension.

† The symbol $\equiv$ means identity, i.e. the notation $a \equiv b$ means that the symbols a and b have the same meaning. We shall also use the symbol $:=$ which denotes the assignment operation, i.e. $a := b$ means that a is assigned the value of b.

Note that properties 3 and 4 imply that

$$\|x-y\| \geqslant \big|\|x\| - \|y\|\big|$$

(the *reverse triangle inequality*).

If properties 1, 3 and 4 are satisfied, then X is called a *semi-normed space* and $\|x\|$ a *semi-norm*. For semi-normed spaces that are not normed, the set $\tilde{\theta} = \{x \in X\colon \|x\| = 0\}$ contains, apart from θ, other elements.

Exercise 1.1.1. Prove that $\tilde{\theta}$ is a linear space.

If X is a semi-normed space, but not a normed space, then it can be partitioned into non-intersecting subsets $\tilde{x}$ of X (classes) satisfying the following condition: if $x_1 \in \tilde{x}$, then $x_2 \in \tilde{x}$ if and only if $x_1 - x_2 \in \tilde{\theta}$. Elements of the same class are called *equivalent*, and the classes $\tilde{x}$ are called *equivalence classes*. The set of all classes $\tilde{x}$, which is the factor space $X/_{\tilde{\theta}}$ of the space X with respect to $\tilde{\theta}$, will be denoted by $\widetilde{X}$.

The set $\widetilde{X}$ becomes a linear space if we define, in a natural way, the operations of addition of classes and of multiplication of a class by a scalar: $\tilde{x} + \tilde{y} := \widetilde{x+y}$, $a\tilde{x} := \widetilde{ax}$, where x and y are any elements in $\tilde{x}$, $\tilde{y}$ respectively, $a \in \mathbb{C}$ and $\widetilde{x+y}$ and $\widetilde{ax}$ are the classes which contain $x+y$, ax respectively. Moreover, $\widetilde{X}$ becomes a normed space if we define the norm $\|x\|$ of a class $\tilde{x}$ by

$$\|\tilde{x}\| \equiv \|\tilde{x}\|_{\widetilde{X}} := \|x\|_X \,,$$

where x is any element in $\tilde{x}$.

Exercise 1.1.2. Verify that, for different $x_1 \in \tilde{x}$ and $x_2 \in \tilde{x}$, the equality $\|x_1\|_X = \|x_2\|_X$ holds, hence the above definition makes sense, and that $\|\tilde{x}\|$ satisfies properties 1–4.

If in the definition of a normed space property 4 is replaced by

4′) there exists* $c \geqslant 1$ such that for all $x, y \in X$

$$\|x+y\| \leqslant c\,(\|x\| + \|y\|)\,,$$

then spaces satisfying conditions 1, 2, 3 and 4′ are called *quasi-normed* and $\|x\|$ a *quasi-norm*. If a space satisfies properties 1, 3 and 4′, it is called a *semiquasi-normed space*.

A sequence $\{x_k\}_{k\in\mathbb{N}}$ of elements $x_k \in X$ is said to *converge* to an element $x \in X$ in a normed space X if

$$\lim_{k\to\infty} \|x_k - x\| = 0. \tag{1.1.1}$$

A normed space X is called *complete* if, for any *Cauchy sequence* $\{x_k\}_{k\in\mathbb{N}}$ of elements $x_k \in X$, i.e. a sequence satisfying

$$\lim_{k,l\to\infty} \|x_k - x_l\| = 0,$$

there exists $x \in X$ such that $\lim\limits_{k\to\infty} x_k = x$ in X.

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* Here and in the sequel, given an inequality, it is assumed that its entries are real numbers, e. g. 'there exists $c \geqslant 1$' means 'there exists a real number $c \geqslant 1$'.

Convergence and completeness for other types of spaces considered above is defined similarly.

A complete normed space is called a *Banach space*, a complete semi-normed space a *semi-Banach space* etc.

The completeness property of real numbers implies that $\mathbb{R}^n$ and $\mathbb{C}^n$ with the standard Euclidean norm are Banach spaces.

A linear operator $T\colon X \to Y$ acting from a semi-quasi-normed space X to a semi-quasi-normed space Y is called *bounded* if there exist $c \geqslant 0$ such that for all $x \in X$

$$\|Tx\|_Y \leqslant c\,\|x\|_X \,.$$

The minimal value of $c \geqslant 0$, in other words, the *best*, or the *sharp*, *constant*, in this inequality is equal to

$$\|T\|_{X\to Y} := \sup_{x\in X,\,\|x\|_X\neq 0} \frac{\|Tx\|_Y}{\|x\|_X} = \sup_{x\in X,\,\|x\|_X=1} \|Tx\|_Y \,. \tag{1.1.2}$$

(We, naturally, assume that $\|x\|_X > 0$ for some $x \in X$. However, if one admits the trivial case in which $\|x\|_X = 0$ for all $x \in X$ which is not excluded by the above definitions, then one should set $\|T\|_{X\to Y} := 0$ in this case.) In general, this quantity is a semi-quasi-norm on the set of all bounded linear operators $T\colon X \to Y$. It is a semi-norm if Y is a semi-normed space, a quasi-norm if Y is a quasi-normed space, and a norm if Y is a normed space. An operator $T\colon X \to Y$ acting from a semi-quasi-normed space X to a semi-quasi-normed space Y is called *continuous* if $Tx_k \to Tx$ in Y for all $x \in X$ and all sequences $\{x_k\}_{k\in\mathbb{N}}$ satisfying $x_k \to x$ in X. Clearly, the boundedness of T implies its continuity. For a linear operator $T\colon X \to Y$ the continuity is equivalent to the boundedness.

Exercise 1.1.3. (Continuity of a semi-norm) Prove that if X is a semi-normed space and $\lim\limits_{k\to\infty} x_k = x$ in X, then $\lim\limits_{k\to\infty} \|x_k\| = \|x\|$.

A linear space X with multiplication by complex numbers is called an *inner-product space* (also known as a *pre-Hilbert space*), if for all $x, y \in X$ a complex number (x, y), the *inner product* of x and y, is defined, and the following properties are satisfied:

1) for all $x \in X$ $\qquad (x,x) \geqslant 0$,

2) if $(x,x) = 0$, then $x = \theta$,

3) for all $x, y \in X$ $\qquad (x,y) = \overline{(y,x)}$,

4) for all $x_1, x_2, y \in X$ and for all $c_1, c_2 \in \mathbb{C}$

$$(c_1x_1 + c_2x_2, y) = c_1\,(x_1,y) + c_2\,(x_2,y)\,.$$

Properties 3 and 4 imply that

$$(y, c_1x_1 + c_2x_2) = \bar{c}_1\,(y,x_1) + \bar{c}_2\,(y,x_2)\,.$$

Elements $x, y \in X$ are said to be *orthogonal*, shorthand notation: $x \perp y$, if $(x,y) = 0$. If $x \perp y$, then

$$\|x+y\|^2 = \|x\|^2 + \|y\|^2$$

where

$$\|x\| := \sqrt{(x,x)} \tag{1.1.3}$$

(the *Pythagoras theorem*). Given $x, y \in X$, $y \neq \theta$, the element $\frac{(x,y)}{\|y\|^2} y$ is the orthogonal projection of x onto y ($\Longleftrightarrow x - \frac{(x,y)}{\|y\|^2} y \perp y$).

Exercise 1.1.4. Let X be an inner product space, $x, y \in X$ and $y \neq \theta$. By applying the Pythagoras theorem, prove that

$$\left\|x - \frac{(x,y)}{\|y\|^2} y\right\|^2 = \|x\|^2 - \frac{|(x,y)|^2}{\|y\|^2}.$$

Hence deduce that for all $x, y \in X$

$$|(x,y)| \leqslant \|x\| \cdot \|y\| \tag{1.1.4}$$

(the *Cauchy—Bunyakovskiĭ inequality*). Deduce also that equality holds if and only if x and y are proportional.

The quantity $\|x\|$ defined above is a norm, and with this norm X becomes a normed space.

If properties 1, 3 and 4 are satisfied, then X is called a *semi-inner-product space*. The Cauchy—Bunyakovskiĭ inequality (1.1.4) also holds for semi-inner-product spaces X.

For a semi-inner-product space X the notation $\lim\limits_{k\to\infty} x_k = x$ in X, where $x \in X$ and for all $k \in \mathbb{N}$ $x_k \in X$ means that equality (1.1.1) holds where $\|\cdot\|$ is defined by (1.1.3).

Exercise 1.1.5. (Continuity of a semi-inner product) Prove that if X is a semi-inner-product space and $\lim\limits_{k\to\infty} x_k = x$, $\lim\limits_{k\to\infty} y_k = y$ in X, then $\lim\limits_{k\to\infty} (x_k, y_k) = (x, y)$.

A normed space X can be made an inner-product space, i.e. it is possible to define an inner product (x,y) on $X \times X$ satisfying $\|x\| = \sqrt{(x,x)}$, if and only if for all $x, y \in X$ the *parallelogram identity*

$$\|x+y\|^2 + \|x-y\|^2 = 2\left(\|x\|^2 + \|y\|^2\right)$$

is satisfied (the *Jordan—Neumann theorem*).

If the parallelogram identity is satisfied, then the inner product (x,y) on $X \times X$ satisfying $\|x\| = \sqrt{(x,x)}$ is defined uniquely by

$$(x,y) := \left\|\frac{x+y}{2}\right\|^2 - \left\|\frac{x-y}{2}\right\|^2 + i\left(\left\|\frac{x+iy}{2}\right\|^2 + \left\|\frac{x-iy}{2}\right\|^2\right).$$

Let X be the space of all vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$, more precisely the space of all classes of vectors. (Each class contains all vectors which are parallel and have the same length and direction.) Then the meaning of the parallelogram identity is that the sum of the squares of lengths of the diagonals of a parallelogram is equal to the sum of the squares of lengths of all its sides, one of the properties of a parallelogram in the planimetry. Thus the statement above means, in particular that the geometry of a normed space, which is not an inner-product space, differs from the standard, i.e. Euclidean geometry. However, some of the properties of the Euclidean geometry are preserved. Say, all balls $B(x,r) = \{y \in X : \|y - x\| < r\}$ where $x \in X, r > 0$ in a normed space X are convex (though not necessarily strictly convex). If X is a semi-normed space, which is not a normed space, this property may not hold.

A complete inner-product space is called a *Hilbert space*. (Sometimes an additional requirement on X is imposed that X is infinite-dimensional.)

For a space X of any of the aforementioned types, the *conjugate space* X' is the space of all continuous linear functionals $l\colon X \to \mathbb{C}$. For $l \in X'$ the quantity

$$\|l\| \equiv \|l\|_{X'} := \sup_{x\in X,\, \|x\|\neq 0} \frac{|l(x)|}{\|x\|} = \sup_{x\in X,\, \|x\|=1} |l(x)| \tag{1.1.5}$$

is a norm. Moreover, for all spaces X under consideration, X' is a Banach space.

A sequence $\{x_k\}_{k\in\mathbb{N}}$ of elements $x_k \in X$ is said to *converge* to an element $x \in X$ *weakly* if for all functionals $l \in X'$

$$\lim_{k\to\infty} l(x_k) = l(x).$$

Given a semi-inner product space X and an element $y \in X$, one can define a functional l by setting for all $x \in X$

$$l(x) := (x, y)\,.$$

By the properties of a semi-inner product l is a continuous linear functional on X and

$$\|l\|_{X'} = \|y\|_X\,.$$

Conversely, if X is a semi-Hilbert space, then for all functionals $l \in X'$ there exists an element $y \in X$ such that for all $x \in X$

$$l(x) = (x, y)$$

(the *Riesz representation theorem*).

