Next we consider functions f, defined on the direct product $\Omega \times G$ of $\Omega \subset \mathbb{R}^n$ and $G \subset \mathbb{R}^m$. We shall also denote them as $f(\cdot,\cdot)$. Theorem 1.3.11 implies the following Theorems 1.3.13 – 1.3.16.

Theorem 1.3.13. ***(Measurability of integrals depending on a parameter)*** *Let Ω be a measurable set in $\mathbb{R}^n$ and G be a measurable set in $\mathbb{R}^m$.*

Assume that a function f is measurable on $\Omega \times G$ and for almost all $y \in G$ the functions $f(\cdot, y)$ are integrable on Ω. Then the function $\int\limits_{\Omega} f(x,\cdot)\,\mathrm{d}x$ is measurable on G.

Theorem 1.3.14. ***(Passing to the limit under the integral sign for integrals depending on a parameter)*** *Let Ω be a measurable set in $\mathbb{R}^n$ and G be an infinite set in $\mathbb{R}^m$.*

Assume that a function f is defined on $\Omega \times G$ and that $y_0 \in \mathbb{R}^m$ is a limit point of G. Furthermore, assume that for almost all $x \in \Omega$ there exists a finite limit $\lim\limits_{y \to y_0, y \in G} f(x,y)$ and that, for some $\delta > 0$, for all $y \in G \cap B(y_0,\delta)$ the functions $f(\cdot,y)$ are measurable on Ω. If there exists a function g non-negative and integrable on G such that for all $y \in G \cap B(y_0,\delta)$ for almost all $x \in G$

$$|f(x,y)| \leqslant g(x)\,, \tag{1.3.23}$$

then the functions $f(\cdot,y)$ and the limit function $\lim\limits_{y \to y_0, y \in G} f(x,y)$ are integrable on Ω for all $y \in G \cap B(y_0,\delta)$ and

$$\lim_{y \to y_0, y \in G} \int\limits_{\Omega} f(x,y)\,\mathrm{d}x = \int\limits_{\Omega} \lim_{y \to y_0, y \in G} f(x,y)\,\mathrm{d}x\,.$$

If G is unbounded, then $y_0 = \infty$ is admissible. In this case $B(y_0,\delta)$ should be everywhere replaced by ${}^cB(y_0,\delta)$.

Theorem 1.3.15. ***(Continuity of integrals depending on a parameter)*** *Let Ω be a measurable set in $\mathbb{R}^n$ and G be a set in $\mathbb{R}^m$.*

Assume that a function f is defined on $\Omega \times G$, for all $y \in G$ the functions $f(\cdot,y)$ are measurable on Ω and for almost all $x \in G$ the functions $f(x,\cdot)$ are continuous on G.

If there exists a function g, non-negative and integrable on G, such that for all $y \in G$ for almost all $x \in G$ inequality (1.3.23) is satisfied, then for all $y \in G$ the functions $f(\cdot,y)$ are integrable on Ω and the function $\int\limits_{\Omega} f(x,\cdot)\,\mathrm{d}x$ is continuous on G.

Theorem 1.3.16. ***(Differentiation of integrals depending on a parameter)*** *Let Ω be a measurable set in $\mathbb{R}^n$ and G be an open set in $\mathbb{R}^m$.*

Assume that a function f is defined on $\Omega \times G$, for all $y \in G$ the functions $f(\cdot,y)$ are integrable on Ω and, for some $j \in \{1,\dots,m\}$, for almost all $x \in G$ for all $y \in G$ there exist the derivative $\frac{\partial f}{\partial y_j}(x,y)$.

If for all compacts $K \subset G$ there exist functions g_K, non-negative and integrable on G, such that for all $y \in K$ for almost all $x \in \Omega$

$$\left|\frac{\partial f}{\partial y_j}(x,y)\right| \leqslant g_K(x)\,,$$

then for all $y \in G$ *the functions* $\frac{\partial f}{\partial y_j}(\cdot, y)$ *are integrable on* Ω *and for all* $y \in G$

$$\frac{\partial}{\partial y_j}\Big(\int\limits_{\Omega} f(x,y)\,\mathrm{d}x\Big) = \int\limits_{\Omega} \frac{\partial f}{\partial y_j}(x,y)\,\mathrm{d}x\,.$$

Corollary 1.3.10. *Let* $l \in \mathbb{N}$, Ω *be a measurable set in* $\mathbb{R}^n$ *and* G *be an open set in* $\mathbb{R}^m$.

Assume that a function f *is defined on* $\Omega \times G$, *for all* $y \in G$ *the functions* $f(\cdot, y)$ *are measurable on* Ω *and for almost all* $x \in G$ $f(x, \cdot) \in C^l(G)$.

If for all compacts $K \subset G$ *there exist functions* g_K, *non-negative and integrable on* G, *such that for all* $\alpha \in \mathbb{N}_0^n$ *satisfying* $|\alpha| \leqslant l$ *and for all* $y \in K$ *for almost all* $x \in \Omega$

$$\big|(D_y^\alpha f)(x,y)\big| \leqslant g_K(x)\,,$$

then for all $y \in G$ *the functions* $\frac{\partial f}{\partial y_j}(\cdot, y)$ *are integrable on* Ω, $\int\limits_{\Omega} f(x,\cdot)\,\mathrm{d}x \in C^l(G)$ *and for all* $\alpha \in \mathbb{N}_0^n$ *satisfying* $|\alpha| \leqslant l$ *for all* $y \in G$

$$D_y^\alpha\Big(\int\limits_{\Omega} f(x,y)\,\mathrm{d}x\Big) = \int\limits_{\Omega} (D_y^\alpha f)(x,y)\,\mathrm{d}x\,.$$

Exercise 1.3.12. Prove theorems 1.3.14 – 1.3.16 by applying Theorem 1.3.11.

Theorem 1.3.17. ***(The Fubini theorem)*** *Let* Ω *be a measurable set in* $\mathbb{R}^n$, G *be a measurable set in* $\mathbb{R}^m$, *and let a function* f *be integrable on* $\Omega \times G$.

Then for almost all $x \in \Omega$ *the functions* $f(x, \cdot)$ *are integrable on* G, *for almost all* $y \in G$ *the functions* $f(\cdot, y)$ *are integrable on* Ω *and*

$$\int\limits_{\Omega\times G} f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int\limits_{G}\Big(\int\limits_{\Omega} f(x,y)\,\mathrm{d}x\Big)\,\mathrm{d}y = \int\limits_{\Omega}\Big(\int\limits_{G} f(x,y)\,\mathrm{d}y\Big)\,\mathrm{d}x\,. \qquad (1.3.24)$$

Corollary 1.3.11. *Let* Ω *be a measurable set in* $\mathbb{R}^n$, G *be a measurable set in* $\mathbb{R}^m$, *a function* f *be measurable on* $\Omega \times G$, *and at least one of the integrals*

$$\int\limits_{G}\Big(\int\limits_{\Omega} |f(x,y)|\,\mathrm{d}x\Big)\,\mathrm{d}y\,, \quad \int\limits_{\Omega}\Big(\int\limits_{G} |f(x,y)|\,\mathrm{d}y\Big)\,\mathrm{d}x$$

be finite. Then all three integrals in (1.3.24) *are finite and equalities* (1.3.24) *hold.*

Corollary 1.3.12. *Let* Ω *be a measurable set in* $\mathbb{R}^n$, G *be a measurable set in* $\mathbb{R}^m$, *a function* f *be non-negative and measurable on* $\Omega \times G$. *Then equalities* (1.3.24) *hold. (In this case the integrals may be infinite.)*

Corollaries 1.3.11 and 1.3.12 are most commonly used sufficient conditions ensuring the possibility of interchanging the order of integration:

$$\int\limits_{G}\Big(\int\limits_{\Omega} f(x,y)\,\mathrm{d}x\Big)\,\mathrm{d}y = \int\limits_{\Omega}\Big(\int\limits_{G} f(x,y)\,\mathrm{d}y\Big)\,\mathrm{d}x\,. \qquad (1.3.25)$$

This equality may also be considered as the formula of integration under integral sign for integrals depending on a parameter.

We would like to emphasize that the assumptions concerning integrability of a function f on $\Omega\times G$ in Corollary 1.3.11 and measurability of f on $\Omega\times G$ in Corollary 1.3.12 are essential, though not necessary, for the validity of equality (1.3.25).

If a function f is not integrable on $\Omega \times G$, then the integrals in (1.3.25) may not exist or they may exist but be not equal. (See Exercise 1.5.10.)

If a function f is non-negative but not measurable on $\Omega \times G$, then it may happen that one of the integrals in (1.3.25) exists and the other one does not. Let, for example, $n = m = 1$, $\Omega = G = (-1,1)$ and $f(x,y) = x\chi(y) + 1$ for $x, y \in (-1,1)$, where χ is the characteristic function of a non-measurable subset of the interval $(-1,1)$. Then $\int\limits_{-1}^{1}\left(\int\limits_{-1}^{1} f(x,y)\,\mathrm{d}x\right)\mathrm{d}y = 4$ whilst $\int\limits_{-1}^{1}\left(\int\limits_{-1}^{1} f(x,y)\,\mathrm{d}y\right)\mathrm{d}x$ does not exist because for all $x \in (-1,1)$ except $x = 0$ the functions $f(x,\cdot)$ are not measurable on $(-1,1)$.

Finally, note that if f is defined not on a direct product $\Omega\times G$ but on an arbitrary measurable set $F \subset \mathbb{R}^n$, then one can choose appropriate Ω and G satisfying $F \subset \Omega \times G$, extend f by 0 outside F and then apply equality (1.3.12) for interchanging the order of integration.

Theorem 1.3.18. ***(The Lebesgue theorem on differentiation of integrals)*** *Let Ω be an open set in $\mathbb{R}^n$ and a function f be* locally integrable *on Ω, i.e. integrable on all compacts $K \subset \Omega$. Then for almost all $x \in \Omega$*

$$\lim_{r\to 0+} \frac{1}{\operatorname{meas} B(x,r)} \int\limits_{B(x,r)} f\,\mathrm{d}y = f(x)\,. \tag{1.3.26}$$

If $n = 1$, $\Omega = (a,b)$, then (1.3.26) takes the form: for almost all $x \in (a,b)$

$$\lim_{r\to 0+} \frac{1}{2r} \int\limits_{x-r}^{x+r} f\,\mathrm{d}y = f(x)\,.$$

In this case a slightly stronger statement holds: for almost all $x \in (a,b)$

$$\left(\int\limits_{x_0}^{x} f\,\mathrm{d}y\right)' = \lim_{r\to 0+} \frac{1}{r} \int\limits_{x}^{x+r} f\,\mathrm{d}y = f(x)\,,$$

where x_0 is a fixed point of the interval (a,b) (see Section 1.4), which explains the name this theorem bears.

Theorem 1.3.19. ***(Substitutions in the Lebesgue integrals)*** *Let Ω be a measurable set in $\mathbb{R}^n$, G an open set in $\mathbb{R}^n$, and $\Omega \subset G$. Assume that a transformation $x := g(y)$ is of class $\overline{C}^1(G)$, i. e for all $j \in \{1,\dots,n\}$ $x_j := g_j(y)$ where $g_j : \Omega \to \mathbb{R}^n$ are of class $\overline{C}^1(G)$, and is a one-to-one map of G onto $g(G)$.*

Then a function f is integrable on $g(\Omega)$ if and only if the function $f(g)\operatorname{Jac}(g)$ is integrable on Ω, where $\operatorname{Jac}(g)$ is the Jacobian determinant of g. Moreover, the equality

$$\int\limits_{g(\Omega)} f\,\mathrm{d}x = \int\limits_{\Omega} f(g)\,|\operatorname{Jac}(g)|\,\mathrm{d}y \tag{1.3.27}$$

holds.

Note that, since the transformation $g \in \overline{C}^1(G)$ is one-to-one, either for all $x \in G$ $\mathrm{Jac}\,(g)(x) \geqslant 0$ or for all $x \in G$ $\mathrm{Jac}\,(g)(x) \leqslant 0$ (Nikol'skiĭ 1983).

Equality (1.3.27) also holds for non-negative measurable functions f and transformations $g \in C^1(G)$, given that the other conditions of the theorem are satisfied. In this case the integrals may be infinite.

If $n = 1$, $\Omega = G = (a, b)$, equality (1.3.27) takes the form

$$\int\limits_{g(a,b)} f\,\mathrm{d}x = \int\limits_{(a,b)} f(g)|g'|\,\mathrm{d}y.$$

Since by the assumptions of Theorem 1.3.19 g is one-to-one, hence either $g'(x) \geqslant 0$ for all $x \in (a, b)$ or $g'(x) \leqslant 0$ for all $x \in (a, b)$, this equality is equivalent to

$$\int\limits_{g(a)}^{g(b)} f\,\mathrm{d}x = \int\limits_{a}^{b} f(g)g'\,\mathrm{d}y.$$

For conditions ensuring the validity of this formula without the assumption that g is one-to-one, see Section 1.4.