One can take $y = \frac{\overline{l(z)}}{\|z\|^2}\, z$, where z is any element in X orthogonal to the set $\{x \in X\colon l(x) = 0\}$, the kernel of l, with $\|z\| \neq 0$. If X is a Hilbert space, then y is defined uniquely.

Given a semi-normed space X and an element $x \in X$, there exists $l \in X'$ such that $\|l\| = 1$ and $l(x) = \|x\|$. (This is a corollary of the Khan—Banach theorem on the extension of linear functionals.)

Exercise 1.1.6. By applying this statement prove that if a sequence $\{x_k\}_{k\in\mathbb{N}}$ of elements $x_k \in X$ converges weakly to an element $x \in X$, then

$$\|x\| \leqslant \liminf_{k\to\infty} \|x_k\|\,.$$

Finally, we recall a typical example showing that, in general, equality in this inequality does not hold. Assume that X is an inner-product space and $\{x_k\}_{k\in\mathbb{N}}$ is an orthonormal sequence of elements $x_k \in X$, i.e. $(x_k, x_l) = 0$ for $k \neq l$ and $\|x_k\| = 1$. Since for all $y \in X$ the Fourier coefficients $(y, x_k) \to 0$ as $k \to \infty$, by the Riesz representation theorem it follows that x_k converges weakly to θ as $k \to \infty$. So, $\|\theta\| = 0 < \lim\limits_{k\to\infty} \|x_k\| = 1$. (This also implies that x_k does not converge to θ in X.)

Exercise 1.1.7. Let X be a semi-inner-product space, a sequence $\{x_k\}_{k\in\mathbb{N}}$ of elements $x_k \in X$ converges weakly to an element $x \in X$ and $\lim\limits_{k\to\infty} \|x_k\| = \|x\|$, then $\lim\limits_{k\to\infty} x_k = x$ in X.

Exposition of the theory of normed and more general spaces considered above, in particular, the proofs of all statements in this section, can be found in standard courses on functional analysis, for example, in [4], [6].

1.2 Spaces of continuous and differentiable functions

Definition 1.2.1. *Let* $\Omega \subset \mathbb{R}^n$. We say that $f \in C(\Omega)$, i.e. f is* continuous *on Ω, if for all $x \in \Omega$ the function f is* continuous *at x* with respect to Ω, *i.e. for all $\varepsilon > 0$ there exists $\delta > 0$ such that for all $y \in \Omega$ satisfying $|x-y| < \delta$ the inequality $|f(x) - f(y)| < \varepsilon$ holds.*

Note that if x is an isolated point of Ω, then any function f is continuous at x. If $x \in \Omega$ is a limit point of Ω, then the continuity of f at x means that $\lim\limits_{y\to x, y\in\Omega} f(y) = f(x)$.

Definition 1.2.2. *Let $\Omega \subset \mathbb{R}^n$. We say that $f \in \overline{C}(\Omega)$ if* 1) *f is bounded on Ω, and* 2) *f is* uniformly continuous *on Ω, i.e., for all $\varepsilon > 0$ there exists $\delta > 0$ such that for all $x, y \in \Omega$ satisfying $|x-y| < \delta$ the inequality $|f(x) - f(y)| < \varepsilon$ holds.*

For a function f defined on a set $\Omega \subset \mathbb{R}^n$ the *modulus of continuity* ω_f of f is defined by: for all $\delta \geqslant 0$

$$\omega_f(\delta) \equiv \omega_{f,\Omega}(\delta) := \sup_{\substack{x,y\in\Omega \\ |x-y|\leqslant\delta}} |f(x) - f(y)|.$$

The function ω_f is non-negative and non-decreasing. If Ω is bounded, then for all $\delta \geqslant \operatorname{diam}\Omega$ $\omega_f(\delta) = \omega_f(\operatorname{diam}\Omega)$.

A function f is uniformly continuous on Ω if and only if $\lim\limits_{\delta\to 0+} \omega_f(\delta) = 0$.

If Ω is a compact and $f \in C(\Omega)$, then 1) f is bounded on Ω, 2) there exist $x_1, x_2 \in \Omega$ such that $f(x_1) = \inf\limits_{x\in\Omega} f(x)$, $f(x_2) = \sup\limits_{x\in\Omega} f(x)$ and 3) f is uniformly continuous on Ω.

So, in general, $\overline{C}(\Omega) \subset C(\Omega)$, but $\overline{C}(\Omega) = C(\Omega)$ if Ω is a compact.

If f is uniformly continuous on Ω and $x \in \partial\Omega$ is a limit point of Ω, then there exists (a finite) $\lim\limits_{y\to x, y\in\Omega} f(y)$, which equals to $f(x)$ if $x \in \Omega$. Therefore, a function f uniformly continuous on Ω may be extended to the boundary in such a way that the extended function is uniformly continuous on the closure $\overline{\Omega}$ of Ω.

Hence, if Ω is bounded, then $f \in \overline{C}(\Omega)$ if and only if there exists $F \in C(\overline{\Omega})$ such that for all $x \in \Omega$ $F(x) = f(x)$. Since any function continuous on a compact may be extended to a function continuous on $\mathbb{R}^n$, functions $f \in \overline{C}(\Omega)$ are restrictions to Ω of functions in $C(\mathbb{R}^n)$:

$$\overline{C}(\Omega) = C(\mathbb{R}^n)\big|_{\Omega}. \tag{1.2.1}$$

* Speaking about sets in $\mathbb{R}^n$ we always assume, without reserve, that they are non-empty.

For any non-empty set $\Omega \subset \mathbb{R}^n$, the quantity

$$\|f\|_{C(\Omega)} := \sup_{x\in\Omega} |f(x)| \tag{1.2.2}$$

is a norm on $\overline{C}(\Omega)$. If Ω is a compact, one can write

$$\|f\|_{C(\Omega)} := \max_{x\in\Omega} |f(x)|.$$

The space $\overline{C}(\Omega)$ with norm (1.2.2) is a Banach space.

Note that $\|f\|_{C(\Omega)}$ makes sense for all functions f defined on Ω, and $\|f\|_{C(\Omega)} < \infty$ means that f is bounded on Ω.

For a sequence $\{f_k\}_{k\in\mathbb{N}}$ of functions f_k and a function f defined on Ω, the equality

$$\lim_{k\to\infty} \|f_k - f\|_{C(\Omega)} = 0$$

means that f_k *converge uniformly* to f on Ω as $k \to \infty$, i.e. for all $\varepsilon > 0$ there exists $m \in \mathbb{N}$ such that for all $k \in \mathbb{N}, k \geqslant m$ and for all $x \in \Omega$ the inequality $|f_k(x) - f(x)| < \varepsilon$ holds.

If Ω is an open set, one may consider a sequence $\{\Omega_m\}_{m\in\mathbb{N}}$ of compacts $\Omega_m \subset \Omega$ satisfying, for all $m \in \mathbb{N}$ $\Omega_m \subset \Omega_{m+1}$, and $\bigcup_{m=1}^{\infty} \Omega_m = \Omega$. Equipped with a countable family of norms $\|f\|_m = \|f\|_{C(\Omega_m)}$, the space $C(\Omega)$ becomes a complete *countably normed* space. A sequence $\{f_k\}_{k\in\mathbb{N}}$ of functions f_k defined on Ω converges to a function f defined on Ω in $C(\Omega)$ if for all $m \in \mathbb{N}$ $\lim_{k\to\infty} \|f_k - f\|_m = 0$.

For all $k \in \mathbb{N}$, let functions f_k be defined on Ω and let x_0 be a limit point of Ω. If for all $k \in \mathbb{N}$ there exist (finite) limits $\lim_{x\to x_0, x\in\Omega} f_k(x)$ and for some $\delta > 0$ f_k converge uniformly on $\Omega \cap B(x_0, \delta)$, then

$$\lim_{\substack{x\to x_0 \\ x\in\Omega}} \lim_{k\to\infty} f_k(x) = \lim_{k\to\infty} \lim_{\substack{x\to x_0 \\ x\in\Omega}} f_k(x).$$

If all f_k are continuous or uniformly continuous on Ω and f_k converge uniformly on Ω to a function f as $k \to \infty$, then f is continuous on Ω, uniformly continuous respectively.

If Ω is an open set, for all $k \in \mathbb{N}$ $f_k \in C(\Omega)$ and f_k converge uniformly to f on every compact $K \subset \Omega$, then $f \in C(\Omega)$.

If for all $k \in \mathbb{N}$ f_k are Riemann integrable on $[a, b]$, where $-\infty < a < b < \infty$, and f_k converge uniformly on $[a, b]$ to a function f, then f is also Riemann integrable and

$$\lim_{k\to\infty} \int_a^b f_k \, dx = \int_a^b \lim_{k\to\infty} f_k \, dx \, .$$

(Much more sophisticated theorems on passing to the limit under the integral sign will be discussed in Section 1.3.4.)

If Ω is an open set, $j \in \{1, \ldots, n\}$, for all $k \in \mathbb{N}$ $f_k, \frac{\partial f_k}{\partial x_j} \in C(\Omega)$, or $f_k, \frac{\partial f_k}{\partial x_j} \in \overline{C}(\Omega)$, f_k converge on Ω, and $\frac{\partial f_k}{\partial x_j}$ converge uniformly on every compact $K \subset \Omega$, or on Ω respectively, then for all $x \in \Omega$

$$\lim_{k\to\infty} \left(\frac{\partial f_k}{\partial x_j}\right)(x) = \frac{\partial}{\partial x_j}\left(\lim_{k\to\infty} f_k\right)(x)$$

and $\lim_{k\to\infty} f_k, \frac{\partial}{\partial x_j}\lim_{k\to\infty} f_k \in C(\Omega)$, $\lim_{k\to\infty} f_k, \frac{\partial}{\partial x_j}\lim_{k\to\infty} f_k \in \overline{C}(\Omega)$ respectively.

Denote by $\mathbb{N}_0$ the set of all nonnegative integers, and, for $n \in \mathbb{N}$, by $\mathbb{N}_0^n$ the set of all $\alpha = (\alpha_1, \ldots, \alpha_n)$ where $\alpha_1, \ldots, \alpha_n \in \mathbb{N}_0$. Elements of ${\mathbb{N}_0}^n$ are often called *multi-indices*. For $\alpha \in {\mathbb{N}_0}^n$ we write* $|\alpha| = \alpha_1 + \cdots + \alpha_n$. Finally, given $\alpha \in \mathbb{N}_0$, we put $D^\alpha := D_1^{\alpha_1} \ldots D_n^{\alpha_n}$ where $D_1 \equiv \frac{\partial}{\partial x_1}, \ldots, D_n \equiv \frac{\partial}{\partial x_n}$. Thus

$$D^\alpha f := \frac{\partial^{\alpha_1+\cdots+\alpha_n} f}{\partial {x_1}^{\alpha_1} \ldots \partial {x_n}^{\alpha_n}} \quad (D^0 f \equiv f).$$

Definition 1.2.3. *Let $\Omega \subset \mathbb{R}^n$ be an open set and $l \in \mathbb{N}$. We say that $f \in C^l(\Omega)$ or $f \in \overline{C}^l(\Omega)$ if for all $\alpha \in \mathbb{N}_0^n$ satisfying $|\alpha| \leqslant l$ the derivatives $D^\alpha f$ exist on Ω and $D^\alpha f \in C(\Omega)$, $D^\alpha f \in \overline{C}(\Omega)$ respectively.*

Let Ω be an open set, $l, m \in \mathbb{N}, m \leqslant l, f \in C^l(\Omega), \beta := (\beta_1, ..., \beta_l)$ where $\beta_1, ..., \beta_m \in \{1, ..., n\}$, and let $\gamma := (\gamma_1, ..., \gamma_m)$ be any vector obtained from β by permutation of its components. Then for all $x \in \Omega$

$$(D_{\beta_1} \cdots D_{\beta_m} f)(x) = (D_{\gamma_1} \cdots D_{\gamma_m} f)(x) = (D^\alpha f)(x),$$

where $\alpha := (\alpha_1, ..., \alpha_n)$ and α_j are the numbers of the components of the vector β equal to j.