Example 1.3.5. (Generalized spherical coordinates) If $n \geqslant 2$, the transformation $x := g(y)$, where $y := (\varrho, \varphi_1, \ldots, \varphi_{n-1})$, defined by

$$\begin{cases} x_n & := \varrho \sin \varphi_{n-1} \\ x_{n-1} & := \varrho \cos \varphi_{n-1} \sin \varphi_{n-2} \\ & \vdots \\ x_2 & := \varrho \cos \varphi_{n-1} \cos \varphi_{n-2} \cdots \cos \varphi_2 \sin \varphi_1 \\ x_1 & := \varrho \cos \varphi_{n-1} \cos \varphi_{n-2} \cdots \cos \varphi_2 \cos \varphi_1 \end{cases}$$

is a one-to-one map of the set

$$\left\{(\varrho, \varphi_1, \ldots, \varphi_{n-1}) \in \mathbb{R}^n : 0 < \varrho < \infty, 0 \leqslant \varphi_1 < 2\pi, -\frac{\pi}{2} \leqslant \varphi_2, \ldots, \varphi_{n-1} \leqslant \frac{\pi}{2}\right\}$$

onto $\mathbb{R}^n \setminus \{0\}$, and

$$\mathrm{Jac}\,(g) = \varrho^{n-1} \cos^{n-2} \varphi_{n-1} \cos^{n-3} \varphi_{n-2} \cdots \cos \varphi_2.$$

For all $r > 0$ and $r = \infty$, for all functions f integrable on B_r ($B_\infty \equiv \mathbb{R}^n$) or non-negative and measurable on B_r, the equality

$$\int\limits_{B_r} f\,\mathrm{d}x = \int\limits_0^r \varrho^{n-1}\,\mathrm{d}\varrho \int\limits_0^{2\pi} \mathrm{d}\varphi_1 \int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos \varphi_2\,\mathrm{d}\varphi_2 \cdots \int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}} F\,\mathrm{d}\varphi_{n-1} \tag{1.3.28}$$

holds, where $F := f(\varrho \cos \varphi_{n-1} \cdots \cos \varphi_1, \ldots, \varrho \sin \varphi_{n-1}) \cos^{n-2} \varphi_{n-1}$.

Exercise 1.3.13. By applying equality (1.3.28), prove that if $0 < r \leqslant \infty$, the function $g(\varrho)\varrho^{n-1}$ is integrable on $(0,r)$ or the function g is nonnegative and measurable on $(0,r)$, then

$$\int\limits_{B_r} g(|x|)\,\mathrm{d}x = \sigma_n \int\limits_0^r g(\varrho)\varrho^{n-1}\,\mathrm{d}\varrho\,, \tag{1.3.29}$$

where $\sigma_n := nv_n$ is the surface area of the unit sphere S_{n-1} in $\mathbb{R}^n$.

Exercise 1.3.14. Let $\alpha \in \mathbb{R}$, $r > 0$. By applying equality (1.3.29), prove that the integral $\int\limits_{B_r} |x|^\alpha\,\mathrm{d}x$ converges if and only if $\alpha > -n$, whereas $\int\limits_{{}^cB_r} |x|^\alpha\,\mathrm{d}x$ if and ondef:measurable function, steply if $\alpha < -n$. Moreover,

$$\int\limits_{B_r} |x|^\alpha\,\mathrm{d}x = \frac{\sigma_n}{\alpha+n} r^{\alpha+n}, \quad \alpha > -n\,, \qquad \int\limits_{{}^cB_r} |x|^\alpha\,\mathrm{d}x = \frac{\sigma_n}{|\alpha+n|} r^{\alpha+n}, \quad \alpha < -n\,.$$

Exercise 1.3.15. Prove Theorem 1.3.13 by reducing it to the case of bounded measurable functions defined on bounded measurable sets, and applying Definition 1.3.12, Dominated Convergence Theorem (see Theorem 1.3.11 and the closedness of the space of measurable functions with respect to passing to the limit.

1.4 Absolute continuity and the Lebesgue integration

In this section we consider, unless specifically stated, complex-valued function of one real variable defined on a bounded closed interval. Proofs of the statements below and further results can be found in (Natanson 1974).

Definition 1.4.1. *Let $-\infty < a < b < \infty$. A function f is* absolutely continuous *on $[a,b]$ if for all $\varepsilon > 0$ there exists $\delta > 0$ such that for all finite families of disjoint intervals $(a_1,b_1),\dots,(a_m,b_m)$, contained in $[a,b]$ and satisfying $\sum\limits_{k=1}^m (b_k - a_k) < \delta$, the inequality*

$$\sum_{k=1}^m |f(b_k) - f(a_k)| < \varepsilon \tag{1.4.1}$$

holds.

The assumption that the intervals (a_k,b_k) are disjoint is essential. (See Exercise 1.5.16 below.)

Clearly each function absolutely continuous on $[a,b]$ is uniformly continuous on $[a,b]$. Converse does not hold. For example, the function f defined by: $f(x) := x\sin\frac{1}{x}$ if $x \in (0,1]$, $f(0) := 0$, is uniformly continuous on $[0,1]$ and infinitely differentiable on $(0,1]$, but is not absolutely continuous on $[0,1]$.

If a function f satisfies the *Lipchitz condition* on $[a,b]$, i.e. there exists $M > 0$ such that for all $x,y \in [a,b]$

$$|f(x) - f(y)| \leqslant M\,|x-y|$$

(in particular, if f is continuous on $[a,b]$ and has a bounded derivative on (a,b)), then f is absolutely continuous on $[a,b]$.

If functions f and g are absolutely continuous on $[a,b]$, then the functions $f \pm g$, fg and, if for all $x \in [a,b]$ $g(x) \neq 0$, $\frac{f}{g}$ are also absolutely continuous on $[a,b]$.

Theorem 1.4.1. *If $-\infty < a < b < \infty$ and a function f is absolutely continuous on $[a,b]$, then for almost all $x \in [a,b]$ there exists the derivative $f'(x)$, the derivative f' is integrable on $[a,b]$ and for all $x \in [a,b]$*

$$f(x) = f(x_0) + \int_{x_0}^{x} f' \, \mathrm{d}y \,, \tag{1.4.2}$$

where x_0 is a fixed point of $[a,b]$.

Theorem 1.4.2. *If a function f is integrable on $[a,b]$, $x_0 \in [a,b]$, then the function $\int_{x_0}^{x} f \, \mathrm{d}y$, $x \in [a,b]$, is absolutely continuous on $[a,b]$ and for almost all $x \in [a,b]$*

$$\left(\int_{x_0}^{x} f \, \mathrm{d}y \right)' = f(x) \,. \tag{1.4.3}$$

In particular, the equality holds in all points of continuity of f.

Corollary 1.4.1. *A function f is absolutely continuous on $[a,b]$ if and only if it is continuous on $[a,b]$, the derivative exists almost everywhere and is integrable on $[a,b]$, and for all $x \in [a,b]$ equality 1.4.2 holds for any fixed $x_0 \in [a,b]$.*

Corollary 1.4.2. *Let a function f be continuous on $[a,b]$ and let the derivative exist everywhere on (a,b). Then f is absolutely continuous on $[a,b]$ if and only if the derivative f' is integrable on (a,b).*

Corollary 1.4.3. ***(The Newton—Leibnitz formula)*** *Let a function f be integrable on $[a,b]$. If a function F absolutely continuous on $[a,b]$ is an* almost antiderivative *of f on $[a,b]$ ($\Longleftrightarrow F'(x) = f(x)$ for almost all $x \in [a,b]$), then*

$$\int_{a}^{b} f \, \mathrm{d}x = F(b) - F(a) \,. \tag{1.4.4}$$

The assumption about the absolute continuity of F is essential as the following example shows.

Example 1.4.1. (Cantor's function) Let D be Cantor's set constructed in Example 1.3.1. On the constituent interval of the open set $[0,1] \setminus D$ of length $\frac{1}{3}$ we set $\theta(x) := \frac{1}{2}$. On the first of the constituent intervals of length $\frac{1}{3^2}$ we set $\theta(x) = \frac{1}{4}$, on the second $\theta(x) = \frac{3}{4}$. In general, on 2^{k-1} intervals of length 3^{-k}, we set $\theta(x)$ to be equal successively to $2^{-k}, 3\,2^{-k}, \ldots, (2^k - 1)\,2^{-k}$. Next $\theta(0) := 0$, $\theta(1) := 1$ and for all $x \in D$, except 0 an 1, $\theta(x) := \sup\limits_{y \in [0,1] \setminus D, y < x} \theta(y)$. The function θ is *Cantor's function*. It is non-decreasing and continuous on $[0,1]$. Moreover, $\theta'(x) = 0$ for almost all $x \in [0,1]$. It is not absolutely continuous on $[0,1]$ because equality (1.4.2) does not hold. The function θ is an almost antiderivative for $f \equiv 0$, but equality (1.4.4) with $a = 1, b = 1$ is not valid.

Theorem 1.4.3. ***(Integration by parts)*** *If functions f and g are absolutely continuous on $[a,b]$, then*

$$\int_a^b fg'\,dx = fg\Big|_a^b - \int_a^b f'g\,dx\,.$$

Theorem 1.4.4. ***(Substitutions in the Lebesgue integrals)*** *Let $-\infty \leqslant a < b \leqslant \infty$ and let a real-valued function g be* locally absolutely continuous *on (a,b) ($\Longleftrightarrow$ g is absolutely continuous on every closed interval $[\alpha,\beta] \subset (a,b)$) and there exist finite or infinite limits $g(a+) = \lim_{x\to a+} g(x)$, $g(b-) = \lim_{x\to b-} g(x)$. Moreover, let a function f be integrable on $g\big((a,b)\big)$ and the function $f(g)g'$ be integrable on (a,b). Then*

$$\int_{g(a+)}^{g(b-)} f\,dx = \int_a^b f(g)g'\,dy\,. \tag{1.4.5}$$

The assumption about local absolute continuity of g is essential. If for example $(a,b) := (-1,1)$, $f := 1$, $g := \theta$, then $g'(x) = 0$ if $x \neq 0$, both integrals in (1.4.5) exist, but equality (1.4.5) does not hold.

Note also that local absolute continuity of g on (a,b) and integrability of f on $g\big((a,b)\big)$ does not imply integrability of the product $f(g)g'$ on (a,b) as shown by Example 1.4.1 below. Under the assumption that the product $f(g)g'$ *locally integrable* on (a,b) equality (1.4.5) holds in the integral in the right-hand side is understood in the improper sense:

$$\int_a^b f(g)g'\,dy = \lim_{\varepsilon_1,\varepsilon_2\to 0+} \int_{a+\varepsilon_1}^{b-\varepsilon_2} f(g)g'\,dy\,.$$

(See Remark 1.3.6.)

Let a real-valued function g be absolutely continuous on $[a,b]$ and f be absolutely continuous on $g\big([a,b]\big)$. If, in addition, f satisfies the Lipschitz condition or g is monotonic, then the composition $f(g)$ is absolutely continuous on $[a,b]$. In general, the composition $f(g)$ would not necessarily absolutely continuous. See Exercise 1.5.16.

Furthermore, for any real-valued function h continuous on $[a,b]$, there exist real-valued functions g_k absolutely continuous on $[a,b]$ and real-valued functions f_k absolutely continuous on $g_k\,([a,b])$, $k \in \{1,2,3\}$, such that $h = f_1(g_1)+f_2(g_2)+f_3(g_3)$, with number 3 not being replaceable by a smaller number (Bari 1930a,1930b).

However, if functions g and g' are both absolutely continuous on $[a,b]$, then the product $f(g)g'$ is absolutely continuous on $[a,b]$, albeit the factor $f(g)$ might be not absolutely continuous (Burenkov 1975).

Exercise 1.4.1. Let $f(0) := g(0) := 0$ and for all $x > 0$ set $f(x) = \frac{1}{\sqrt{x}}$, $g(x) := x^6\left(\sin\frac{1}{x^3}+2\right)$. Prove that function g is absolutely continuous on $[0,1]$, f is integrable on $g\,([0,1])$ but $f(g)g'$ is not integrable on $[0,1]$.

Exercise 1.4.2. Let $f(0) := g(0) := 0$ and for all $x > 0$ let $f(x) := \sqrt[3]{x}$ and $g(x) := x^6\left(\sin\frac{1}{x^3}+2\right)$. Prove that g is absolutely continuous on $[0,1]$ and that f is absolutely continuous on $g([0,1])$ but $f(g)$ is not absolutely continuous on $[0,1]$.

1.5 Further exercises

Exercise 1.5.1. Prove that if $\Omega \subset \mathbb{R}^n$ is a bounded set and a function f is uniformly continuous on Ω, then f is bounded on Ω.