Without the assumption concerning the continuity of the derivatives, this equality may not hold, see Exercise 1.2.2.

The quantity†

$$\|f\|_{C^l(\Omega)} := \sum_{|\alpha| \leqslant l} \|D^\alpha f\|_{C(\Omega)} \tag{1.2.3}$$

is a norm on $\overline{C}^l(\Omega)$. The space $\overline{C}^l(\Omega)$ with this norm is a Banach space.

Definition 1.2.3 of $C^l(\Omega)$ given for open sets $\Omega \in \mathbb{R}^n$ can be extended to a wider class of sets Ω in the following way.

Definition 1.2.4. *Let $\Omega \subset \mathbb{R}^n$ be such that $\partial\Omega = \partial\underline{\Omega}$, where $\underline{\Omega}$ is the set of inner points of Ω, and $l \in \mathbb{N}$. We say that $f \in C^l(\Omega)$ if $f \in C^l(\underline{\Omega})$, and for all $x \in \Omega \setminus \underline{\Omega}$ and for all $\alpha \in \mathbb{N}_0^n$ satisfying $|\alpha| \leqslant l$ there exists a (finite) limit $\lim_{y\to x, y \in \underline{\Omega}} (D^\alpha f)(y)$. For $x \in \Omega \setminus \underline{\Omega}$, $(D^\alpha f)(x)$ is defined to be equal to this limit.*

If $n = 1$, $-\infty < a < b < \infty$, then the space $C^l([a, b])$ can be defined in an equivalent way as the set of functions f defined on $[a, b]$ for which for all $k \in \{1, \ldots, l\}$ and for all $x \in [a, b]$ the derivative $f^{(k)}$ exists and $f^l \in C([a, b])$. It is assumed that $f^{(k)}(a)$ is the right derivative and $f^{(k)}(b)$ is the left derivative.

For a function f defined on $\Omega \subset \mathbb{R}^n$, the set

$$\operatorname{supp} f := \overline{\{x \in \Omega \colon f(x) \neq 0\}}$$

is called the *support* of f.

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* For $x \in \mathbb{R}^n$ we write $|x| = (x_1^2 + \cdots + x_n^2)^{\frac{1}{2}}$ (the Euclidean distance).

† Due to the above property, the information about all derivatives of order less than or equal to l is taken into account. One can verify that

$$\sum_{m=0}^{l} \sum_{\beta_1=1}^{n} \cdots \sum_{\beta_m}^{n} \|D_{\beta_1} \cdots D_{\beta_m} f\|_{C(\Omega)} = \sum_{|\alpha| \leqslant l} \frac{|\alpha|!}{\alpha_1! \cdots \alpha_n!} \|D^\alpha f\|_{C(\Omega)}.$$

Definition 1.2.5. *Let $\Omega \subset \mathbb{R}^n$ be an open set. We say that $f \in C_0(\Omega)$ if $f \in C(\Omega)$,* $\operatorname{supp} f$ *is compact and* $\operatorname{supp} f \subset \Omega$. *Moreover, for $l \in \mathbb{N}$, $C_0^l(\Omega) := C^l(\Omega) \cap C_0(\Omega)$.*

Each function $f \in C_0(\Omega)$ vanish in a certain 'strip' along the boundary $\partial\Omega$, depending on f, namely, on $\Omega \setminus \operatorname{supp} f$.

Definition 1.2.6. *For open sets $\Omega \subset \mathbb{R}^n$*

$$C^\infty(\Omega) := \bigcap_{l=0}^{\infty} C^l(\Omega), \qquad C_0^\infty(\Omega) := C^\infty(\Omega) \bigcap C_0(\Omega).$$

Note that, for $n > 1$, a function f may have all derivatives of any order on $\mathbb{R}^n$, but not belong to $C^\infty(\mathbb{R}^n)$. (See Exercise 1.5.4.)

The following function is an important example of a function* in $C_0^\infty(\mathbb{R}^n)$:

$$h(x) := \begin{cases} e^{\frac{1}{|x|^2-1}} & \text{if } x \in B_1, \\ 0 & \text{if } x \in {}^cB_1. \end{cases} \tag{1.2.4}$$

Exercise 1.2.1. Prove that for all $\alpha \in \mathbb{N}_0^n$ there exist a polynomial P_α such that for all $x \in B_1$

$$(D^\alpha h)(x) = \frac{P_\alpha(x)}{(1-|x|^2)^{2\alpha}}\, e^{\frac{1}{|x|^2-1}}.$$

Hence deduce that $h \in C_0^\infty(\mathbb{R}^n)$.

For all $\Omega \subset \mathbb{R}^n$ and for all $\delta > 0$ there exists a function $\eta \in C^\infty(\mathbb{R}^n)$ such that

$$0 \leqslant \eta(x) \leqslant 1 \text{ if } x \in \mathbb{R}^n, \qquad \eta(x) = 0 \text{ if } x \notin \Omega^\delta,$$

where Ω^δ is the δ-neighbourhood of Ω:

$$\Omega^\delta = \bigcup_{x \in \Omega} B(x, \delta).$$

Functions of such type are called 'hat-like functions' and they play an important role in some constructions.

For all $\Omega \subset \mathbb{R}^n$ the set $C^\infty(\mathbb{R}^n) \bigcap \overline{C}(\Omega)$ is dense in $\overline{C}(\Omega)$, i.e. for all $f \in \overline{C}(\Omega)$ and for all $\varepsilon > 0$ there exists $\varphi \in C^\infty(\mathbb{R}^n) \bigcap \overline{C}(\Omega)$ such that $\|f - \varphi\|_{C(\Omega)} < \varepsilon$.

This statement and formula (1.2.1) explain why the space $\overline{C}(\Omega)$ are normally used rather than the Banach space $C_b(\Omega)$ of all functions bounded and continuous on Ω with the same norm.

Exercise 1.2.2. Prove that if a bounded set Ω is not compact, then $C_b(\Omega) \neq \overline{C}(\Omega)$ and the set $C^\infty(\Omega) \cap C_b(\Omega)$ is not dense in $C_b(\Omega)$.

1.3 Basic facts in the theory of the Lebesgue integral

In this section we recall the definitions of the Lebesgue measure and the definition of the Lebesgue integral based on the scheme 'measure—integral', and we state, without proofs, the main theorems in the theory of the Lebesgue integral discussing in more detail those of them which will be used in the sequel.

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* Here and in the sequel, for $x \in \mathbb{R}^n$ and $r > 0$, $B(x, r)$ denotes an open ball in $\mathbb{R}^n$ of radius r centered at x, i.e. $B(x,r) = \{y \in \mathbb{R}^n : |x - y| < r\}$, and $B_r \equiv B(0, r)$. ${}^c\Omega$ denotes the complement of Ω.

1.3.1 Lebesgue measure

The theory of the Lebesgue integration in $\mathbb{R}^n$, constructed by using the scheme 'measure – integral', is based on the notion of the Lebesgue measure of a set $\Omega \subset \mathbb{R}^n$ which generalizes the notions of the length of an interval, area of a rectangle, the volume of a cuboid* etc. The proofs of the statements below may be found in [??].

Definition 1.3.1. *The measure of a cuboid*†

$$Q := \{x \in \mathbb{R}^n \colon a_j < x_j < b_j, j \in \{1, \dots, n\}\} ,$$

where $-\infty < a_j < b_j < \infty$, *and also the measure of its closure, is*

$$\operatorname{meas} Q := \operatorname{meas} \overline{Q} := \prod_{j=1}^{n} (b_j - a_j) . \tag{1.3.1}$$

Each open set $\Omega \in \mathbb{R}^n$ may be represented (in many ways) as

$$\Omega = \bigcup_{k=1}^{s} \overline{Q}_k , \tag{1.3.2}$$

where $s \in \mathbb{N}$ or $s = \infty$ and Q_k are disjoint cuboids.

Definition 1.3.2. *The measure of a bounded open set* $\Omega \subset \mathbb{R}^n$ *is*

$$\operatorname{meas} \Omega := \sum_{k=1}^{s} \operatorname{meas} Q_k .$$

It is also assumed that $\operatorname{meas} \emptyset = 0$.

One can prove that for different representations (1.3.2), the sum of the measures of Q_k is the same.

Definition 1.3.3. *The measure of a compact* $\Omega \subset \mathbb{R}^n$ *is*

$$\operatorname{meas} \Omega := \operatorname{meas} Q - \operatorname{meas} (Q \setminus \Omega) ,$$

where Q *is an arbitrary cuboid containing* Ω.

The definition does not depend on the choice of Q.
Next we pass to the case of an arbitrary bounded set $\Omega \subset \mathbb{R}^n$.

Definition 1.3.4. *The outer measure of a bounded set* $\Omega \subset \mathbb{R}^n$ *is*

$$\operatorname{meas}^* \Omega := \inf_{G \supset \Omega} \operatorname{meas} G$$

where the infimum is taken with respect to all bounded open sets $G \supset \Omega$.

---

* Here and in the sequel by cuboid we always mean a bounded open cuboid, whose faces are parallel to the coordinate planes.

† Here and in the sequel by cuboid we always mean a bounded open cuboid, whose faces are parallel to the coordinate planes.

Definition 1.3.5. *The inner measure of a bounded set* $\Omega \subset \mathbb{R}^n$ *is*

$$\operatorname{meas}_* \Omega := \sup_{F \subset \Omega} \operatorname{meas} F$$

where the supremum is taken with respect to all compacts $F \subset \Omega$.

Definition 1.3.6. *A bounded set* $\Omega \subset \mathbb{R}^n$ *is said to be* Lebesgue measurable *if* $\operatorname{meas}_* \Omega = \operatorname{meas}^* \Omega$, *and its* measure *is*

$$\operatorname{meas} \Omega := \operatorname{meas}_* \Omega = \operatorname{meas}^* \Omega .$$

Definition 1.3.7. *An unbounded set* $\Omega \subset \mathbb{R}^n$ *is said to be* Lebesgue measurable *if for all* $k \in \mathbb{N}$ *the set* $\Omega \cap B_k$ *is measurable, and its* measure *is*

$$\operatorname{meas} \Omega := \lim_{k \to \infty} \Omega \cap B_k .$$

(It may happen that $\operatorname{meas} \Omega = \infty$.*)*

Any open set $\Omega \subset \mathbb{R}$ may be represented uniquely as a union of a finite or countable family of disjoint intervals:

$$\Omega = \bigcup_{k=1}^{s} (a_k, b_k)$$

where $s \in \mathbb{N}$ or $s = \infty$. The intervals (a_k, b_k) are called *constituent intervals*. Moreover,

$$\operatorname{meas} \Omega = \sum_{k=1}^{s} (b_k - a_k) .$$

Note that a set $\Omega \subset \mathbb{R}^n$ is of *zero Lebesgue measure* if and only if for all $\varepsilon > 0$ there exists a finite or countable family of cuboids or balls covering Ω such that the sum of measures of those cuboids, balls respectively, is less than ε.