Next, let Ω be an unbounded set, and let a function f be, for all $r > 0$, uniformly continuous on $\Omega \cap B_r$. Prove that if there exists a finite limit $\lim\limits_{y\to\infty, y\in\Omega} f(y)$, then $f \in \overline{C}(\Omega)$. For any unbounded set Ω, give an example showing that the converse does not hold.

Exercise 1.5.2. Let f be uniformly continuous on $\mathbb{R}^n$. Prove that there exist $A, B > 0$ such that for all $x \in \mathbb{R}^n$

$$|f(x)| \leqslant A\,|x| + B.$$

Exercise 1.5.3. Let

$$f(x_1, x_2) = \begin{cases} \frac{x_1 x_2 (x_1^2 - x_2^2)}{x_1^2 + x_2^2} & \text{if } x_1^2 + x_2^2 \neq 0, \\ 0 & \text{if } x_1^2 + x_2^2 = 0. \end{cases}$$

Prove that

$$\frac{\partial^2 f}{\partial x_1 \partial x_2}(0,0) = 1, \quad \frac{\partial^2 f}{\partial x_2 \partial x_1}(0,0) = -1\,.$$

Exercise 1.5.4. Let

$$f(x_1, x_2) = \begin{cases} e^{-\frac{1}{x_1^2} - \frac{1}{x_2^2}} \left(e^{-\frac{1}{x_1^2}} + e^{-\frac{1}{x_2^2}} \right) & \text{if } x_1 x_2 \neq 0, \\ 0 & \text{if } x_1 x_2 = 0. \end{cases}$$

Prove that

1) if $x_1 x_2 \neq 0$, then for all $\alpha \in \mathbb{N}_0^2$

$$D^\alpha f(x_1, x_2) = \Bigg(\sum_{\beta \in \mathbb{N}:\ |\beta| = |\alpha|} Q_{\alpha,\beta}\left(\frac{1}{x_1}, \frac{1}{x_2}\right) e^{-\frac{1+2\beta_1}{x_1^2} - \frac{1+2\beta_2}{x_2^2}} \Bigg) \left(e^{-\frac{1}{x_1^2}} + e^{-\frac{1}{x_2^2}} \right),$$

where $Q_{\alpha,\beta}$ are some polynomials,

2) if $x_1 x_2 = 0$, then for all $\alpha \in \mathbb{N}_0^2$ $D^\alpha f(x_1, x_2) = 0\,,$

3) f is discontinuous at $(0,0)\,.$

Exercise 1.5.5. Let $\Omega \subset \mathbb{R}^n$ be a measurable set.

1. If $\operatorname{meas}\Omega = \infty$, prove that there exist disjoint subsets Ω_k, $k \in \mathbb{N}$, such that $\Omega = \bigcup\limits_{k=1}^{\infty} \Omega_k$ and $\operatorname{meas}\Omega_k = 1$ for all $k \in \mathbb{N}$.

2. If $0 < \operatorname{meas}\Omega < \infty$, prove that a) there exist disjoint subsets Ω_1 and Ω_2 such that $\Omega = \Omega_1 \cup \Omega_2$ and $\operatorname{meas}\Omega_1 = \operatorname{meas}\Omega_2$, and hence b) there exist disjoint subsets Ω_k, $k \in \mathbb{N}$, such that $\Omega = \bigcup\limits_{k=1}^{\infty} \Omega_k$ and $\operatorname{meas}\Omega_k = 2^{-k}\operatorname{meas}\Omega$ for all $k \in \mathbb{N}$.

Exercise 1.5.6. By appropriately modifying the method described in Example 1.3.1, for all $a \in (0,1)$ construct a subset of $[0,1]$, which is perfect, nowhere dense in $[0,1]$ and whose measure is equal to a.

Exercise 1.5.7. Prove that there are no subsets of $[0,1]$, which are perfect, nowhere dense in $[0,1]$ and whose measure is equal to 1.

Exercise 1.5.8. Let $\Omega \subset \mathbb{R}^n$ be an unbounded set. Assume that a property is satisfied almost everywhere on $\Omega \cap B_k$ for all $k \in \mathbb{N}$. Prove that it is satisfied almost everywhere on Ω.

Exercise 1.5.9. Prove Theorem 1.3.7 by applying the equality $N(x) = \sum\limits_{k=1}^{s} \chi_k(x)$ for all $x \in \bigcup\limits_{m=1}^{s} \Omega_m$, where χ_k denotes the characteristic function of the set Ω_k, and Corollary 1.3.8.

Exercise 1.5.10. Prove that

$$\int_0^1 \left(\int_0^1 \frac{x^2 - y^2}{(x^2 + y^2)^2}\, dx \right) dy = -\frac{\pi}{4}, \qquad \int_0^1 \left(\int_0^1 \frac{x^2 - y^2}{(x^2 + y^2)^2}\, dy \right) dx = \frac{\pi}{4}.$$

Exercise 1.5.11. By applying equality (1.3.28), or otherwise, prove that for all $r > 0$ $\quad \operatorname{meas} B_r = v_r r^n$, where *

$$v_n = \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2} + 1)},$$

the volume of the unit ball in $\mathbb{R}^n$. In particular, for all $m \in \mathbb{N}$

$$v_{2m} = \frac{\pi^m}{m!}, \qquad v_{2m-1} = \frac{\pi_{m-1} 2^m}{(2m-1)!!}.$$

Exercise 1.5.12. Let the set $\Omega \subset \mathbb{R}^n$ be defined with the help of the spherical coordinates by the inequalities

$$0 \leqslant \varphi_1 < 2\pi, -\frac{\pi}{2} \leqslant \varphi_2, \dots, \varphi_{n-1} \leqslant \frac{\pi}{2}, \quad 0 \leqslant \varrho < \Phi(\varphi_1, \dots, \varphi_{n-1}),$$

where Φ is a function non-negative and measurable on the unit sphere S_{n-1}. Then

$$\operatorname{meas} \Omega = \frac{1}{n} \int_0^{2\pi} d\varphi_1 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos \varphi_2\, d\varphi_2 \dots \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \Phi^n(\varphi_1, \dots, \varphi_{n-1}) \cos^{n-2} \varphi_{n-1}\, d\varphi_{n-1}.$$

Exercise 1.5.13. Let $0 < a < b < \infty$ and let a non-negative function f be measurable on $(0, \infty)$. By applying Theorem 1.3.7 prove that

$$\sum_{k=-\infty}^{\infty} \int_{2^k a}^{2^k b} f\, dx \leqslant \mu \int_0^{\infty} f\, dx,$$

where μ is the smallest integer greater than or equal to $\log_2 \frac{b}{a}$. Prove also that μ cannot be replaced by a smaller number.

* Γ is the Euler gamma-function: for $\alpha > 0$ $\Gamma(\alpha) := \int_0^{\infty} e^{-x} x^{\alpha - 1}\, dx$.

Let a function f be integrable on a measurable set $\Omega \subset \mathbb{R}^n$. For $\delta > 0$ set

$$\Lambda_f(\delta) := \sup_{\omega \subset \Omega:\ \mathrm{meas}\,\omega \leqslant \delta} \left| \int\limits_\omega f \, \mathrm{d}x \right| . \qquad (1.5.1)$$

Theorem 1.3.8 can be reformulated in terms of this function in the following way: for all functions f integrable on Ω $\lim\limits_{\delta \to 0+} \Lambda_f(\delta) = 0$. The following exercise show that Λ_f can tend to 0 arbitrarily slowly.

Exercise 1.5.14. Let a positive function ψ defined on $(0,1)$ is such that $\lim\limits_{\delta \to 0+} \psi(\delta) = 0$. Construct a function f integrable on $(0,1)$ such that

$$\lim_{\delta \to 0+} \frac{\int\limits_0^\delta f \, \mathrm{d}x}{\psi(\delta)} = \infty .$$

Exercise 1.5.15. Let a function f be defined on $[a,b]$ and $a < c < b$. Prove that if f is absolutely continuous on $[a,c]$ and $[c,b]$, then it is absolutely continuous on $[a,b]$.

Exercise 1.5.16. If $f : [a,b] \to \mathbb{C}$ and for all $\varepsilon > 0$ there exists $\delta > 0$ such that for any family of intervals $(a_1, b_1), \dots, (a_m, b_m)$, $m \in \mathbb{N}$, contained in $[a,b]$, for which $\sum\limits_{k=1}^m (b_k - a_k) < \delta$, the inequality $\sum\limits_{k=1}^m |f(b_k) - f(a_k)| < \varepsilon$ holds, then f satisfies the Lipschitz condition.

Exercise 1.5.17. Prove Theorem 1.4.3 under the following weaker assumptions: the derivatives f' and g' exist almost everywhere on $[a,b]$, at least one of the products $f'g$ or fg' is integrable on $[a,b]$, and the product fg is absolutely continuous on $[a,b]$.

Exercise 1.5.18. Using Exercise 1.5.17 and the theorem on absolute continuity of the product $f(g)g'$ quoted in Section 1.4, prove that, given $\alpha > 0$, $u \in C^2([a,b])$ such that for all $x \in (a,b)$ $u(x) \neq 0$,

$$\int\limits_a^b u^\alpha u'' \, \mathrm{d}x = u^\alpha u' \big|_a^b - \alpha \int\limits_a^b u^{\alpha-1} (u')^2 \, \mathrm{d}x .$$

If $0 < \alpha < 1$, one cannot apply Theorem 1.4.3, since the function u^α is, in general, not absolutely continuous on $[a,b]$. (See Exercise 1.4.1).

Chapter 2

Spaces $L_p(\Omega)$

2.1 Definitions and basic properties

2.1.1 Spaces $L_p(\Omega)$, $0 < p < \infty$

Definition 2.1.1. *Let $0 < p < \infty$, Ω be a measurable set in $\mathbb{R}^n$, and a function $f\colon \Omega \to \mathbb{C}$. The function $f \in L_p(\Omega)$ if f is measurable on Ω and*

$$\|f\|_{L_p(\Omega)} := \Big(\int\limits_{\Omega} |f|^p \, \mathrm{d}x \Big)^{\frac{1}{p}} < \infty \,. \tag{2.1.1}$$

Note that if $\operatorname{meas}\Omega > 0$, then the conditions $f \in L_p(\Omega)$ and $\|f\|_{L_p(\Omega)} < \infty$ are not equivalent. If, for example, $f := 1$ on a non-measurable subset G of the set $\Omega \cap B_r$ where $r > 0$ is such that $\operatorname{meas}(\Omega \cap B_r) > 0$, $f := -1$ on $(\Omega \cap B_r) \setminus G$, and $f := 0$ on $\Omega \setminus B_r$, then f is not measurable on Ω, hence does not belong to $L_p(\Omega)$ for any $0 < p < \infty$, but $\|f\|_{L_p(\Omega)} < \infty$.

Example 2.1.1. Let $0 < p < \infty$, $\gamma \in \mathbb{R}$, $0 < r < \infty$. The function $|x|^\gamma \in L_p(B_r)$ if and only if $\gamma > -\frac{n}{p}$. The same function $|x|^\gamma \in L_p({}^cB_r)$ if and only if $\gamma < -\frac{n}{p}$ (see Exercise 1.3.14). Moreover,

$$\||x|^\gamma\|_{L_p(B_r)} = (n + \gamma p)^{-\frac{1}{p}} \sigma_n^{\frac{1}{p}} r^{\gamma + \frac{n}{p}} \,, \quad \gamma > -\frac{n}{p},$$

$$\||x|^\gamma\|_{L_p({}^cB_r)} = |n + \gamma p|^{-\frac{1}{p}} \, \sigma_n^{\frac{1}{p}} r^{\gamma + \frac{n}{p}}, \quad \gamma < -\frac{n}{p} \,.$$

Exercise 2.1.1. Let $0 < p < \infty$, $\delta, \gamma \in \mathbb{R}$, $0 < r < \infty$. Prove that

$$|x|^\gamma \,(1 + |\ln |x||)^\delta \in L_p(B_r) \Longleftrightarrow \gamma > -\frac{n}{p} \,, \quad \text{or } \gamma = -\frac{n}{p} \text{ and } \delta < -\frac{1}{p}$$

and

$$|x|^\gamma \,(1 + |\ln |x||)^\delta \in L_p({}^cB_r) \Longleftrightarrow \gamma < -\frac{n}{p} \,, \quad \text{or } \gamma = -\frac{n}{p} \text{ and } \delta < -\frac{1}{p} \,.$$

Let $0 < p < \infty$ and $f \in L_p(\mathbb{R}^n) \cap C(\mathbb{R}^n)$. This clearly implies that the function f should be 'small' at infinity. However, without additional assumptions, one cannot claim that $f(x) \to 0$ as $x \to \infty$. Moreover, along a fixed sequence of points x_k, $k \in \mathbb{N}$, satisfying $x_k \to \infty$ as $k \to \infty$, it may tend to infinity arbitrarily quickly as the following example shows.

Exercise 2.1.2. Let $0 < p < \infty$, $\{\varrho_k\}_{k\in\mathbb{N}}$ be a sequence of positive numbers ϱ_k satisfying $\lim_{k\to\infty} \varrho_k = \infty$, and let $\{x_k\}_{k\in\mathbb{N}}$ be a sequence of $x_k \in \mathbb{R}^n$ satisfying $\lim_{k\to\infty} x_k = \infty$. Moreover, let $h \in C_0^\infty(\mathbb{R}^n)$ be the function defined by (1.2.4). Prove that for sufficiently small positive δ_k, $k \in \mathbb{N}$, the function f defined for all $x \in \mathbb{R}^n$ by

$$f(x) := 3\sum_{k=1}^{\infty} \varrho_k\, h\Big(\frac{x - x_k}{\delta_k}\Big)$$

is such that $f \in L_p(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n)$ and $f(x_k) > \varrho_k$ for all $k \in \mathbb{N}$.

A similar example can be constructed if $\lim_{k\to\infty} x_k = x_0$ where $x_0 \in \mathbb{R}^n$ and $f \in L_p(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n \setminus \{x_0\})$.

Under additional assumptions of monotonicity type the condition $f \in L_p(\mathbb{R}^n)$ implies that $f(x) \to 0$ as $x \to \infty$. Moreover, f must tend to 0 not too slowly.

Exercise 2.1.3. Let $0 < p < \infty$, $f \in L_p(\mathbb{R}^n)$ and $f(x) = g(|x|)$ for all $x \in \mathbb{R}^n$ where g is a positive non-increasing function on $(0, \infty)$. Prove that $f(x) = o(|x|^{-\frac{n}{p}})$ as $x \to \infty$ and as $x \to 0$.

Finally, let us consider the set $\widehat{L}_p((0,1))$ consisting of all function $f \in L_p((0,1))$ which are positive and non-increasing. The following exercise shows that in $\widehat{L}_p((0,1))$ with $0 < p < \infty$ there are no 'extreme functions', i.e. functions f such that any function g positive non-increasing and satisfying

$$\lim_{x\to 0+} \frac{g(x)}{f(x)} = \infty \tag{2.1.2}$$

does not belong to $\widehat{L}_p((0,1))$.

Exercise 2.1.4. Let $0 < p < \infty, f \in \widehat{L}_p((0,1))$, and $0 < \varepsilon < 1$. Define for all $x \in (0,1)$

$$g(x) := \left(\frac{\|f\|_{L_p((0,1))}}{\|f\|_{L_p((0,x))}}\right)^{\varepsilon} f(x).$$

Prove that $g(x) > f(x)$ for all $x \in (0,1)$, condition (2.1.2) is satisfied, but $g \in \widehat{L}_p((0,1))$. (If $\varepsilon \geqslant 1$, then $g \notin \widehat{L}_p((0,1))$.)

2.1.2 Spaces of sequences l_p. Jensen's inequality

A definition, similar to Definitions 2.1.1, can be introduced also for sequences $a := \{a_k\}_{k\in\mathbb{N}}$, where a_k are complex numbers.

Definition 2.1.2. *Let $0 < p < \infty$. A sequence $a \in l_p$ if*

$$\|a\|_{l_p} := \Big(\sum_{k=1}^{\infty} |a_k|^p\Big)^{\frac{1}{p}} < \infty. \tag{2.1.3}$$

Exercise 2.1.5. Let $0 < p < \infty$ and $\delta, \gamma \in \mathbb{R}$. Prove that

$$k^\gamma (\ln k)^\delta \in l_p \Longleftrightarrow \gamma < -\frac{1}{p}, \quad \text{or } \gamma = -\frac{1}{p} \text{ and } \delta < -\frac{1}{p}.$$

Example 2.1.2. If $\Omega \subset \mathbb{R}^n$ is a measurable set and $f : \Omega \to \mathbb{C}$ is a simple function: $\Omega = \bigcup_{k=1}^{\infty} \Omega_k$, where Ω_k are disjoint measurable subsets of Ω, and $f(x) = a_k$ for $x \in \Omega_k$, then $\|f\|_{L_p(\Omega)} = \|b\|_{l_p}$, where $b := \{b_k\}_{k\in\mathbb{N}}$ and $b_k = a_k(\operatorname{meas}\Omega_k)^{\frac{1}{p}}, k \in \mathbb{N}$. If, in particular, $\Omega = (0, \infty)$ and $\Omega_k = (k-1, k], k \in \mathbb{N}$, then $\|f\|_{L_p(\Omega)} = \|a\|_{l_p}$.

Theorem 2.1.1. **(Jensen's inequality)** *Let $0 < p < q < \infty$ and $a := \{a_k\}_{k\in\mathbb{N}}$ where $a_k \in \mathbb{C}, k \in \mathbb{N}$, then*

$$l_p \subset l_q \tag{2.1.4}$$

and

$$\|a\|_{l_q} \leqslant \|a\|_{l_p}. \tag{2.1.5}$$

Moreover, for $a \in l_p$ equality holds if and only if all a_k except possibly one are equal to 0.

Idea of the proof If $\|a\|_{l_p} = 1$, apply the implication $|a_k| < 1 \Longrightarrow |a_k|^q < |a_k|^p$. If $\|a\|_{l_p} < 1$, consider the sequence $b := \{b_k\}_{k\in\mathbb{N}}$, where $b_k := \frac{a_k}{\|a\|_{l_p}}, k \in \mathbb{N}$. □

Proof If all a_k except possibly one are equal to 0, then clearly inequality (2.1.5) turns into an equality.

Assume that at least two of a_k are non-zero. If $\|a\|_{l_p} = 1$, then $|a_k| < 1$ for all $k \in \mathbb{N}$. Hence, $\sum_{k=1}^{\infty} |a_k|^q < \sum_{k=1}^{\infty} |a_k|^p = 1$ and $\|a\|_{l_q} < 1$, which was required to prove in this case.

If $\|a\|_{l_p} = \infty$, then inequality (2.1.5) is trivial. If $\|a\|_{l_p} \neq 1, \infty$, then $\|b\|_{l_p} = \left(\|a\|_{l_p}\right)^{-1}\left(\sum_{k=1}^{\infty} |a_k|^p\right)^{\frac{1}{p}} = 1$. Hence, $\|b\|_{l_q} = \left(\|a\|_{l_p}\right)^{-1}\|a\|_{l_q} < 1$ by the previous argument, which implies (2.1.5) with the strict inequality. □

Note the following simplest particular case of inequality (2.1.5): if $0 < p < 1$, then for all $a, b \geqslant 0$

$$(a+b)^p \leqslant a^p + b^p. \tag{2.1.6}$$

(It is equivalent to the one-dimensional inequalities $\frac{(x+1)^p}{x^p+1} \leqslant 1$ and $(x+1)^p - x^p \leqslant 1$ for $x \geqslant 0$, which are easily proved by finding the maximum of $\frac{(x+1)^p}{x^p+1}$, $(x+1)^p - x^p$ respectively.)

Exercise 2.1.6. Prove that for $0 < q < p$ inclusion (2.1.4) does not hold.

Lemma 2.1.1. *For finite sequences*

$$\lim_{p\to\infty} \|a\|_{l_p} = \sup_{k\in\mathbb{N}} |a_k|. \tag{2.1.7}$$

If $\|a\|_{l_\infty} = \infty$, this equality also holds. If $\|a\|_{l_\infty} < \infty$, it holds if and only if $a \in l_q$ for some $q > 0$.

Idea of the proof For finite sequences apply the inequality

$$\max_{k\in\{1,\dots,m\}} |a_k| \leqslant \left(\sum_{k=1}^{m} |a_k|^p\right)^{\frac{1}{p}} \leqslant m^{\frac{1}{p}} \max_{k\in\{1,\dots,m\}} |a_k| . \tag{2.1.8}$$

For infinite sequences $a \in l_q$ deduce that for all $p > \max\{1, q\}$ and for all $m \in \mathbb{N}$

$$\max_{k\in\{1,\dots,m\}} |a_k| \leqslant \|a\|_{l_p} \leqslant m^{\frac{1}{p}} \max_{k\in\{1,\dots,m\}} |a_k| + \left(\sum_{k=m+1}^{\infty} |a_k|^q\right)^{\frac{1}{q}} . \tag{2.1.9}$$

□

Proof Let $m \in \mathbb{N}$ and $a_k = 0$ for all $k \geqslant m+1$, then the statement follows by passing to the limit in (2.1.8) since $\lim\limits_{m\to\infty} m^{\frac{1}{p}} = 1$.

If, in the case of infinite sequences, $\|a\|_{l_\infty} = \infty$, it follows that also $\lim\limits_{k\to\infty} \|a\|_{l_p} = \infty$ since the left inequality in (2.1.8) holds for all $m \in \mathbb{N}$. Assume that $\|a\|_{l_\infty} < \infty$ and equality (2.1.7) holds. Then, by the properties of limits, there exists $q > 0$ such that $a \in l_q$. Conversely, let $a \in l_q$ for some $q > 0$. Taking into account inequality (2.1.6), we get

$$\left(\sum_{k=1}^{m} |a_k|^p\right)^{\frac{1}{p}} \leqslant \|a\|_{l_p} = \left(\sum_{k=1}^{m} |a_k|^p + \sum_{k=m+1}^{\infty} |a_k|^p\right)^{\frac{1}{p}}$$

$$\leqslant \left(\sum_{k=1}^{m} |a_k|^p\right)^{\frac{1}{p}} + \left(\sum_{k=m+1}^{\infty} |a_k|^p\right)^{\frac{1}{p}}$$

for all $p > 1$ and for all $m \in \mathbb{N}$. Assume also that also $p > q$. Then by applying inequalities (2.1.5) and (2.1.8), we arrive at inequality (2.1.9).

Since $\|a\|_{l_q} < \infty$, inequality (2.1.5) implies that the limit $\lim\limits_{p\to\infty} \|a\|_{l_p}$ exists. Therefore, passing in inequality (2.1.9) to the limit as $p \to \infty$, we get

$$\max_{k\in\{1,\dots,m\}} |a_k| \leqslant \lim_{p\to\infty} \|a\|_{l_p} \leqslant \max_{k\in\{1,\dots,m\}} |a_k| + \left(\sum_{k=m+1}^{\infty} |a_k|^q\right)^{\frac{1}{q}} .$$

Finally, passing to the limit as $m \to \infty$ and taking into account that

$$\lim_{m\to\infty} \max_{k\in\{1,\dots,m\}} |a_k| = \sup_{k\in\mathbb{N}} |a_k| , \quad \lim_{m\to\infty} \left(\sum_{k=m+1}^{\infty} |a_k|^q\right)^{\frac{1}{q}} = 0$$

because $a \in l_q$, we arrive at equality (2.1.7). □

Note the following simplest particular case of (2.1.7): for $a, b \geqslant 0$

$$\lim_{p\to\infty} (a^p + b^p)^{\frac{1}{p}} = \max\{a, b\} . \tag{2.1.10}$$

Lemma 2.1.1 justifies the following definition of l_∞.

Definition 2.1.3. *A sequence $a \in l_\infty$ if*

$$\|a\|_{l_\infty} := \sup_{k\in\mathbb{N}} |a_k| < \infty\,, \tag{2.1.11}$$

i.e. if it is bounded.

Note that if at least two members of a sequence a are non-zero, then $\lim\limits_{p\to 0+} \|a\|_{l_p} = \infty$. Furthermore, $\|a\|_{l_p}$ makes sense for negative p. If at least one of $a_k = 0$, we assume that $\|a\|_{l_p} = 0$. With this convention each sequence $a \in l_p$, $\|a\|_{l_q} \leqslant \|a\|_{l_p}$ if $q < p < 0$, and

$$\lim_{p\to-\infty} \|a\|_{l_p} = \inf_{k\in\mathbb{N}} |a_k| =: \|a\|_{l_{-\infty}}\,. \tag{2.1.12}$$

This follows since $\|a\|_{l_p} = \left(\|a\|_{l_{-p}}\right)^{-1}$ and $\left(\sup\limits_{k\in\mathbb{N}} \frac{1}{|a_k|}\right)^{-1} = \inf\limits_{k\in\mathbb{N}} |a_k|\,.$

For finite sequences $a = \{a_k\}_{k=1}^m$ of non-zero complex numbers a_k it is also of interest to consider the following expressions

$$\|a\|_{l_p}^* := \left(\frac{1}{m}\sum_{k=1}^m |a_k|^p\right)^{\frac{1}{p}}, \tag{2.1.13}$$

where $p \neq 0$. In particular,

$$\|a\|_{l_1}^* := \frac{1}{m}\sum_{k=1}^m |a_k|\,, \quad \|a\|_{l_{-1}}^* := m\left(\sum_{k=1}^m \frac{1}{|a_k|}\right)^{-1}$$

are *arithmetic, harmonic* respectively, *means* of the collection $|a_1|, \ldots, |a_m|$.