Each finite or countable set has zero Lebesgue measure. However not every set of zero Lebesgue measure is finite or countable as the following example shows.

Example 1.3.1. (Cantor's set) From the closed interval $[0,1]$, let an open interval $\left(\frac{1}{3}, \frac{2}{3}\right)$ of length $\frac{1}{3}$ be cut out. Next from each of the two remaining closed intervals, let open intervals centered at their midpoints of length $\frac{1}{3^2}$ be cut out, and so on. The set obtained by cutting out all such open intervals which is a *perfect set*, i.e. a closed set without isolated points, and nowhere dense* in $[0,1]$, is *Cantor's set* D, which has zero Lebesgue measure and is uncountable. It consists of those, and only those, points in $[0,1]$, for which the corresponding real numbers may be written as infinite fractions to base 3, whose entries are not equal to 1.

The union of a finite or countable family of sets of zero Lebesgue measure is also a set of zero Lebesgue measure.

A set $\Omega \subset \mathbb{R}^n$ is called a *Borel set* if it can be constructed by finite or countable number of operations of union and intersection from open and closed sets. Each Borel set is Lebesgue measurable. However, there exist measurable sets which are not Borel sets.

There exist non-measurable sets. Moreover, each measurable set of positive measure contains a non-measurable subset.

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* A set $B \subset [0,1]$ is *dense* in $[0,1]$ if $\overline{B} \supset [0,1]$. A set $B \subset [0,1]$ is *nowhere dense* in $[0,1]$ if it is not dense in any closed interval $[\alpha, \beta] \subset [0,1]$.

Remark 1.3.1. This fact is based on the *of choice axiom*, which is assumed in this book. However, if the axiom of choice is replaced by the so-called *determination axiom* (Definition 3.2.2), then all sets in $\mathbb{R}^n$ are Lebesgue measurable. See Appendix 3.2.2 for details.

Given a measurable set $\Omega \subset \mathbb{R}^n$, any set congruent to Ω has the same measure. In particular, $\operatorname{meas}(\Omega + h) = \operatorname{meas}\Omega$, where $h \in \mathbb{R}^n$, and $\Omega + h = \{x + h \colon x \in \Omega\}$ is the translation of the set Ω.

If Ω_1 and Ω_2 are measurable sets in $\mathbb{R}^n$ and $\Omega_2 \subset \Omega_1$, then $\operatorname{meas}\Omega_2 \leqslant \operatorname{meas}\Omega_1$. If, further, $\operatorname{meas}\Omega_2 < \infty$, then $\operatorname{meas}(\Omega_1 \setminus \Omega_2) = \operatorname{meas}\Omega_1 - \operatorname{meas}\Omega_2$.

If for all $k \in \mathbb{N}$ sets $\Omega_k \subset \mathbb{R}^n$ are measurable, then the set $\bigcup\limits_{k=1}^{\infty} \Omega_k$ is also measurable, and

$$\operatorname{meas} \bigcup_{k=1}^{\infty} \Omega_k \leqslant \sum_{k=1}^{\infty} \operatorname{meas}\Omega_k \,.$$

If, further, Ω_k are disjoint, then

$$\operatorname{meas} \bigcup_{k=1}^{\infty} \Omega_k = \sum_{k=1}^{\infty} \operatorname{meas}\Omega_k \tag{1.3.3}$$

(*countable additivity* of the Lebesgue measure).

For all $k \in \mathbb{N}$ let the sets Ω_k be measurable. If for all $k \in \mathbb{N}$ $\Omega_k \subset \Omega_{k+1}$, then

$$\operatorname{meas} \bigcup_{k=1}^{\infty} \Omega_k = \lim_{k\to\infty} \operatorname{meas}\Omega_k. \tag{1.3.4}$$

If for all $k \in \mathbb{N}$ $\quad \Omega_k \supset \Omega_{k+1}$ and $\operatorname{meas}\Omega_1 < \infty$, then

$$\operatorname{meas} \bigcap_{k=1}^{\infty} \Omega_k = \lim_{k\to\infty} \operatorname{meas}\Omega_k. \tag{1.3.5}$$

Let $\Omega \subset \mathbb{R}^n$. A property is said to be satisfied *for almost all* $x \in \Omega$ ($\equiv$ *almost everywhere* on Ω) if the subset of all $x \in \Omega$, for which it is not satisfied, has zero measure.

Let $n \geqslant 2$, $\Omega \subset \mathbb{R}^n$ be a measurable set, $m \in \mathbb{N}$, $m < n$, and let Ω' denote the projection of Ω onto the m-dimensional coordinate plane $\{x \in \mathbb{R}^n \colon x_{m+1} = \cdots = x_n = 0\}$.

Then for almost all $(x_1, \ldots, x_m) \in \Omega'$ in the sense of the m-dimensional Lebesgue measure, the intersection $\Omega \cap \mathbb{R}^{n-m}_{x_1,\ldots,x_m}$, where $\mathbb{R}^{n-m}_{x_1,\ldots,x_m} := \{y \in \mathbb{R}^n \colon y_1 = x_1, \ldots, y_m = x_m\}$, is measurable in the sense of the $(n-m)$-dimensional Lebesgue measure. The converse does not hold.

If $m \in \mathbb{N}$, $m < n$, a set $\Omega_1 \subset \mathbb{R}^m$ is measurable in the sense of the m-dimensional measure, and the set $\Omega_2 \subset \mathbb{R}_{n-m}$ is measurable in the sense of the $(n-m)$-dimensional measure, then the set $\Omega_1 \times \Omega_2 \subset \mathbb{R}^n$ is measurable in the sense of the n-dimensional measure, and

$$\operatorname{meas}_n(\Omega_1 \times \Omega_2) = \operatorname{meas}_m \Omega_1 \cdot \operatorname{meas}_{n-m} \Omega_2 \,.$$

1.3.2 Measurable functions

We start with the following basic definition.

Definition 1.3.8. 1. *Let $\Omega \subset \mathbb{R}^n$ be a measurable set and $f\colon \Omega \to \mathbb{R}$. The function f is* measurable *on Ω if for all $a \in \mathbb{R}$ the set $f^{-1}((a,\infty)) = \{x \in \Omega\colon f(x) > a\}$ is measurable.*

2. *Let $\Omega \subset \mathbb{R}^n$ be a measurable set and $f\colon \Omega \to \mathbb{C}$. The function f is* measurable *on Ω if the functions $\Re f$ and $\Im f$ are measurable on Ω.*

Next we recall several other definitions equivalent to Definition 1.3.8.

Two functions $f, g\colon \Omega \to \mathbb{C}$ are said to be *equivalent* on Ω, shorthand: $f \sim g$, if $f(x) = g(x)$ for almost all $x \in \Omega$.

Definition 1.3.9. *Let $\Omega \subset \mathbb{R}^n$ be a measurable set and $f\colon \Omega \to \mathbb{C}$. The function f is* measurable on Ω *if there exists a sequence of functions, continuous on Ω, converging to f almost everywhere on Ω.*

Definition 1.3.10. *Let $\Omega \subset \mathbb{R}^n$ be a measurable set and $f\colon \Omega \to \mathbb{C}$. The function f is* measurable on Ω *if for all $\varepsilon > 0$ there exists $g \in C(\mathbb{R}^n)$ such that* $\operatorname{meas}\{f(x) \neq g(x)\} < \varepsilon$.

The equivalence of Definitions 1.3.8 and 1.3.9 is the *Fréchet theorem*, of Definitions 1.3.8 and 1.3.10 the *Luzin* theorem* respectively.

A function $f\colon \Omega \to \mathbb{C}$ is called a *function simple*, if its range $f(\Omega)$ is a finite or countable set.

Definition 1.3.11. 1. *Let $\Omega \subset \mathbb{R}^n$ be a measurable set and let $f\colon \Omega \to \mathbb{C}$ be a simple function. The function f is* measurable *if for all $a \in f(\Omega)$ the set $\{x \in \Omega\colon f(x) = a\}$ is measurable.*

2. *Let $\Omega \subset \mathbb{R}^n$ be a measurable set and let $f\colon \Omega \to \mathbb{C}$. The function f is* measurable on Ω *if there exists a sequence of measurable simple functions uniformly convergent to f on Ω.*

Exercise 1.3.1. Let $\Omega \subset \mathbb{R}^n$ and $f\colon \Omega \to \mathbb{C}$. Prove that the sequence of simple functions

$$f_k(x) = \frac{1}{k}\big([k\Re f(x)] + i\,[k\Im f(x)]\big), \quad x \in \Omega,\ k \in \mathbb{N}$$

converges uniformly to f on Ω.

A function $f\colon \Omega \to \mathbb{C}$ is called a *step-function* if its range $f(\mathbb{R}^n)$ is a finite set and for all $a \in f(\mathbb{R}^n)$, $a \neq 0$, the set $\{x \in \mathbb{R}^n\colon f(x) = a\}$ is a cuboid whose faces are parallel to the coordinate planes.

Definition 1.3.12. *Let $\Omega \subset \mathbb{R}^n$ be a measurable set and $f\colon \Omega \to \mathbb{C}$. The function f is measurable on Ω if there exists a sequence of step-functions (defined in $\mathbb{R}^n$) convergent to f almost everywhere on Ω.*

Definition 1.3.13. 1. *Let $\Omega \subset \mathbb{R}^n$ be a measurable set and $f\colon \Omega \to \mathbb{R}$. The function f is measurable on Ω if there exists functions f_1 and f_2 such that $f = f_1 - f_2$ and* non-decreasing *sequences of step-functions (defined in $\mathbb{R}^n$) convergent to f_1, f_2 respectively, almost everywhere on Ω.*

2. *Let $\Omega \subset \mathbb{R}^n$ be a measurable set and $f\colon \Omega \to \mathbb{C}$. The function f is* measurable *on Ω if the functions $\Re f$ and $\Im f$ are measurable on Ω.*

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* Also commonly transliterated as 'Lusin'.

Note that for $\Omega = \mathbb{R}^n$ Definitions 1.3.12 and 1.3.13 only require that the Lebesgue measure be defined for cuboids and sets of zero measure. Also note that, given a measurable set $\Omega \subset \mathbb{R}^n$, a function $f\colon \Omega \to \mathbb{C}$ is measurable on Ω if and only if its extension to $\mathbb{R}^n$ by zero

$$f_0(x) = \begin{cases} f(x) & \text{if } x \in \Omega, \\ 0 & \text{if } x \in {}^c\Omega \end{cases}$$

is measurable on $\mathbb{R}^n$.

Let $\Omega \subset \mathbb{R}^n$ and let $f\colon \Omega \to \mathbb{C}$ be a non-negative function. The set

$$T_f := \{(x, x_{n+1}) \in \mathbb{R}^{n+1} : 0 \leqslant x_{n+1} \leqslant f(x)\} \tag{1.3.6}$$

is called the *subgraph* of f.