Lemma 2.1.2. *For a sequence $a = \{a_k\}_{k=1}^m$ of non-zero complex numbers a_k numbers*

$$\lim_{p\to 0} \|a\|_{l_p}^* = \left(\prod_{k=1}^m |a_k|\right)^{\frac{1}{m}} =: \|a\|_{l_0}^*\,, \tag{2.1.14}$$

the geometric mean *of the collection $|a_1|, \ldots, |a_m|$.*

Idea of the proof Write $\|a\|_{l_p}^*$ as $\exp\left(\ln\left(\|a\|_{l_p}^*\right)\right)$ and apply the equivalences $\ln(1+x) \sim x$, $\frac{\sigma^x - 1}{x} \sim x\ln\sigma$ as $x \to 0$ $(\sigma > 0)$. □

Proof Indeed,

$$\lim_{p\to 0} \|a\|_{l_p}^* = \lim_{p\to 0} \exp\left(\frac{1}{p}\ln\left(\frac{1}{m}\sum_{k=1}^m |a_k|^p\right)\right)$$

$$= \exp\left(\lim_{p\to 0}\frac{1}{p}\ln\left(1 + \frac{1}{m}\sum_{k=1}^m \left(|a_k|^p - 1\right)\right)\right) = \exp\left(\lim_{p\to 0}\frac{1}{m}\sum_{k=1}^m\left(\frac{|a_k|^p - 1}{p}\right)\right)$$

$$= \exp\left(\frac{1}{m}\sum_{k=1}^m \ln|a_k|\right) = \left(\prod_{k=1}^m |a_k|\right)^{\frac{1}{m}}.$$

□

Note that for finite sequences a

$$\|a\|^*_{l_p} \leqslant \|a\|^*_{l_q} \tag{2.1.15}$$

if $-\infty \leqslant p < q \leqslant \infty$. (See (2.1.15) below.) In particular,

$$\|a\|^*_{l_{-1}} \leqslant \|a\|^*_{l_0} \leqslant \|a\|^*_{l_1} \, . \tag{2.1.16}$$

This means that the harmonic mean does not exceed the geometric mean which in its turn does not exceed the arithmetic mean.

2.1.3 Properties of essential infima and suprema

In order to extend Definition 2.1.1 to the case $p = \infty$ in the spirit of Definition 2.1.3, one would need the notion of the *essential supremum.*

Definition 2.1.4. *Let $\Omega \subset \mathbb{R}^n$ and $f : \Omega \to \mathbb{R}$. Then*

$$\sup_{\Omega}\text{vrai}\, f \equiv \sup_{x\in\Omega}\text{vrai}\, f(x) := \inf_{\substack{\omega\subset\Omega:\\ \text{meas}\,\omega=0}} \big(\sup_{x\in\Omega\setminus\omega} f(x)\big)\,. \tag{2.1.17}$$

(The infimum is taken with respect to all subsets $\omega \subset \Omega$ of zero measure.)

The *essential infimum* is defined similarly:

$$\inf_{\Omega}\text{vrai}\, f \equiv \inf_{x\in\Omega}\text{vrai}\, f(x) := \sup_{\substack{\omega\subset\Omega:\\ \text{meas}\,\omega=0}} \big(\inf_{x\in\Omega\setminus\omega} f(x)\big)\,. \tag{2.1.18}$$

Clearly

$$\sup_{x\in\Omega}\text{vrai}\, f(x) \leqslant \sup_{x\in\Omega} f(x). \tag{2.1.19}$$

However, it may be that the inequality is strict.

Example 2.1.3. Let $e \subset \mathbb{R}^n$ be a set of zero measure, and let $f\colon \mathbb{R}^n \to \mathbb{R}$ is such that $f(x) > 0$ for all $x \in e$, and $f(x) = 0$ for all $x \notin e$. Then

$$\sup_{x\in\mathbb{R}^n} f(x) = \sup_{x\in e} f(x) > 0, \quad \sup_{x\in\mathbb{R}^n}\text{vrai}\, f(x) = 0.$$

The second equality follows, since

$$0 \leqslant \sup_{x\in\mathbb{R}^n}\text{vrai}\, f(x) = \inf_{\substack{\omega\subset\mathbb{R}^n:\\ \text{meas}\,\omega=0}} \big(\sup_{x\in\mathbb{R}^n\setminus\omega} f(x)\big) \leqslant \sup_{x\in\mathbb{R}^n\setminus e} f(x) = 0.$$

Note that $\sup\limits_{\Omega}\text{vrai}\, f$, finite or infinite, is defined for all $\Omega \subset \mathbb{R}^n$ and $f\colon \Omega \to \mathbb{R}$. If $\sup\limits_{\Omega}\text{vrai}\, f < \infty$, then the function f is called *essentially bounded above.* Note also that if $\text{meas}\,\Omega = 0$, then for all functions $f\colon \Omega \to \mathbb{R}$

$$\sup_{x\in\Omega}\text{vrai}\, f(x) = \sup_{x\in\emptyset} f(x) = \sup\emptyset = -\infty\,.$$

For a real-valued function f defined on $\Omega \subset \mathbb{R}^n$ and $a \in \mathbb{R}$, define

$$\Omega_a \equiv \Omega_a(f) := \{x \in \Omega\colon f(x) > a\}\,. \tag{2.1.20}$$

Definition 2.1.5. *Let $\Omega \subset \mathbb{R}^n$ and $f : \Omega \to \mathbb{R}$. Then*

$$\sup_{x\in\Omega}\operatorname{vrai} f(x) := \inf\{a \in \mathbb{R}\colon\ \operatorname{meas}\Omega_a = 0\}. \tag{2.1.21}$$

Lemma 2.1.3. *Definitions* 2.1.4 *and* 2.1.5 *are equivalent.*

Proof Let $\mathfrak{M} := \{a \in \mathbb{R}\colon\ \operatorname{meas}\Omega_a = 0\}$ and $M := \inf\mathfrak{M}$. Since $M_1 := \sup\limits_{x\in\Omega}\operatorname{vrai} f(x)$ in the sense of Definition 2.1.4 is defined for all $f\colon \Omega \to \mathbb{R}$, it suffices to prove that $M_1 = M$.

First, let $\mathfrak{M} \neq \emptyset$ and $\mathfrak{M} \neq \mathbb{R}$. The definition of infimum implies that for all $k \in \mathbb{N}$ there exist $a_k \in \mathfrak{M}$ such that $M \leqslant a_k \leqslant M + \frac{1}{k}$. So $\Omega_M = \bigcup\limits_{k=1}^{\infty} \Omega_{a_k}$, hence $\operatorname{meas}\Omega_M = 0$, because for all $k \in \mathbb{N}$ $\operatorname{meas}\Omega_{a_k} = 0$. Consequently, $M_1 \leqslant \sup\limits_{x\in\Omega\setminus\Omega_M} f(x) \leqslant M$, because, by the definition of the set Ω_M, $f(x) \leqslant M$ on $\Omega\setminus\Omega_M$. On the other hand, if $M_1 < M$, then there exists $\omega_1 \subset \Omega$ satisfying $\operatorname{meas}\omega_1 = 0$ such that $\sigma_1 := \sup\limits_{x\in\Omega} f(x) < M$. This implies $\Omega_{\sigma_1} \subset \omega_1$. Hence $\operatorname{meas}\Omega_{\sigma_1} = 0$. Therefore $\sigma_1 \in \mathfrak{M}$, which is impossible as $M = \inf\mathfrak{M}$.

Now let $\mathfrak{M} = \emptyset$, then $\inf\mathfrak{M} = +\infty$. On the other hand, $M_1 = +\infty$ as well. Indeed, if $M_1 < +\infty$, then there exists $\omega_2 \subset \Omega$ satisfying $\operatorname{meas}\omega_2 = 0$ such that $\sigma_2 := \sup\limits_{x\in\Omega\setminus\omega_2} f(x) < +\infty$. This implies $\Omega_{\sigma_2} \subset \omega_2$. Hence $\operatorname{meas}\Omega_{\sigma_2} = 0$. Therefore $\sigma_2 \in \mathfrak{M}$, which is impossible as $\mathfrak{M} = \emptyset$.

Finally, let $\mathfrak{M} = \mathbb{R}$. Then $\inf\mathfrak{M} = -\infty$. Note that $\Omega = \bigcup\limits_{k=1}^{\infty} \Omega_{-k}$ and $\operatorname{meas}\Omega = 0$ since for all $k \in \mathbb{N}$ $\operatorname{meas}\Omega_{-k} = 0$. So, $M_1 = -\infty$ as well. □

Corollary 2.1.1. *Let $\Omega \subset \mathbb{R}^n$,* $\operatorname{meas}\Omega \neq 0$ *and** $f\colon \Omega \to \mathbb{R}$.

1. *If $M := \sup\limits_{\Omega}\operatorname{vrai} f < \infty$, then*

$$\operatorname{meas}\Omega_M = 0\,, \tag{2.1.22}$$

which is equivalent to

$$\text{for almost all } x \in \Omega \quad f(x) \leqslant M\,, \tag{2.1.23}$$

and

$$\text{for all } \varepsilon > 0 \quad \operatorname{meas}\Omega_{M-\varepsilon} \neq 0\,. \tag{2.1.24}$$

2. *If $M = \infty$, then*

$$\text{for all } N > 0 \quad \operatorname{meas}\Omega_N \neq 0\,. \tag{2.1.25}$$

Proof The equality (2.1.22) was obtained in the first part of the proof of Lemma 2.1.11. Inequality (2.1.24) holds because otherwise $\sup\limits_{x\in\Omega}\operatorname{vrai} f(x) \leqslant M - \varepsilon$ which is impossible. Similarly, if $M = \infty$, then inequality (2.1.24) holds because otherwise $\sup\limits_{x\in\Omega}\operatorname{vrai} f(x) \leqslant N$. Finally, inequality (2.1.23) is equivalent to equality (2.1.22) because the set of all $x \in \Omega$ for which this inequality is not satisfied is exactly Ω_M. □

*In Definitions 2.1.4 and 2.1.5 it is not assumed that Ω is measurable. For this reason we write $\operatorname{meas}\Omega \neq 0$ rather than $\operatorname{meas}\Omega > 0$, which means that either Ω is measurable and $\operatorname{meas}\Omega > 0$ or Ω is non-measurable.

Remark 2.1.1. The equality $\operatorname{meas}\Omega_M = 0$ implies that in (2.1.21) inf may be replaced by min if $\operatorname{meas}\Omega \neq 0$ and $\sup\operatorname{vrai}_{x\in\Omega} f(x) < \infty$. □

Equality (2.1.22) together with Definition 2.1.5 provide convenient tools for proving the properties of the essential supremum and they will be used in the proofs below.

We start with some properties of the essential supremum similar to the appropriate properties of 'ordinary' supremum.

Lemma 2.1.4. *Let* $\Omega_1, \Omega_2 \subset \mathbb{R}^n$ *and* $f\colon \Omega_1 \cup \Omega_2 \to \mathbb{R}$. *Then*

$$\sup_{\Omega_1\cup\Omega_2}\operatorname{vrai} f = \max\{\sup_{\Omega_1}\operatorname{vrai} f, \sup_{\Omega_2}\operatorname{vrai} f\}. \tag{2.1.26}$$

Idea of the proof Apply the equality

$$(\Omega_1 \cup \Omega_2)_a(f) = (\Omega_1)_a(f) \cup (\Omega_2)_a(f) \tag{2.1.27}$$

where $a \in \mathbb{R}$. □

Proof If $(\Omega_1)_a(f)$ and $(\Omega_1 \cup \Omega_2)_a(f)$ are measurable, then (2.1.27) implies that

$$\max\{\operatorname{meas}((\Omega_1)_a(f)), \operatorname{meas}((\Omega_2)_a(f))\} \leqslant \operatorname{meas}((\Omega_1 \cup \Omega_2)_a(f))$$
$$\leqslant \operatorname{meas}((\Omega_1)_a(f)) + \operatorname{meas}((\Omega_2)_a(f). \tag{2.1.28}$$

Let $M_k := \sup_{\Omega_k}\operatorname{vrai} f, k = 1, 2$, and let, say, $M_1 \leqslant M_2$. Assume that $M_2 < \infty$. Equality (2.1.22) and the second of inequalities (2.1.28) with $a := M_2$ imply that $\operatorname{meas}((\Omega_1 \cup \Omega_2)_{M_2}(f)) = 0$. Also, for all $\varepsilon > 0$, the first of inequalities (2.1.28) with $a := M_2 - \varepsilon$ implies that $\operatorname{meas}((\Omega_1 \cup \Omega_2)_{M_2-\varepsilon}(f)) \neq 0$. Indeed, if $\operatorname{meas}((\Omega_1 \cup \Omega_2)_{M_2-\varepsilon}(f)) = 0$, then by (2.1.28) $\operatorname{meas}((\Omega_2)_{M_2-\varepsilon}(f)) = 0$. Hence by Definition 2.1.5 $M_2 = \sup_{\Omega_2}\operatorname{vrai} f \leqslant M_2 - \varepsilon$. Consequently, by Definition 2.1.5 $\sup_{\Omega_1\cup\Omega_2}\operatorname{vrai} f = M_2$.