Given a function $f\colon \Omega \to \mathbb{R}$, define the *positive part* f_+ of f and the *negative part* f_- of f by

$$f_+(x) := \max\{f(x), 0\}, \quad f_-(x) := \max\{-f(x), 0\}, \quad x \in \Omega.$$

Then

$$f = f_+ - f_-, \quad |f| = f_+ + f_-.$$

Definition 1.3.14. 1. *Let $\Omega \subset \mathbb{R}^n$ be a measurable set and $f\colon \Omega \to \mathbb{R}$. The function f is* measurable *on Ω in the sense of the n-dimensional measure if the sets T_{f_+} and T_{f_-} are measurable in the sense of the $(n+1)$-dimensional measure.*

2. *Let $\Omega \subset \mathbb{R}^n$ be a measurable set and $f\colon \Omega \to \mathbb{C}$. The function f is* measurable *on Ω if the functions $\Re f$ and $\Im f$ are measurable on Ω.*

The characteristic function $\chi(\Omega)$ of a set $\Omega \subset \mathbb{R}^n$, defined by $\chi(\Omega)(x) = 1$ for $x \in \Omega$ and $\chi(\Omega)(x) = 0$ for $x \in {}^c\Omega$, is measurable on $\mathbb{R}^n$ if and only if the set Ω is measurable. Hence there exist non-measurable functions. For example, the characteristic function of a non-measurable set $\Omega \subset \mathbb{R}^n$ is non-measurable on $\mathbb{R}^n$.

Remark 1.3.2. The existence of non-measurable functions is based on the existence of non-measurable sets, hence on the axiom of choice. See Remark 1.3.1. If the axiom of choice is replaced by the determination axiom, then all functions defined on any sets in $\mathbb{R}^n$ are measurable. See Appendix 3.2.2 for details.

If $\Omega \subset \mathbb{R}^n$, $f\colon \Omega \to \mathbb{R}$ is measurable on Ω and $A \subset \mathbb{R}$ is a Borel set, then the set $f^{-1}(A)$ is measurable. In particular, for all $a, b \in \mathbb{R}$ the sets $\{x \in \Omega\colon f(x) \geqslant a\}$, $\{x \in \Omega\colon f(x) = a\}$, $\{x \in \Omega\colon f(x) < a\}$ and $\{x \in \Omega\colon a < f(x) < b\}$ are all measurable. (In the last two cases each of the signs $<$ can be replaced by $\leqslant$.) However, given a measurable set $A \subset \Omega$, the set $f^{-1}(A)$ is not necessarily measurable.

Exercise 1.3.2. For any measurable set $\Omega \subset \mathbb{R}^n$ with $\operatorname{meas}\Omega > 0$ construct a function $f\colon \Omega \to \mathbb{R}$ such that for all $a \in \mathbb{R}$ the sets $\{x \in \Omega\colon f(x) = a\}$ are measurable, but f is not measurable on Ω.

Assume that $\Omega \subset \mathbb{R}^n$ is a measurable set. Then the following statements hold.

If a function $f\colon \Omega \to \mathbb{C}$ is measurable, then the function $|f|$ is also measurable on Ω. (Converse does not hold, say for the function f which is equal to 1 on a non-measurable subset of Ω with $\operatorname{meas}\Omega > 0$ and equal to -1 on $\Omega \setminus \Omega_1$.)

If $f\colon \Omega \to \mathbb{C}$ is measurable and $g \sim f$ on Ω, then g is also measurable on Ω.

A function on $f\colon \Omega \to \mathbb{C}$ continuous for almost all $x \in \Omega$ with respect to Ω, or a function equivalent to f on Ω, is measurable on Ω.

If $f, g\colon \Omega \to \mathbb{C}$ are measurable, then the functions $f \pm g$, fg and (under assumption that for all* $x \in \Omega$ $g(x) \neq 0$) $\frac{f}{g}$ are measurable on Ω.

If for all $k \in \mathbb{N}$ functions $f_k\colon \Omega \to \mathbb{C}$ are measurable on Ω, $f\colon \Omega \to \mathbb{C}$ and $f(x) = \lim\limits_{k\to\infty} f_k(x)$ for almost all $x \in \Omega$, then f is also measurable on Ω.

Thus, the arithmetic operations and the operation of passing to the limit are closed in the space of functions measurable on Ω.

Exercise 1.3.3. Let $\Omega \subset \mathbb{R}^n$ be a measurable set, functions $f_k\colon \Omega \to \mathbb{R}$, $k \in \mathbb{N}$ be measurable on Ω and $f\colon \Omega \to \mathbb{R}$. If, for almost all $x \in \Omega$, $f(x) = \sup\limits_{k\in\mathbb{N}} f_k(x)$, then f is also measurable on Ω. (A similar statement holds if sup is replaced by inf, lim sup or lim inf.)

Let $\Omega \subset \mathbb{R}^n$ be a measurable set and $f\colon \Omega \to \mathbb{R}^m$ be measurable. If a function $\varphi\colon f(\Omega) \to \mathbb{C}$ is continuous on $f(\Omega)$, then the composition $\varphi(f)$ is also measurable on $f(\Omega)$. If a function $\varphi\colon f(\Omega) \to \mathbb{C}$ is measurable on $f(\Omega)$, then the composition $\varphi(f)$ may not be measurable on $f(\Omega)$ even if f is continuous on Ω.

In particular, if $f\colon \Omega \to \mathbb{C}$ is measurable on Ω and $0 < p < \infty$, then $|f|^p$ is also measurable on Ω.

Exercise 1.3.4. By applying Definition 1.3.8 give a direct proof of this statement. Furthermore, assuming that $f\colon \Omega \to \mathbb{R}$, prove that the function $\operatorname{sgn} f$ is measurable on Ω.

Let $n \geqslant 2$, $\Omega \subset \mathbb{R}^n$ be a measurable set, $m \in \mathbb{N}$, $m < n$, and let Ω' denote the projection of Ω onto the m-dimensional coordinate plane $\{x \in \mathbb{R}^n\colon x_{m+1} = \cdots = x_n = 0\}$. Furthermore, let a function $f\colon \Omega \to \mathbb{C}$ be measurable. Then for almost all $(x_1, \dots, x_m) \in \Omega'$ in the sense of the m-dimensional measure, the restriction of f to the set $\Omega \cap \mathbb{R}^{n-m}_{x_1,\dots,x_m}$ is measurable in the sense of the $(n-m)$-dimensional measure. The converse does not hold.

If $\Omega \subset \mathbb{R}^n$ is a measurable set, $\operatorname{meas}\Omega < \infty$, functions $f\colon \Omega \to \mathbb{C}$ and $f_k : \Omega \to \mathbb{C}$, $k \in \mathbb{N}$, are measurable on Ω and $\lim\limits_{k\to\infty} f_k(x) = f(x)$ almost everywhere on Ω, then for all $\varepsilon > 0$ there exists a closed set $F \subset \Omega$ such that $\operatorname{meas}(\Omega \setminus F) < \varepsilon$ and f_k converge uniformly to f on F (the *Egorov theorem*).

Proofs of the equivalence of the definitions formulated above and other statements in this section can be found in [??].

Exercise 1.3.5. Let Ω be a measurable set in $\mathbb{R}^n$, a function $\varphi\colon \Omega \to \mathbb{R}^m$ be continuous on Ω, and a function $f\colon \varphi(\Omega) \to \mathbb{C}$ be measurable on $\varphi(\Omega)$. Assume that for all sets $\omega \subset \varphi(\Omega)$ satisfying $\operatorname{meas}_m \omega = 0$ one has $\operatorname{meas}_n \varphi^{-1}(\omega) = 0$. (In other words, the inverse map φ^{-1} possesses the *N-property.*) By applying Definition 1.3.9 prove that the function $f(\varphi)\colon \Omega \to \mathbb{C}$ is measurable on Ω.

Exercise 1.3.6. Let a function $f\colon \mathbb{R}^n \to \mathbb{C}$ be measurable on $\mathbb{R}^n$. Define the function $F\colon \mathbb{R}^{2n} \to \mathbb{C}$ by setting $F(x, y) := f(x - y)$ for all $x, y \in \mathbb{R}^n$. By applying Exercise 1.3.5 prove that the function F is measurable on $\mathbb{R}^{2n}$.

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* One may admit that $g(x) \neq 0$ for almost all $x \in \Omega$ and define $\frac{f}{g}(x)$ in an arbitrary way for those $x \in \Omega$ for which $g(x) = 0$.

Exercise 1.3.7. Let a function $f\colon (0,\infty) \to \mathbb{C}$ be measurable on $\mathbb{R}^n$. Define the function $F\colon (0,\infty)\times(0,\infty) \to \mathbb{C}$ by setting $F(x,y) := f(xy)$ for all $x, y \in (0,\infty)$. By applying Exercise 1.3.5 prove that the function F is measurable on $(0,\infty)\times(0,\infty)$.

1.3.3 Definition of the Lebesgue integral

Definition 1.3.15. *Let $\Omega \subset \mathbb{R}^n$ be a measurable set and let a function $f : \Omega \to \mathbb{R}$ be non-negative and measurable on Ω. The* Lebesgue integral *of f over Ω is the measure of its subgraph* (1.3.6)*:*

$$\int\limits_\Omega f\,\mathrm{d}x \equiv \int\limits_\Omega f(x)\,\mathrm{d}x := \mathrm{meas}_{n+1} T_f.$$

(It is not ruled out that $\int\limits_\Omega f\,\mathrm{d}x = \infty$.)

If $n = 1$, $\Omega = [a,b]$, then the subgraph of f is a 'curvilinear trapezium', and the Lebesgue integral is the 'area of the curvilinear trapezium'.

Definition 1.3.16. *Let $\Omega \subset \mathbb{R}^n$ be a measurable set and let a function $f : \Omega \to \mathbb{R}$ be measurable on Ω. If at least one of the integrals $\int\limits_\Omega f_+\,\mathrm{d}x$ or $\int\limits_\Omega f_-\,\mathrm{d}x$ is finite, then the* Lebesgue integral *over Ω is*

$$\int\limits_\Omega f\,\mathrm{d}x := \int\limits_\Omega f_+\,\mathrm{d}x - \int\limits_\Omega f_-\,\mathrm{d}x.$$

(It is assumed that, for $a \in \mathbb{R}$, $+\infty - a = +\infty$ and $a - (+\infty) = -\infty$.) If both integrals are finite, it is said that f is Lebesgue integrable *on Ω (or* summable *on Ω).*

Definition 1.3.17. *Let $\Omega \subset \mathbb{R}^n$ be a measurable set and let a function $f : \Omega \to \mathbb{C}$ be measurable on Ω. The function f is said to be Lebesgue integrable on Ω if both $\Re f$ and $\Im f$ are Lebesgue integrable on Ω. If f is integrable on Ω, then*

$$\int\limits_\Omega f\,\mathrm{d}x := \int\limits_\Omega \Re f\,\mathrm{d}x + i\int\limits_\Omega \Im f\,\mathrm{d}x.$$

If $n = 1$, $-\infty \leqslant a < b \leqslant \infty$, then $\int\limits_a^b f\,\mathrm{d}x := \int\limits_{[a,b]} f\,\mathrm{d}x$. Conventionally $\int\limits_a^a f\,\mathrm{d}x := 0$, and, for $-\infty \leqslant b < a \leqslant \infty$, $\int\limits_a^b f\,\mathrm{d}x := -\int\limits_b^a f\,\mathrm{d}x$.

Next we formulate two other definitions equivalent to Definition 1.3.15.

Assume that $\Omega \subset \mathbb{R}^n$ is a bounded measurable set, and a function $f : \Omega \to \mathbb{R}$ is measurable and bounded on Ω, say, for all $x \in \Omega$ $A < f(x) < B$, where $A, B \in \mathbb{R}$, $A < B$. For all $m \in \mathbb{N}$ let $y_0^{(m)}, y_1^{(m)}, \ldots, y_m^{(m)} \in \mathbb{R}$ and

$$y_0^{(m)} := A < y_1^{(m)} < y_2^{(m)} < \cdots < y_m^{(m)} := B\,. \tag{1.3.7}$$

Moreover, for all $k \in \{0, \ldots, m-1\}$, let

$$\Omega_k^{(m)} := \left\{ x \in \Omega : y_k^{(m)} \leqslant f(x) < y_{k+1}^{(m)} \right\}.$$

($\Omega_k^{(m)}$ are bounded measurable sets — see Section 1.3.2.) The *lower* and *upper Lebesgue sums* s and S are defined as

$$s := \sum_{k=0}^{m-1} y_k^{(m)} \operatorname{meas} \Omega_k^{(m)}, \qquad S := \sum_{k=0}^{m-1} y_{k+1}^{(m)} \operatorname{meas} \Omega_k^{(m)}.$$

(Note that $A \operatorname{meas} \Omega \leqslant s \leqslant S \leqslant B \operatorname{meas} \Omega$.)

Definition 1.3.18. *Let $\Omega \subset \mathbb{R}^n$ be a measurable set and let a function $f : \Omega \to \mathbb{R}$ be non-negative and measurable on Ω.*

1. *If Ω and f are bounded, the* Lebesgue integral *of f over Ω is*

$$\int_\Omega f \, dx := \sup_{y_0^{(m)}, y_1^{(m)}, \ldots, y_m^{(m)}} s, \tag{1.3.8}$$

where the supremum is taken over all $m \in \mathbb{N}$ and over all $y_0^{(m)}, y_1^{(m)}, \ldots, y_m^{(m)}$ satisfying (1.3.7).

2. *If Ω is bounded and f is unbounded, then*

$$\int_\Omega f \, dx := \lim_{k \to \infty} \int_\Omega [f]_k \, dx,$$

where for all $k \in \mathbb{N}$ and for all $x \in \Omega$ $[f]_k(x) := f(x)$ if $f(x) \leqslant k$ and $[f]_k(x) := k$ if $f(x) > k$.

3. *If Ω is unbounded, then*

$$\int_\Omega f \, dx := \lim_{k \to \infty} \int_{\Omega \cap B_k} f \, dx. \tag{1.3.9}$$

One can prove that definition (1.3.8) is independent of the choice of A and B, satisfying for all $x \in \Omega$ $A < f(x) < B$, and

$$\sup_{y_0^{(m)}, y_1^{(m)}, \ldots, y_m^{(m)}} s = \inf_{y_0^{(m)}, y_1^{(m)}, \ldots, y_m^{(m)}} S.$$

Let $f : \Omega \to \mathbb{R}$ be a simple function: $f(\Omega) = \{y_k\}_{k=1}^{s}$ where $s \in \mathbb{N}$ or $s = \infty$, and let $\Omega_k := \{x \in \Omega : f(x) = y_k\}$ for all $k \in \{1, \ldots, s\}$.

Definition 1.3.19. *Let $\Omega \subset \mathbb{R}^n$ be a measurable set and let a function $f : \Omega \to \mathbb{R}$ be non-negative and measurable on Ω.*

1. *If Ω is bounded and $f : \Omega \to \mathbb{R}$ is a simple function, then the* Lebesgue integral *of f over Ω is*

$$\int_\Omega f \, dx := \sum_{k=1}^{s} y_k \operatorname{meas} \Omega_k.$$

2. *If Ω is bounded and f is not a simple function, then the* Lebesgue integral *of f over Ω is*

$$\int_{\Omega} f \, dx := \lim_{k\to\infty} \int_{\Omega} f_k \, dx \,, \tag{1.3.10}$$

where for all $k \in \mathbb{N}$ the non-negative measurable simple functions f_k are defined for all $x \in \Omega$ by

$$f_k(x) := \frac{[kf(x)]}{k} \,.$$

3. *If Ω is unbounded, then the* Lebesgue integral *of f over Ω is defined by* (1.3.9).

One can prove that in (1.3.10) the limit, finite or infinite, always exists and has the same value if the sequence $\{f_k\}_{k\in\mathbb{N}}$ is replaced by any other sequence of non-negative measurable simple functions converging uniformly to f on Ω.

Remark 1.3.3. The equivalence of Definitions 1.3.15, 1.3.18 and 1.3.19 is proved in [??]. Some of the formulated definitions make also sense for a wider class of functions. For example, part 1 of Definition 1.3.18 is applicable to all functions $f : \Omega \to \mathbb{R}$ measurable and bounded on a measurable bounded set Ω. Part 1 of Definition 1.3.19 is applicable to all simple functions $f : \Omega \to \mathbb{C}$ for which the series $\sum\limits_{k=1}^{\infty} y_k \operatorname{meas} \Omega_k$ converges. Part 2 of Definition 1.3.19 is applicable to all measurable functions $f : \Omega \to \mathbb{C}$ if the functions f_k are defined for all $x \in \Omega$ by

$$f_k(x) := \frac{[k\Re f(x)]}{k} + i\,\frac{[k\Im f(x)]}{k}$$

and both limits $\lim\limits_{k\to\infty} \int\limits_{\Omega} \Re f_k \, dx$ and $\lim\limits_{k\to\infty} \int\limits_{\Omega} \Im f_k \, dx$ are finite.

Note that each bounded measurable function defined on a bounded measurable set $\Omega \subset \mathbb{R}^n$ is Lebesgue integrable on Ω. For non-measurable functions the Lebesgue integral is not defined.

Remark 1.3.4. By Remarks 1.3.1 and 1.3.2 each bounded function defined on any bounded set $\Omega \subset \mathbb{R}^n$ is Lebesgue integrable on Ω if the axiom of choice is replaced by the determination axiom. See Appendix 3.2.2 for details.

Note also that Definition 1.3.15 clearly implies that for all measurable sets $\Omega \subset \mathbb{R}^n$

$$\int_{\Omega} dx = \operatorname{meas} \Omega \,. \tag{1.3.11}$$

Remark 1.3.5. If $n = 1, -\infty < a < b < \infty$ and $f : [a,b] \to \mathbb{C}$ is Riemann integrable on $[a,b]$, then f is also Lebesgue integrable on $[a,b]$ and both integrals have the same value.

Recall that $f \colon [a,b] \to \mathbb{C}$ is Riemann integrable on $[a,b]$ if and only if it is bounded on $[a,b]$ and the set of all points of discontinuity has zero Lebesgue measure. (For a proof see, for example, (Nikol'skiı,19??).) Hence the set of all functions Lebesgue integrable on $[a,b]$ is much wider than the set of all functions Riemann integrable on $[a,b]$. A typical example of a bounded function, which is not Riemann integrable but is Lebesgue integrable, is the Dirichlet function, that is equal to 1 in all rational points of $[a,b]$ and to 0 in all irrational points of $[a,b]$. (Its Lebesgue integral is equal to 0 because it is equivalent to 0 on $[a,b]$.)

Remark 1.3.6. In a number of cases there arises a necessity in introducing an improper Lebesgue integral. If, say, $n=1$, $a \in \mathbb{R}$, then for functions $f:(a,\infty) \to \mathbb{C}$, Lebesgue integrable on (a,A) for all $A > a$, the *improper Lebesgue integral* is defined, similarly to the improper Riemann integral, by

$$\int_a^\infty f\,\mathrm{d}x := \lim_{A\to\infty} \int_a^A f\,\mathrm{d}x\,.$$

If f is Lebesgue integrable on (a,∞), then this integral coincides with the 'ordinary' Lebesgue integral. However, there exist functions, which are not Lebesgue integrable on (a,∞), but their improper Lebesgue integrals exist and are finite. For example, the function $\frac{\sin x}{x}$ is not Lebesgue integrable on $(0,\infty)$, but is Lebesgue integrable in improper sense. (The value of the improper integral over $(0,\infty)$ is equal to $\frac{\pi}{2}$).

In some other cases, it becomes necessary to introduce Lebesgue integration in the sense of the principal value. For example, if a function $f:\mathbb{R}\to\mathbb{R}$, for all $\varepsilon > 0$, Lebesgue integrable on $(-\infty,\varepsilon)\cup(\varepsilon,\infty)$, then the *Lebesgue integral in the sense of the principal value* is defined as

$$\text{v. p.} \int_{-\infty}^{\infty} f\,\mathrm{d}x := \lim_{\varepsilon\to 0+} \left(\int_{-\infty}^{-\varepsilon} + \int_{\varepsilon}^{\infty} \right) f\,\mathrm{d}x\,.$$

The function $f(x) = \frac{1}{x^3}$ for $x \neq 0$, $f(0) = 0$ is not Lebesgue integrable on $(-\infty,\infty)$, is not Lebesgue integrable in improper sense on $(-\infty,0)$ and $(0,\infty)$, but v. p. $\int\limits_{-\infty}^{\infty} f\,\mathrm{d}x = 0$.

Remark 1.3.7. It is also possible to define the Lebesgue integral of a function f non-negative and measurable on a measurable set $\Omega \subset \mathbb{R}^n$ as the improper Riemann integral of its *non-increasing rearrangement* f^*, which is a non-negative non-increasing function on $(0,\infty)$ equimeasurable with f, i.e. $\mathrm{meas}_1\{t \in (0,\infty)\colon f^*(t) > a\} = \mathrm{meas}_n\{x \in \Omega\colon f(x) > a\}$ for all $a \in \mathbb{R}$, by setting

$$(L)\int_\Omega f\,\mathrm{d}x := (R)\int_0^\infty f^*\,\mathrm{d}t := \lim_{\substack{A\to+\infty \\ a\to 0+}} (R)\int_a^A f^*\,\mathrm{d}t\,.$$

Remark 1.3.8. We have outlined the scheme for constructing the Lebesgue measure and integral 'measure—integral'. In this scheme, first, the measure is defined, and then, the integral is defined essentially as a measure of some set, or in some other equivalent way. There exists another equivalent scheme for constructing the Lebesgue measure and integral, 'integral—measure'. We give its short description. (For detailed exposition, see [??].) First, the measure of a cuboid is defined. Then the sets of zero measure are defined as the sets, for which for all $\varepsilon > 0$ there exists a finite or countable family of cuboids covering them, the sum of whose measures is less than ε. For real-valued step-functions in $\mathbb{R}^n$, which have constant values a_k on a finite family of cuboids Δ_k, $k \in \{1, \dots, m\}$, and are equal to zero elsewhere, the integral is defined by setting

$$\int\limits_{\mathbb{R}^n} f \, dx := \sum_{k=1}^{m} a_k \operatorname{meas} \Delta_k .$$

Next, the integral over $\mathbb{R}^n$ is defined for real-valued functions f, for which there exist non-decreasing sequences $\{f_k\}_{k \in \mathbb{N}}$ of step-functions convergent to f almost everywhere, by setting

$$\int\limits_{\mathbb{R}^n} f \, dx := \lim_{k \to \infty} \int\limits_{\mathbb{R}^n} f_k \, dx.$$

Finally, if a function $f\colon \Omega \to \mathbb{R}$ is measurable on $\mathbb{R}^n$, hence * there exist functions f_1 and f_2 such that $f = f_1 - f_2$ and non-decreasing sequences of real valued step-functions defined in $\mathbb{R}^n$ convergent to f_1, f_2 respectively, almost everywhere on Ω, then by definition

$$\int\limits_{\mathbb{R}^n} f \, dx := \int\limits_{\mathbb{R}^n} f_1 \, dx - \int\limits_{\mathbb{R}^n} f_2 \, dx$$

if at least one of the integrals in the right-hand side is finite. If $f\colon \Omega \to \mathbb{C}$, then Definition 1.3.17 should be applied.