The case $M_2 = \infty$ is considered similarly. □

Corollary 2.1.2. *Let* $G \subset \Omega \subset \mathbb{R}^n$ *and* $f\colon \Omega \to \mathbb{R}$. *Then*

$$\sup_G\operatorname{vrai} f \leqslant \sup_\Omega\operatorname{vrai} f. \tag{2.1.29}$$

Idea of the proof Apply the lemma with $\Omega_1 = G$, $\Omega_2 = \Omega \setminus G$. □

Lemma 2.1.5. *Let f and g be real-valued functions defined on a set $\Omega \subset \mathbb{R}^n$. If $f \leqslant g$ almost everywhere on Ω, then*

$$\sup_{x\in\Omega}\operatorname{vrai} f(x) \leqslant \sup_{x\in\Omega}\operatorname{vrai} g(x). \tag{2.1.30}$$

Idea of the proof Apply equality (2.1.22) and Definition 2.1.5. □

Proof Let $M := \sup_\Omega\operatorname{vrai} g$. The case $M = \infty$ is trivial. Let $M < \infty$. Then by Corollary 2.1.1 $\operatorname{meas}\Omega_M(g) = 0$. Let $\omega := \{x \in \Omega\colon f(x) > g(x)\}$, then $\operatorname{meas}\omega = 0$. Since $\Omega_M(f) \subset \Omega_M(g) \cup \omega$, it follows that $\operatorname{meas}\Omega_M(f) = 0$. Hence, by Definition 2.1.5, $\sup_\Omega\operatorname{vrai} f \leqslant M$. □

The following several statements deal with the case of equality in inequality (2.1.19).

Lemma 2.1.6. *Let* $\Omega \subset \mathbb{R}^n, \operatorname{meas}\Omega \neq 0$ *and* $f\colon \Omega \to \mathbb{R}$.

1. *There exists a subset* $G \subset \Omega$ *such that* $\operatorname{meas} G = \operatorname{meas}\Omega$ *and*

$$\sup_{x\in\Omega}\operatorname{vrai} f(x) = \sup_{x\in G} f(x)\,.$$

2. *There exists a function* g *equivalent to* f *on* Ω *such that*

$$\sup_{x\in\Omega}\operatorname{vrai} f(x) = \sup_{x\in\Omega} g(x)\,.$$

Idea of the proof If $M = \infty$, then by (2.1.19) $\sup\limits_{x\in\Omega} f(x) = \infty$. If $M < \infty$, then take $G := \Omega_M$ and set $g(x) := f(x)$ for $x \in G$ and $g(x) = \frac{M}{2}$ for $x \in \Omega \setminus G$. □

Lemma 2.1.7. *If* $\Omega \subset \mathbb{R}^n$ *is an open set and a real-valued function* $f \in C(\Omega)$, *then*

$$\sup_{x\in\Omega}\operatorname{vrai} f(x) = \sup_{x\in\Omega} f(x)\,. \tag{2.1.31}$$

Idea of the proof Let $M := \sup\limits_{x\in\Omega} f(x)$. By (2.1.19) $\sup\limits_{x\in\Omega}\operatorname{vrai} f(x) \leqslant M$. To prove the converse in the case $M < +\infty$ choose for all $k \in \mathbb{N}$ balls $B(x_k, \delta_k)$ on which $f(x) > M - \frac{1}{k}$. □

Proof First, let $M < +\infty$. Then for all $k \in \mathbb{N}$ there exists $x_k \in \Omega$ such that $f(x_k) > M - \frac{1}{k}$. Since Ω is open and $f \in C(\Omega)$, there exists $\delta_k > 0$ such that $B(x_k, \delta_k) \subset \Omega$ and $f(x) > M - \frac{1}{k}$ for all $x \in B(x_k, \delta_k)$. Therefore, by inequality (2.1.29), $\sup\limits_{x\in\Omega}\operatorname{vrai} f(x) \geqslant \sup\limits_{x\in B(x_k,\delta_k)}\operatorname{vrai} f(x) \geqslant M - \frac{1}{k}$. Passing to the limit as $k \to \infty$, we get $\sup\limits_{x\in\Omega}\operatorname{vrai} f(x) \geqslant M$. Hence $\sup\limits_{x\in\Omega}\operatorname{vrai} f(x) = M$.

Next, let $M = +\infty$. Then for all $k \in \mathbb{N}$ there exists $x_k \in \Omega$ such that $f(x_k) > k$. Since Ω is open and $f \in C(\Omega)$, there exists $\delta_k > 0$ such that $B(x_k, \delta_k) \subset \Omega$ and $f(x) > k$ for all $x \in B(x_k, \delta_k)$. Hence $\sup\limits_{x\in\Omega}\operatorname{vrai} f(x) = +\infty$. □

Remark 2.1.2. Since $\inf\limits_{x\in\Omega}\operatorname{vrai} f(x) = -\sup\limits_{x\in\Omega}\operatorname{vrai}(-f(x))$, the above properties of the essential supremum imply similar properties for the essential infimum.

Exercise 2.1.7. Let $\Omega \subset \mathbb{R}^n, f\colon \Omega \to \mathbb{R}$ and $f(x) \neq 0$ for all $x \in \Omega$. Prove that

$$\inf_{x\in\Omega}\operatorname{vrai} f(x) = \frac{1}{\sup\limits_{x\in\Omega}\operatorname{vrai} \frac{1}{f(x)}}\,. \tag{2.1.32}$$

Inequality (2.1.23) implies the following variant of the mean value theorem stated in Corollary 1.3.4.

Lemma 2.1.8. ***(Mean value theorem)*** *Let* $\Omega \subset \mathbb{R}^n$ *be a measurable set, a real-valued function* f *be measurable on* Ω, *and a real-valued function* g *be non-negative and integrable on* Ω. *Then*

$$\inf_{\Omega}\operatorname{vrai} f \int\limits_{\Omega} g\,\mathrm{d}x \leqslant \int\limits_{\Omega} fg\,\mathrm{d}x \leqslant \sup_{\Omega}\operatorname{vrai} f \int\limits_{\Omega} g\,\mathrm{d}x\,.$$

Idea of the proof Apply inequality (2.1.30), a similar inequality for inf vrai and Corollary 1.3.4. □

2.1.4 Space $L_\infty(\Omega)$

Definition 2.1.6. *Let Ω be a measurable set in $\mathbb{R}^n$ and a function $f\colon \Omega \to \mathbb{C}$. If* $\operatorname{meas}\Omega > 0$, *then the function $f \in L_\infty(\Omega)$ if f is measurable on Ω and*

$$\|f\|_{L_\infty(\Omega)} := \sup_{x\in\Omega} \operatorname{vrai} |f(x)| < \infty\,. \tag{2.1.33}$$

If $\operatorname{meas}\Omega = 0$, *then any function $f\colon \Omega \to \mathbb{C}$ belongs to $L_\infty(\Omega)$, and $\|f\|_{L_\infty(\Omega)} = 0$. (In this case* $\sup\operatorname{vrai}_{x\in\Omega} |f(x)| = -\infty$.*)*

Note that for any non-decreasing function $\varphi : [0,\infty) \to [0,\infty)$

$$\|\varphi(|f|)\|_{L_\infty(\Omega)} = \varphi(\|f\|_{L_\infty(\Omega)})\,. \tag{2.1.34}$$

This follows since $\sup_{x\in\Omega\setminus\omega} \varphi(|f(x)|) = \varphi(\sup_{x\in\Omega\setminus\omega} |f(x)|)$ for all $\omega \subset \Omega$ satisfying $\operatorname{meas}\omega = 0$.

Lemma 2.1.9. *Let $\Omega \subset \mathbb{R}^n$ be a measurable set and functions f, g be measurable on Ω. Then*

$$\left|\int_\Omega fg\,\mathrm{d}x\right| \leqslant \int_\Omega |fg|\,\mathrm{d}x \leqslant \|f\|_{L_\infty(\Omega)}\,\|g\|_{L_1(\Omega)}\,. \tag{2.1.35}$$

Idea of the proof Apply inequality (2.1.23). □

Proof The statement follows by Theorem 1.3.5, Definition 2.1.6, and Theorem 1.3.4 since, by (2.1.23), $|f(x)|\,|g(x)| \leqslant \|f\|_{L_\infty(\Omega)}\,|g(x)|$ for almost all $x \in \Omega$. □

Corollary 2.1.3. *Let $\Omega \subset \mathbb{R}^n$ be a measurable set,* $\operatorname{meas}\Omega < \infty$, *a function f be measurable on Ω, and $0 < p < \infty$. Then*

$$\|f\|_{L_p(\Omega)} \leqslant (\operatorname{meas}\Omega)^{\frac{1}{p}}\,\|f\|_{L_\infty(\Omega)}\,. \tag{2.1.36}$$

Idea of the proof Apply inequality (2.1.35) and equality (2.1.34). □

Proof Indeed, by (2.1.35) and (2.1.34) with $\varphi(t) := t^p$

$$\|f\|_{L_p(\Omega)} = \left(\int_\Omega |f|^p \cdot 1\,\mathrm{d}x\right)^{\frac{1}{p}}$$

$$\leqslant \left(\|\,|f|^p\|_{L_\infty(\Omega)}\right)^{\frac{1}{p}} \left(\|1\|_{L_p(\Omega)}\right)^{\frac{1}{p}} = \left(\operatorname{meas}\Omega\right)^{\frac{1}{p}}\,\|f\|_{L_\infty(\Omega)}\,.$$

□

The following theorem justifies Definition 2.1.6.

Theorem 2.1.2. ***(The Riesz theorem)*** *If $\Omega \subset \mathbb{R}^n$ is a set of finite measure and a function f is measurable on Ω, then*

$$\lim_{p\to\infty} \|f\|_{L_p(\Omega)} = \|f\|_{L_\infty(\Omega)}\,. \tag{2.1.37}$$

Idea of the proof Let $M := \|f\|_{L_\infty(\Omega)}$. For $0 < M < \infty$, by passing to the upper limit in inequality (2.1.36), obtain that $\limsup_{p\to\infty} \|f\|_{L_p(\Omega)} \leqslant M$. For $a \geqslant 0$ we consider the sets $G_a := \Omega_a(|f|) = \{x \in \Omega\colon |f(x)| > a\}$ and by passing to the lower limit in the inequality $\|f\|_{L_p(\Omega)} \geqslant (M-\varepsilon)\big(\operatorname{meas} G_{M-\varepsilon}\big)^{\frac{1}{p}}$ obtain that $\liminf\limits_{p\to\infty} \|f\|_{L_p(\Omega)} \geqslant M$. □

Proof If $\operatorname{meas}\Omega = 0$, then $\|f\|_{L_p(\Omega)} = \|f\|_{L_\infty(\Omega)} = 0$, and equality (2.1.37) is trivial. Let $\operatorname{meas}\Omega > 0$. If $M = 0$, then by Lemma 2.1.10 $f \sim 0$ on Ω, and equality (2.1.37) is again trivial.

Next let $0 < M < \infty$. By passing to the upper limit in inequality (2.1.36) we get, taking into account that $\operatorname{meas}\Omega < \infty$, that*

$$\limsup_{p\to\infty} \|f\|_{L_p(\Omega)} \leqslant M \limsup_{p\to\infty} \big(\operatorname{meas}\Omega\big)^{\frac{1}{p}} = M\,. \tag{2.1.38}$$

By Corollary 1.3.4

$$\|f\|_{L_p(\Omega)} \geqslant \Big(\int\limits_{G_{M-\varepsilon}} |f|^p \,\mathrm{d}x\Big)^{\frac{1}{p}} \geqslant (M-\varepsilon)\big(\operatorname{meas} G_{M-\varepsilon}\big)^{\frac{1}{p}}\,. \tag{2.1.39}$$

Since by (2.1.24) $\operatorname{meas} G_{M-\varepsilon} > 0$ for all $\varepsilon > 0$, we have

$$\liminf_{p\to\infty} \|f\|_{L_p(\Omega)} \geqslant (M-\varepsilon) \liminf_{p\to\infty} \big(\operatorname{meas} G_{M-\varepsilon}\big)^{\frac{1}{p}} = M - \varepsilon\,. \tag{2.1.40}$$

Passing here to the limit as $\varepsilon \to 0+$ we get

$$\liminf_{p\to\infty} \|f\|_{L_p(\Omega)} \geqslant M\,. \tag{2.1.41}$$

Inequalities (2.1.38) and (2.1.41) imply equality (2.1.37).