A set $\Omega \subset \mathbb{R}^n$ is said to be measurable if its characteristic function $\chi(\Omega)$ is measurable on $\mathbb{R}^n$. A function $f\colon \Omega \to \mathbb{C}$ defined on a measurable set Ω is said to be measurable on Ω if its extension by zero f_0 is measurable on $\mathbb{R}^n$ and

$$\int\limits_{\Omega} f \, dx := \int\limits_{\mathbb{R}^n} f_0 \, dx .$$

For a measurable set Ω, the measure is defined by

$$\operatorname{meas} \Omega := \int\limits_{\Omega} dx := \int\limits_{\mathbb{R}^n} \chi(\Omega) \, dx .$$

(Compare with formula (1.3.11).)

Remark 1.3.9. In many cases the Lebesgue integration with respect to a general measure is widely used. Let S be a *σ-ring* of sets, i.e. a family of sets satisfying the following conditions: 1) $\emptyset \in S$, 2) if $\Omega_1, \Omega_2 \in S$, then $\Omega_1 \setminus \Omega_2 \in S$, and 3) if $\Omega_k \in S$ $k \in \mathbb{N}$, then $\bigcup\limits_{k=1}^{\infty} \Omega_k \in S$. A *measure* is a non-negative function of sets μ such that $\mu(\emptyset) = 0$ and the property of *countable additivity* is satisfied, i.e. for all sequences $\{\Omega_k\}_{k \in \mathbb{N}}$ of disjoint sets $\Omega_k \in S$

$$\mu\Big(\bigcup_{k=1}^{\infty} \Omega_k\Big) = \sum_{k=1}^{\infty} \mu(\Omega_k) .$$

* In this scheme Definition 1.3.13 is the main definition of measurability.

The sets in S are called μ-measurable sets. For a function $f\colon \Omega \to \mathbb{C}$ defined on a μ-measurable set, the Lebesgue integral with respect to the measure μ, $\int\limits_\Omega f\mu(\mathrm{d}x)$, is defined as in Definitions 1.3.18, 1.3.16 and 1.3.17 replacing in the lower and upper Lebesgue sums $\operatorname{meas}\Omega_k^{(m)}$ by $\mu(\Omega_k^{(m)})$.

Let $n = 1$ and let g be a non-decreasing function defined on $\mathbb{R}$. The *Lebesgue—Stiltjes measure* is the completion of the measure defined on semi-closed intervals by $\mu([a,b)) := g(b) - g(a)$. (Completion in the sense of the measure theory.) The integral with respect to this measure is called the *Lebesgue—Stiltjes integral* and is denoted by $\int\limits_\Omega f\,\mathrm{d}g(x)$ where $\Omega \subset \mathbb{R}$. If g is non-increasing, then $\int\limits_\Omega f\,\mathrm{d}g(x) = -\int\limits_\Omega f\,\mathrm{d}(-g(x))$.

Example 1.3.2. Let $g(x) := \sum\limits_{k=1}^{\infty} \theta(x-k), x \in \mathbb{R}$, where $\theta(x) = 0$ for $x < 0$ and $\theta(x) = 1$ for $x \geqslant 0$. Then for any function f continuous on $\mathbb{R}$

$$\int\limits_0^\infty f\,\mathrm{d}g(x) = \sum_{k=1}^{\infty} f(k)\,.$$

Example 1.3.3. Let $m \in \mathbb{N}, a_0 = 0, 0 < a_1 < a_2 < \cdots < a_m$, $A_1 \geqslant A_2 \geqslant \ldots \geqslant A_m$, $A_{m+1} = 0$ and

$$g(x) = \sum_{k=1}^{m} A_k \chi_{[a_{k-1},a_k)}\,.$$

Then for any function f continuous on $(0,\infty)$

$$\int\limits_0^\infty f\,\mathrm{d}g(x) = \sum_{k=1}^{m} f(a_k)\left(g(a_k+0) - g(a_k-0)\right) = \sum_{k=1}^{m} f(a_k)\left(A_{k+1} - A_k\right). \tag{1.3.12}$$

For our purposes it suffices to consider the case in which S is the set of all Lebesgue measurable sets in $\mathbb{R}^n$ and $\mu(\Omega) = \int\limits_\Omega \varrho\,\mathrm{d}x$, where ϱ is a function non-negative and measurable on $\mathbb{R}^n$. In this case

$$\int\limits_\Omega f\mu(\mathrm{d}x) = \int\limits_\Omega f\varrho\,\mathrm{d}x\,.$$

1.3.4 Properties of the Lebesgue integral

Everywhere in this section, unless specifically stated, functions under consideration, are complex-valued functions defined on measurable sets in $\mathbb{R}^n$. Speaking about integrable functions, we always mean functions Lebesgue integrable. Proofs of the theorems below can be found in [??].

Theorem 1.3.1. *If functions f_1 and f_2 are integrable on a measurable set Ω, $c_1, c_2 \in \mathbb{C}$, then the function $c_1f_1 + c_2f_2$ is also integrable on Ω and*

$$\int\limits_\Omega (c_1f_1 + c_2f_2)\,\mathrm{d}x = c_1\int\limits_\Omega f_1\,\mathrm{d}x + c_2\int\limits_\Omega f_2\,\mathrm{d}x\,.$$

Theorem 1.3.2. *If a function f is integrable on a measurable set Ω and a function g is equivalent to f on Ω, then g is also integrable on Ω and*

$$\int_{\Omega} f \,\mathrm{d}x = \int_{\Omega} g \,\mathrm{d}x .$$

Corollary 1.3.1. *If* $\operatorname{meas}\Omega = 0$, *then, for any function f defined on Ω,* $\int_{\Omega} f \,\mathrm{d}x = 0$.

Corollary 1.3.2. *If a function is integrable on a measurable set Ω_1 and a set $\Omega_2 \subset \Omega_1$ is such that* $\operatorname{meas}(\Omega_1 \setminus \Omega_2) = 0$, *then* $\int_{\Omega_2} f \,\mathrm{d}x = \int_{\Omega_1} f \,\mathrm{d}x$.

Theorem 1.3.3. *If a function f is integrable on a measurable set Ω_1, then it is integrable on any of its measurable subsets Ω_2. If, further, it is real-valued and non-negative, then*

$$\int_{\Omega_2} f \,\mathrm{d}x \leqslant \int_{\Omega_1} f \,\mathrm{d}x .$$

Theorem 1.3.4. 1. *If real-valued functions f and g are integrable on a measurable set Ω, and $f \leqslant g$ almost everywhere on Ω, then**

$$\int_{\Omega} f \,\mathrm{d}x \leqslant \int_{\Omega} g \,\mathrm{d}x . \tag{1.3.13}$$

2. *If, further, $f < g$ on a measurable subset of Ω of positive measure, then*

$$\int_{\Omega} f \,\mathrm{d}x < \int_{\Omega} g \,\mathrm{d}x . \tag{1.3.14}$$

3. *Inequality* (1.3.13) *also holds if functions f and g are non-negative and measurable on a measurable set Ω, and $f \leqslant g$ almost everywhere on Ω. (In this case the integrals may be infinite.) If $\int_{\Omega} g \,\mathrm{d}x < \infty$, then also $\int_{\Omega} f \,\mathrm{d}x < \infty$. If $\int_{\Omega} f \,\mathrm{d}x = \infty$, then also $\int_{\Omega} g \,\mathrm{d}x = \infty$.*

Corollary 1.3.3. *If a non-negative function f is measurable on a measurable set Ω of positive measure, then $\int_{\Omega} f \,\mathrm{d}x = 0$ if and only if the function $f \sim 0$ on Ω.*

Corollary 1.3.4. ***(Mean value theorem)*** *If a non-negative function g if integrable on a measurable set Ω, a real-valued function f is measurable on Ω, and, for some $a, b \in \mathbb{R}$, $a \leqslant f(x) \leqslant b$ for almost all $x \in \Omega$, then*

$$a \int_{\Omega} g \,\mathrm{d}x \leqslant \int_{\Omega} f \,\mathrm{d}x \leqslant b \int_{\Omega} g \,\mathrm{d}x .$$

* Sometimes verifying the measurability of the entries of the inequality $f \leqslant g$ may be quite tedious (see, for example, Sections ?? and ??), and many authors do not do this assuming that, roughly speaking, all functions under consideration are measurable. In this book we try to be persistent in verifying the measurability everywhere where required. (See also Remark 1.3.2.)

Theorem 1.3.5. *A function f measurable on a measurable set Ω is integrable on Ω if and only if the function $|f|$ is integrable on Ω. If f is integrable on Ω, then*

$$\left|\int\limits_{\Omega} f \, \mathrm{d}x\right| \leqslant \int\limits_{\Omega} |f| \, \mathrm{d}x .$$

Corollary 1.3.5. *If a function f is measurable on a measurable set Ω, a function g is non-negative and integrable on Ω, and $|f| \leqslant g$ almost everywhere on Ω, then f is also integrable on Ω.*

Without the assumption on measurability of f, the statement of Theorem 1.3.5 is not true. The integrability of f implies the integrability of $|f|$. However, the converse does not hold, as shown by the appropriate example in Section 1.3.2.

Theorem 1.3.6. *Let $\{\Omega_k\}_{k\in\mathbb{N}}$ be a finite or countable family of disjoint measurable sets ($s \in \mathbb{N}$ or $s = \infty$). If a function f is integrable on each of the sets Ω_k and, in the case $s = \infty$, $\sum\limits_{k=1}^{\infty} \int\limits_{\Omega_k} |f| \, \mathrm{d}x < \infty$, then f is integrable on $\bigcup\limits_{k=1}^{s} \Omega_k$ and*

$$\int\limits_{\bigcup\limits_{k=1}^{s} \Omega_k} f \, \mathrm{d}x = \sum_{k=1}^{s} \int\limits_{\Omega_k} f \, \mathrm{d}x$$

(countable additivity *of the Lebesgue integral*).

Let M be a set of indices. By the *multiplicity of covering* of a family of sets $\{\Omega_\mu\}_{\mu\in M}$ we mean the expression

$$\varkappa \equiv \varkappa\left(\{\Omega_\mu\}_{\mu\in M}\right) := \sup_{x\in\mathbb{R}^n} N(x),$$

where $N(x)$ is the number sets Ω_μ containing x. (If $x \in {}^c\big(\bigcup\limits_{\mu\in M} \Omega_\mu\big)$, then $N(x) := 0$.)

Theorem 1.3.7. *Let $\{\Omega_k\}_{k=1}^{s}$ be a finite or countable family of measurable sets ($s \in \mathbb{N}$ or $s = \infty$) and, in the case $s = \infty$, $\varkappa \equiv \varkappa(\{\Omega_k\}_{k=1}^{s}) < \infty$. Then for any function f non-negative and measurable on $\bigcup\limits_{k=1}^{s} \Omega_k$,*

$$\frac{1}{\varkappa} \sum_{k=1}^{s} \int\limits_{\Omega_k} f \, \mathrm{d}x \leqslant \int\limits_{\bigcup\limits_{k=1}^{s} \Omega_k} f \, \mathrm{d}x \leqslant \sum_{k=1}^{s} \int\limits_{\Omega_k} f \, \mathrm{d}x .$$

Theorem 1.3.8. ***(Absolute continuity of the Lebesgue integral)*** *Let a function f be integrable on a measurable set Ω. Then for all $\varepsilon > 0$ there exists $\delta > 0$ such that for all measurable sets $\omega \subset \Omega$ satisfying* $\operatorname{meas} \omega < \delta$ *the inequality*

$$\left|\int\limits_{\omega} f \, \mathrm{d}x\right| < \varepsilon$$

holds.

This statement can be briefly written as $\lim\limits_{\text{meas}\,\omega\to 0+} \int\limits_{\omega} f\,\mathrm{d}x = 0$.

Corollary 1.3.6. *If a function f is integrable on a measurable set Ω and sets $\Omega_k \subset \Omega$, $k \in \mathbb{N}$, are such that for all $r > 0$*

$$\lim_{k\to\infty} \text{meas}((\Omega \setminus \Omega_k) \cap B_r) = 0$$

(in particular, if for all $k \in \mathbb{N}$ $\Omega_k \subset \Omega_{k+1}$ and $\bigcup\limits_{k=1}^{\infty} \Omega_k = \Omega$), then

$$\lim_{k\to\infty} \int\limits_{\Omega_k} f\,\mathrm{d}x = \int\limits_{\Omega} f\,\mathrm{d}x\,.$$

Theorem 1.3.9. ***(The Fatou theorem)*** *Let for all $k \in \mathbb{N}$ functions f_k be non-negative and measurable on a measurable set Ω. Assume that for almost all $x \in \Omega$ there exists a finite or infinite limit $\lim\limits_{k\to\infty} f_k(x)$. Then*

$$\int\limits_{\Omega} \lim_{k\to\infty} f_k\,\mathrm{d}x \leqslant \liminf_{k\to\infty} \int\limits_{\Omega} f\,\mathrm{d}x\,. \tag{1.3.15}$$

(If $\lim\limits_{k\to\infty} f_k(x) = \infty$ on a subset of Ω of positive measure, it is assumed that $\int\limits_{\Omega} \lim\limits_{k\to\infty} f_k\,\mathrm{d}x = \infty$.)

In general, the inequality sign cannot be replaced by the equality sign, as shown by Example 1.3.4 below.

For an arbitrary sequence $\{f_k\}_{k\in\mathbb{N}}$ of functions f_k non-negative and measurable on a measurable set Ω the Fatou theorem can be formulated in the following way:

$$\int\limits_{\Omega} \liminf_{k\to\infty} f_k\,\mathrm{d}x \leqslant \liminf_{k\to\infty} \int\limits_{\Omega} f_k\,\mathrm{d}x\,. \tag{1.3.16}$$

Exercise 1.3.8. It is clear that inequality (1.3.15) implies that

$$\int\limits_{\Omega} \lim_{k\to\infty} f_k\,\mathrm{d}x \leqslant \sup_{k\in\mathbb{N}} \int\limits_{\Omega} f_k\,\mathrm{d}x\,. \tag{1.3.17}$$

Prove that, if in the statement of Theorem 1.3.9 inequality (1.3.15) is replaced by this inequality, then the amended theorem implies inequality (1.3.15).

Theorem 1.3.10. *For all $k \in \mathbb{N}$, let functions f_k be non-negative and measurable on a measurable set Ω. Assume that for almost all $x \in \Omega$ there exists a finite or infinite limit $\lim\limits_{k\to\infty} f_k(x) =: F(x)$, and $f_k(x) \leqslant F(x)$ for all $k \in \mathbb{N}$ and for almost all $x \in \Omega$. Then*

$$\lim_{k\to\infty} \int\limits_{\Omega} f_k\,\mathrm{d}x = \int\limits_{\Omega} \lim_{k\to\infty} f_k\,\mathrm{d}x\,. \tag{1.3.18}$$

It is not ruled out that both parts of this equality are infinite.

Exercise 1.3.9. Prove Theorem 1.3.10 by applying Theorem 1.3.9.

Corollary 1.3.7. *(Monotone Convergence Theorem*) For all $k \in \mathbb{N}$, let functions f_k be non-negative and measurable on a measurable set Ω. Assume that for all $k \in \mathbb{N}$ and for almost all $x \in \Omega$ $f_k(x) \leqslant f_{k+1}(x)$. Then equality* (1.3.18) *holds.*

Corollary 1.3.8. *For all $k \in \mathbb{N}$, let functions f_k be non-negative and measurable on a measurable set Ω. Then*

$$\int\limits_\Omega \left(\sum_{k=1}^{\infty} f_k\right) \mathrm{d}x = \sum_{k=1}^{\infty}\left(\int\limits_\Omega f_k \,\mathrm{d}x\right).$$

Corollary 1.3.9. *For all $k \in \mathbb{N}$, let functions f_k be measurable on a measurable set Ω. If*

$$\sum_{k=1}^{\infty} \int\limits_\Omega |f_k| \,\mathrm{d}x < \infty .$$

then the series $\sum\limits_{k=1}^{\infty} f_k(x)$ converges for almost all $x \in \Omega$.

Next we formulate one of the most usable sufficient conditions ensuring the possibility of passing to the limit under the integral sign.

Theorem 1.3.11. *(Dominated Convergence Theorem) For all $k \in \mathbb{N}$, let functions f_k be measurable on a measurable set Ω. Assume that for almost all $x \in \Omega$ there exists a finite limit $\lim\limits_{k\to\infty} f_k(x)$ and there exists a function g non-negative, integrable on Ω and such that for all $k \in \mathbb{N}$ and for almost all $x \in \Omega$*

$$|f_k(x)| \leqslant g(x) . \tag{1.3.19}$$

Then for all $k \in \mathbb{N}$ the functions f_k and the limit function $\lim\limits_{k\to\infty} f_k(x)$ are integrable on Ω and equality (1.3.18) *holds.*

Note that the smallest function g satisfying inequality (1.3.19) is the function g_0 defined for all $x \in \Omega$ by $g_0(x) := \sup\limits_{k\in\mathbb{N}} |f_k(x)|$.

Example 1.3.4. Let $n = 1$, $\mu \geqslant 0$ and, for all $k \in \mathbb{N}$, functions f_k be defined by: $f_k(x) := k^\mu$ if $0 < x < \frac{1}{k}$ and $f_k(x) := 0$ if $\frac{1}{k} \leqslant x \leqslant 1$. In this case, for all $x \in (0,1)$ $\lim\limits_{k\to\infty} f_k(x) = 0$ and $2^{-\mu}x^{-\mu} \leqslant g_0(x) \leqslant x^{-\mu}$. If $0 \leqslant \mu < 1$, then the function g_0 is integrable on $(0,1)$ and, by Theorem 1.3.11, one can state that $\lim\limits_{k\to\infty} \int\limits_0^1 f_k \,\mathrm{d}x = 0$. If $\mu \geqslant 1$, then the function g_0 is not integrable on $(0,1)$. One can easily verify directly that in this case it is not possible to pass to the limit under integral sign.

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* Known also as the Levi theorem.

Remark 1.3.10. If a sequence $\{f_k\}_{k\in\mathbb{N}}$ of functions f_k, measurable on a measurable set Ω of finite measure, converges uniformly on Ω to a function integrable on Ω, then there exists a function g, non-negative and integrable on Ω, for which inequality (1.3.19) holds, hence so does equality (1.3.18). The converse is not true as shown by the above example with $0 < \mu < 1$.

Exercise 1.3.10. Prove the statement of Remark 1.3.10.

Condition (1.3.19) implies that there exists $A > 0$ such that for all $k \in \mathbb{N}$

$$\int\limits_{\Omega} |f_k| \,\mathrm{d}x \leqslant A. \tag{1.3.20}$$

Condition (1.3.20) is not sufficient for the validity of equality (1.3.18), as shown by example 1.3.4 with $\mu = 1$. However, a condition slightly stronger than (1.3.20) is already sufficient for the validity of equality (1.3.18). This is shown by the following theorem.

Theorem 1.3.12. ***(The Vitali—Vallée Poussin theorem)*** *For all $k \in \mathbb{N}$, let functions f_k be measurable on a measurable set Ω. Assume that for almost all $x \in \Omega$ there exists a finite limit $\lim\limits_{k\to\infty} f_k(x)$. If there exist $A > 0$ and a positive non-decreasing function Φ defined on $[0,\infty)$ satisfying $\lim\limits_{t\to\infty} \Phi(t) = \infty$, such that for all $k \in \mathbb{N}$*

$$\int\limits_{\Omega} |f_k|\, \Phi(|f_k|) \,\mathrm{d}x \leqslant A, \tag{1.3.21}$$

then the functions f_k and the limit function $\lim\limits_{k\to\infty} f_k(x)$ are integrable on Ω, and equality (1.3.18) *holds.*

Under some additional assumptions, condition (1.3.21) is also necessary for the validity of equality (1.3.18). See (Natanson 1974) for details.

Remark 1.3.11. Condition (1.3.21) is clearly satisfied if there exist $A > 0$ and $\varepsilon > 0$ such that

$$\int\limits_{\Omega} |f_k|^{1+\varepsilon} \,\mathrm{d}x \leqslant A\,. \tag{1.3.22}$$

Exercise 1.3.11. Let $n = 1$, $\mu \geqslant 0$ and, for all $k \in \mathbb{N}$, functions f_k be defined by: $f_k(x) := \varphi(k)$ if $0 < x < \frac{1}{k}$ and $f_k(x) := 0$ if $\frac{1}{k} \leqslant x < 1$, where φ is a positive non-decreasing function on $[1,\infty)$ such that $\lim\limits_{k\to\infty} \varphi(x) = \infty$. Clearly, $\lim\limits_{k\to\infty} f_k(x) = 0$ for all $x \in (0,1)$ and $\lim\limits_{k\to\infty} \int\limits_0^1 f_k \,\mathrm{d}x = 0 \Leftrightarrow \lim\limits_{x\to\infty} \frac{\varphi(x)}{x} = 0$. Prove that the assumptions of Theorem 1.3.11 are satisfied if and only if $\int\limits_1^\infty \frac{\varphi(x)}{x^2}\,\mathrm{d}x < \infty$ (not always when $\lim\limits_{k\to\infty} \int\limits_0^1 f_k \,\mathrm{d}x = 0$), whilst the assumptions of Theorem 1.3.12 are satisfied if and only if $\lim\limits_{x\to\infty} \frac{\varphi(x)}{x} = 0$ (always when $\lim\limits_{k\to\infty} \int\limits_0^1 f_k \,\mathrm{d}x = 0$).