Finally, let $M = \infty$. Then by (2.1.25) $\operatorname{meas} G_N > 0$ for all $N > 0$. Therefore, similarly to the first part of the proof,

$$\|f\|_{L_p(\Omega)} \geqslant \Big(\int\limits_{G_N} |f|^p \,\mathrm{d}x\Big)^{\frac{1}{p}} \geqslant N\big(\operatorname{meas} G_N\big)^{\frac{1}{p}}$$

and

$$\liminf_{p\to\infty} \|f\|_{L_p(\Omega)} \geqslant N\,.$$

Since this inequality holds for all $N > 0$, it implies that $\liminf\limits_{p\to\infty} \|f\|_{L_p(\Omega)} = \infty$. Consequently $\lim\limits_{p\to\infty} \|f\|_{L_p(\Omega)} = \infty$. □

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* In the case under consideration $\limsup\limits_{p\to\infty} \varphi(p) \equiv \limsup\limits_{p\to+\infty} \varphi(p)$ is the upper limit of $\varphi(p)$ as $p \to +\infty$, i.e. the greatest of the partial limits of $\varphi(p)$ as $p \to +\infty$ ($\Longleftrightarrow$ $\limsup\limits_{p\to+\infty} \varphi(p)$ is the supremum of all ξ for which there exist $p_k > 0$ satisfying $\lim\limits_{k\to\infty} p_k = +\infty$ and $\lim\limits_{k\to\infty} \varphi(p_k) = \xi$.) Recall that $\lim\limits_{p\to+\infty} \varphi(p)$ exists if and only if $\liminf\limits_{p\to+\infty} \varphi(p) = \limsup\limits_{p\to+\infty} \varphi(p)$, and if this equality holds, then $\lim\limits_{p\to+\infty} \varphi(p)$ is equal the common value of the lower and the upper limits.

Exercise 2.1.8. Assume that $\Omega \subset \mathbb{R}^n$ and $\operatorname{meas}\Omega = \infty$. In this case, in general, equality (2.1.37) does not hold. For example, it does not hold for $f \equiv 1$ on Ω. Prove that if $\|f\|_{L_\infty(\Omega)} = \infty$, then equality (2.1.37) holds. If $\|f\|_{L_\infty(\Omega)} < \infty$, then it holds if and only if $\limsup\limits_{p\to\infty} \|f\|_{L_p(\Omega)} < \infty$.

Theorem 2.1.3. *If $\Omega \subset \mathbb{R}^n$ is a measurable set, $0 < \operatorname{meas}\Omega < \infty$, and a function f is measurable on Ω, is not equivalent to 0 and $\|f\|_{L_p(\Omega)} < \infty$ for all sufficiently large p, then*

$$\lim_{p\to\infty} \frac{\int\limits_\Omega |f|^{p+1}\,dx}{\int\limits_\Omega |f|^p\,dx} = \|f\|_{L_\infty(\Omega)}\,. \tag{2.1.42}$$

Idea of the proof Let $M := \|f\|_{L_\infty(\Omega)}$ and G_a have the same meaning as in the proof of the previous theorem. By applying Lemma 2.1.9 prove that

$$\liminf_{p\to\infty} \frac{\int\limits_\Omega |f|^{p+1}\,dx}{\int\limits_\Omega |f|^p\,dx} \leqslant M\,. \tag{2.1.43}$$

For $0 < \delta < \varepsilon < M$ obtain the estimate

$$\frac{\int\limits_\Omega |f|^{p}\,dx}{\int\limits_\Omega |f|^{p+1}\,dx} \leqslant \frac{1}{\varepsilon}\left[\left(\frac{\delta}{\varepsilon}\right)^p \frac{\operatorname{meas}(\Omega\setminus G_\delta)}{\operatorname{meas} G_\varepsilon} + \frac{\operatorname{meas}(G_\delta\setminus G_\varepsilon)}{\operatorname{meas} G_\varepsilon} + 1\right]. \tag{2.1.44}$$

By passing here to the limit as, successively, $p\to\infty$, $\delta\to\varepsilon-$ and $\varepsilon\to M-$ prove that

$$\liminf_{p\to\infty} \frac{\int\limits_\Omega |f|^{p+1}\,dx}{\int\limits_\Omega |f|^p\,dx} \geqslant M\,. \tag{2.1.45}$$

□

Proof The assumptions of the theorem imply that there exists $p_0 > 0$ such that $0 < \int\limits_\Omega |f|^p\,dx < \infty$ for all $p \geqslant p_0$ and that $M > 0$.

Assume that $p \geqslant p_0$ and $M < \infty$. By Lemma 2.1.9

$$\int\limits_\Omega |f|^{p+1}\,dx \leqslant \|f\|_{L_\infty(\Omega)} \int\limits_\Omega |f|^p\,dx = M \int\limits_\Omega |f|^p\,dx\,.$$

Hence inequality (2.1.43) follows.

Further, let $0 < \delta < \varepsilon < M$. Then

$$\int\limits_\Omega |f|^{p+1}\,dx \geqslant \int\limits_{G_\varepsilon} |f|^{p+1}\,dx \geqslant \varepsilon \int\limits_{G_\varepsilon} |f|^p\,dx\,, \quad \int\limits_{G_\varepsilon} |f|^p\,dx \geqslant \varepsilon^p \operatorname{meas} G_\varepsilon$$

and

$$\int_\Omega |f|^p \, dx = \int_{\Omega\setminus G_\delta} |f|^p \, dx + \int_{G_\delta \setminus G_\varepsilon} |f|^p \, dx + \int_{G_\varepsilon} |f|^p \, dx$$

$$\leqslant \delta^p \operatorname{meas}(\Omega \setminus G_\delta) + \varepsilon^p \operatorname{meas}(G_\delta \setminus G_\varepsilon) + \int_{G_\varepsilon} |f|^p \, dx .$$

Consequently, taking into account that by (2.1.24) $\operatorname{meas} G_\varepsilon > 0$ for all $0 < \varepsilon < M$, we get

$$\frac{\int_\Omega |f|^p \, dx}{\int_\Omega |f|^{p+1} \, dx} \leqslant \frac{1}{\varepsilon}\left[\frac{\delta^p \operatorname{meas}(\Omega \setminus G_\delta) + \varepsilon^p \operatorname{meas}(G_\delta \setminus G_\varepsilon)}{\int_{G_\varepsilon} |f|^p \, dx} + 1\right],$$

which implies inequality (2.1.44).

Now we pass to the upper limit in this inequality as $p \to \infty$. Since $\frac{\delta}{\varepsilon} < 1$, we get

$$\limsup_{p\to\infty} \frac{\int_\Omega |f|^p \, dx}{\int_\Omega |f|^{p+1} \, dx} \leqslant \frac{1}{\varepsilon}\left[\frac{\operatorname{meas}(G_\delta \setminus G_\varepsilon)}{\operatorname{meas} G_\varepsilon} + 1\right].$$

Next we pass to the limit as $\delta \to \varepsilon-$. Since $G_{\delta_1} \setminus G_\varepsilon \supset G_{\delta_2} \setminus G_\varepsilon$ for $\delta_1 < \delta_2 < \varepsilon$ and $\bigcap_{0<\delta<\varepsilon} (G_\delta \setminus G_\varepsilon) = \emptyset$, it follows that $\lim_{\delta\to\varepsilon-} \operatorname{meas}(G_\delta \setminus G_\varepsilon) = 0$. (See Section 1.3.1.) Therefore

$$\limsup_{p\to\infty} \frac{\int_\Omega |f|^{p+1} \, dx}{\int_\Omega |f|^p \, dx} \leqslant \frac{1}{\varepsilon}.$$

Finally passing to the limit as $\varepsilon \to M-$, we establish

$$\limsup_{p\to\infty} \frac{\int_\Omega |f|^p \, dx}{\int_\Omega |f|^{p+1} \, dx} \leqslant \frac{1}{M}.$$

Consequently, by equality (2.1.32) inequality (2.1.45) follows which, together with inequality (2.1.43), implies the desired statement. □

Exercise 2.1.9. Complete the proof by considering the case $M = \infty$.

Remark 2.1.3. Recall that if, for a sequence $\{a_k\}_{k\in\mathbb{N}}$ of positive numbers $a_k > 0$, the limit $\lim_{k\to\infty} \frac{a_{k+1}}{a_k}$ exists, finite or infinite, then $\lim_{k\to\infty} (a_k)^{\frac{1}{k}} = \lim_{k\to\infty} \frac{a_{k+1}}{a_k}$. Therefore (2.1.42) implies (2.1.37) where p is replaced by $k \in \mathbb{N}$. Furthermore, the existence of the limit in (2.1.37) can be established independently since $\|f\|_{L_p(\Omega)} = (\operatorname{meas}\Omega)^{\frac{1}{p}} \|f\|^*_{L_p(\Omega)}$ if $\operatorname{meas}\Omega > 0$, where $\|f\|^*_{L_p(\Omega)}$ defined by (2.1.58) is a non-decreasing function of p. (See (2.1.59).) Hence the limit in (2.1.37) is equal to its partial limit when p is replaced by $k \in \mathbb{N}$. So, Theorem 2.1.2 can be derived from Theorem 2.1.3. However, the straightforward proof of Theorem 2.1.2 given above is much shorter.

Remark 2.1.4. Expression (2.1.1) makes sense also for $-\infty < p < 0$. However, in this case, if f vanishes at some $x \in \Omega$, we should agree how to understand $\|f\|_{L_p(\Omega)}$. Let $N := \{x \in \Omega\colon f(x) = 0\}$. We assume that $\int\limits_\Omega |f|^p\, dx = \int\limits_{\Omega\setminus N} |f|^p\, dx$ if $\operatorname{meas} N = 0$ and $\int\limits_\Omega |f|^p\, dx = +\infty$ if $\operatorname{meas} N > 0$. We also assume that $(+\infty)^p = 0$. With these conventions $\|f\|_{L_p(\Omega)}$ makes sense for all functions f measurable on Ω. Thus, $L_p(\Omega)$ for all $-\infty < p < 0$ is the set of all functions f measurable on Ω. Furthermore, $\|f\|_{L_p(\Omega)} = 0$ if and only if $\operatorname{meas} N > 0$, or $\operatorname{meas} N = 0$ and the integral $\int\limits_{\Omega\setminus N} |f|^p\, dx$ diverges.

Exercise 2.1.10. Let $\Omega \subset \mathbb{R}^n$, $0 < \operatorname{meas}\Omega < \infty$ and a function f be measurable on Ω. Then

$$\lim_{p\to-\infty} \|f\|_{L_p(\Omega)} = \inf_{x\in\Omega} \operatorname{vrai} |f(x)| \,. \tag{2.1.46}$$

Taking into account equality (2.1.46) and the above comments we may define $L_{-\infty}(\Omega)$ as the set of all functions measurable on Ω and set $\|f\|_{L_{-\infty}(\Omega)} := \inf\limits_{\Omega} \operatorname{vrai} |f|$ if $\operatorname{meas}\Omega > 0$. (If $\operatorname{meas}\Omega = 0$, we assume that $\|f\|_{L_{-\infty}(\Omega)} := 0$.)

2.1.5 Basic properties of the spaces $L_p(\Omega), 0 < p \leqslant \infty$

By Definitions 2.1.1 and 2.1.6 it follows, for all $0 < p \leqslant \infty$, that if $f \in L_p(\Omega)$ and $\alpha \in \mathbb{C}$, then

$$\|\alpha f\|_{L_p(\Omega)} = |\alpha| \cdot \|f\|_{L_p(\Omega)} \,. \tag{2.1.47}$$

Moreover, if $\beta > 0$, then

$$\left\| |f|^\beta \right\|_{L_p(\Omega)} = \|f\|^\beta_{L_{\beta p}(\Omega)} \,. \tag{2.1.48}$$

If $p = \infty$, this is a particular case of much more general equality (2.1.34). It also follows that if $\operatorname{meas}\Omega = 0$, then $\|f\|_{L_p(\Omega)} = 0$ for all $0 < p \leqslant \infty$ and for all $f\colon \Omega \to \mathbb{C}$.

Lemma 2.1.10. *Let $\Omega \subset \mathbb{R}^n$ be a measurable set,* $\operatorname{meas}\Omega > 0$*, a function f be measurable on Ω and $0 < p \leqslant \infty$. Then $\|f\|_{L_p(\Omega)} = 0$ if and only if f is equivalent to 0 on Ω.*

Lemma 2.1.11. *Let $\Omega \subset \mathbb{R}^n$ be a measurable set, $0 < p \leqslant \infty$ and $f \in L_p(\Omega)$. If $g \sim f$ on Ω, then $g \in L_p(\Omega)$ and $\|g\|_{L_p(\Omega)} = \|f\|_{L_p(\Omega)}$.*

Lemma 2.1.12. *Let $\Omega \subset \mathbb{R}^n$ be an unbounded measurable set, a function f be measurable on Ω and $0 < p \leqslant \infty$. Assume that measurable sets Ω_k are such that $\Omega_k \subset \Omega_{k+1}$ and $\bigcup\limits_{k=1}^{\infty} \Omega_k = \Omega$. Then $\lim\limits_{k\to\infty} \|f\|_{L_p(\Omega_k)} = \|f\|_{L_p(\Omega)}$.*

Exercise 2.1.11. If $0 < p < \infty$, then Lemmas 2.1.10, 2.1.11 and 2.1.12 follow by the properties of the Lebesgue integral. See Theorem 1.3.2, Corollary 1.3.3 and part 3 of Definition 1.3.18. Prove them for $p = \infty$.

Lemma 2.1.13. *For all $0 < p \leqslant \infty$ and all measurable sets $\Omega \subset \mathbb{R}^n$ the space $L_p(\Omega)$ is a linear (vector) space with respect to point-wise addition and multiplication by complex numbers.*

Idea of the proof If $p < \infty$, apply the elementary numerical inequalities (2.1.6) and*

$$(a+b)^p \leqslant 2^{p-1}(a^p + b^p)\,, \tag{2.1.49}$$

where $1 \leqslant p < \infty$ and $a, b \geqslant 0$. If $p = \infty$ apply equality (2.1.22) of Corollary 2.1.1 and Definition 2.1.5. □

Proof *Step* 1. Equality (2.1.47) implies that $\alpha f \in L_p(\Omega)$ for all $f \in L_p(\Omega)$ and all $\alpha \in \mathbb{C}$.

Step 2. Let $0 < p < \infty$ and $f, g \in L_p(\Omega)$. By (2.1.6) and (2.1.49) we get that for all $x \in \Omega$

$$|f(x) + g(x)|^p \leqslant A_p\left(|f(x)|^p + |g(x)|^p\right), \tag{2.1.50}$$

where $A_p := 2^{p-1}$ if $1 < p < \infty$ and $A_p = 1$ if $0 < p \leqslant 1$. Since the function in the right-hand side of this inequality is integrable on Ω and the function in the left-hand side is measurable on Ω, by Theorem 1.3.4 the function in the left-hand side is also integrable on Ω, hence $f_1 + f_2 \in L_p(\Omega)$, and

$$\int\limits_\Omega |f+g|^p \,\mathrm{d}x \leqslant A_p\bigg(\int\limits_\Omega |f(x)|^p \,\mathrm{d}x + \int\limits_\Omega |g(x)|^p \,\mathrm{d}x\bigg)\,. \tag{2.1.51}$$

Step 3. Let $p = \infty$, $f, g \in L_\infty(\Omega)$ and $M := \|f\|_{L_\infty(\Omega)}$, $N := \|g\|_{L_\infty(\Omega)}$. Note that by Corollary 2.1.1 $\operatorname{meas}\Omega_M(|f|) = \operatorname{meas}\{x \in \Omega\colon |f(x)| > M\} = 0$, $\operatorname{meas}\Omega_N(|f|) = \operatorname{meas}\{x \in \Omega\colon |g(x)| > N\} = 0$ If $|f(x) + g(x)| > M + N$, then either $|f(x)| > M_1$ or $|g(x)| > N$. (Otherwise $|f(x) + g(x)| \leqslant |f(x)| + |g(x)| \leqslant M + N$.) Therefore

$$\Omega_{M+N}(|f+g|) \subset \Omega_M(|f|) \cup \Omega_N(|g|)\,,$$

hence $\operatorname{meas}\Omega_{M+N}(|f+g|) = 0$. So, by Definition 2.1.5 $\|f+g\|_{L_\infty(\Omega)} \leqslant M + N$. Thus $f + g \in L_\infty(\Omega)$ and

$$\|f+g\|_{L_\infty(\Omega)} \leqslant \|f\|_{L_\infty(\Omega)} + \|g\|_{L_\infty(\Omega)}\,. \tag{2.1.52}$$

□

Remark 2.1.5. A similar argument shows that the spaces $l_p, 0 < p \leqslant \infty$, with term by term addition and multiplication by a scalar are linear spaces as well.

Lemma 2.1.14. *If $\Omega \subset \mathbb{R}^n$ is a measurable set, $0 < p \leqslant \infty$, $f \in L_p(\Omega)$, and $h \in \mathbb{R}^n$, then*

$$\|f(x+h)\|_{L_p(\Omega)} = \|f(x)\|_{L_p(\Omega+h)}\,, \tag{2.1.53}$$

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* The inequality is equivalent to the one-dimensional inequalities $\frac{(x+1)^p}{x^p+1} \leqslant 2^{p-1}$ and $(x+1)^p - 2^{p-1}x^p \leqslant 2^{p-1}$, which are easily proved by finding the maximum of $\frac{(x+1)^p}{x^p+1}$, $(x+1)^p - 2^{p-1}x^p \leqslant 2^{p-1}$ respectively. It may also be treated as the simplest particular cases of Hölder's inequality for sequences. (See Section 2.2.3). Since (in both cases) the only point of maximum is $x = 1$, inequality (2.1.49) turns into an equality if and only if $a = b$.

where $\Omega + h := \{x + h : x \in \Omega\}$ is a translation of the set Ω. In particular

$$\|f(x+h)\|_{L_p(\mathbb{R}^n)} = \|f(x)\|_{L_p(\mathbb{R}^n)} \tag{2.1.54}$$

(translation invariance *of* $\|f\|_{L_p(\mathbb{R}^n)}$).

Lemma 2.1.15. *If $\Omega \subset \mathbb{R}^n$ is a measurable set, $0 < p \leqslant \infty$, $f \in L_p(\Omega)$, and $\varepsilon > 0$, then*

$$\|f(\varepsilon x)\|_{L_p(\Omega)} = \varepsilon^{-\frac{n}{p}} \|f(x)\|_{L_p(\varepsilon\Omega)}\,, \tag{2.1.55}$$

where $\varepsilon\Omega := \{\varepsilon x : x \in \Omega\}$ is a dilation of the set Ω. In particular

$$\|f(\varepsilon x)\|_{L_p(\mathbb{R}^n)} = \varepsilon^{-\frac{n}{p}} \|f\|_{L_p(\mathbb{R}^n)}\,. \tag{2.1.56}$$

Exercise 2.1.12. If $0 < p < \infty$, the statements of Lemmas 2.1.14 and 2.1.15 are simple particular cases of the general Theorem 1.3.19. Prove them for $p = \infty$.

2.1.6 Space $L_0(\Omega)$

In the previous section passing to the limit in $\|f\|_{L_p(\Omega)}$ as $p \to \infty$ or $p \to -\infty$ was discussed in detail. Another limiting case arises if $p \to 0+$. This case differs essentially from the previous one though also makes a certain sense if the setting of the problem is slightly altered. Amendments are required for the following reason. First note that if $f \in L_{p_0}(\Omega)$ for some $p_0 > 0$, then

$$\lim_{p\to 0+} \int_\Omega |f|^p \, dx = \operatorname{meas}(\Omega \setminus N) \tag{2.1.57}$$

where $N := \{x \in \Omega\colon f(x) = 0\}$. Indeed, this follows by Theorem 1.3.11 because $\lim\limits_{p\to 0+} |f(x)|^p = 1$ for all $x \in \Omega \setminus N$, $\lim\limits_{p\to 0+} |f(x)|^p = 0$ for all $x \in N$, and $|f(x)|^p \leqslant |f(x)|^{p_0} + 1$ for all $x \in \Omega$ and for all $0 < p \leqslant p_0$. Consequently,

$$\lim_{p\to 0+} \|f\|_{L_p(\Omega)} = 0 \ \text{ if} \operatorname{meas}(\Omega \setminus \mathrm{N}) < 1\,, \quad \lim_{p\to 0+} \|f\|_{L_p(\Omega)} = \infty \ \text{ if} \operatorname{meas}(\Omega \setminus \mathrm{N}) > 1\,.$$

With this in mind, it is natural to consider, for measurable sets $\Omega \subset \mathbb{R}^n$ satisfying $0 < \operatorname{meas}\Omega < \infty$, the quantity

$$\|f\|^*_{L_p(\Omega)} := \left(\frac{1}{\operatorname{meas}\Omega} \int_\Omega |f|^p \, dx\right)^{\frac{1}{p}}, \tag{2.1.58}$$

which differs from $\|f\|_{L_p(\Omega)}$ only by a multiple. Another reason for considering this quantity is that for $p = 1$ it is the *mean value* of $|f|$ over Ω, an integral analogue of the *arithmetic mean*, and for $p = -1$ it is an integral analogue of the *harmonic mean*.

Example 2.1.4. Let $m \in \mathbb{N}$, $\Omega = \bigcup\limits_{k=1}^{m} \Omega_k$ where $\Omega_k \subset \mathbb{R}^n$ are disjoint measurable sets of equal measure, and $f(x) = a_k$ for all $x \in \Omega_k$ where a_k are positive numbers. Then

$$\|f\|^*_{L_1(\Omega)} = \|a\|^*_{l_1} = \frac{1}{m}\sum_{k=1}^{m} a_k\,, \quad \|f\|^*_{L_{-1}(\Omega)} = \|a\|^*_{l_{-1}} = m\left(\sum_{k=1}^{m} \frac{1}{a_k}\right)^{-1}.$$

Note that

$$\|f\|^*_{L_p(\Omega)} \leqslant \|f\|^*_{L_q(\Omega)} \tag{2.1.59}$$

if $0 < p < q < \infty$. (See Corollary 2.2.3 below.)

Definition 2.1.7. *Let Ω be a measurable set in $\mathbb{R}^n$, $0 < \operatorname{meas}\Omega < \infty$, and a function $f\colon \Omega \to \mathbb{C}$. The function $f \in L_0(\Omega)$ if f is measurable on Ω and*

$$\|f\|^*_{L_0(\Omega)} := \mathrm{e}^{\frac{1}{\operatorname{meas}\Omega}\int\limits_\Omega \ln|f|\,dx} < \infty\,. \tag{2.1.60}$$

(We assume that $\mathrm{e}^{+\infty} := +\infty$, $\mathrm{e}^{-\infty} := 0$.*)*

The quantity $\|f\|^*_{L_0(\Omega)}$ is an integral analogue of the *geometric mean* as the following example shows.

Example 2.1.5. Under the assumptions of Exercise 2.1.4

$$\|f\|^*_{L_0(\Omega)} = \|a\|^*_{l_0} = \left(\prod_{k=1}^m a_k\right)^{\frac{1}{m}}.$$

Note that

$$\|f\|^*_{L_0(\Omega)} = 0 \Longleftrightarrow \int\limits_\Omega \ln|f|\,dx = -\infty\,.$$

In particular, $\|f\|^*_{L_0(\Omega)} = 0$ if $\operatorname{meas} N > 0$ and $\int\limits_\Omega (\ln|f|)_+\,dx < \infty$.

Exercise 2.1.13. Prove that $L_0(\Omega)$ is a linear space.

Example 2.1.6. Let $\gamma \in \mathbb{R}$ and $0 < r < \infty$. The function $\mathrm{e}^{|x|^\gamma}$ belongs to $L_0(B_r)$ if and only if $\gamma > -n$. Moreover,

$$\left\|\mathrm{e}^{|x|^\gamma}\right\|^*_{L_0(B_r)} = \mathrm{e}^{\frac{n}{n+\gamma}r^\gamma}, \quad \gamma > -n\,.$$

If, in particular, $-n < \gamma < 0$, then $\mathrm{e}^{|x|^\gamma} \notin L_p(B_r)$ for any $0 < p \leqslant \infty$, but $\mathrm{e}^{|x|^\gamma} \in L_0(B_r)$.

Exercise 2.1.14. Prove that $\||x|^\gamma\|^*_{L_0(B_r)} = \mathrm{e}^{-\frac{\gamma}{n}}r^\gamma$ for all $\gamma \in \mathbb{R}$ and $r > 0$.

The following theorem justifies the notation used in (2.1.60).

Theorem 2.1.4. *If $\Omega \subset \mathbb{R}^n$ is a measurable set,* $0 < \operatorname{meas}\Omega < \infty$ *and $f \in L_{p_0}(\Omega)$ for some $p_0 > 0$, then $f \in L_0(\Omega)$ and*

$$\lim_{p\to 0+} \|f\|^*_{L_p(\Omega)} = \|f\|^*_{L_0(\Omega)}\,. \tag{2.1.61}$$

Idea of the proof Follow the proof of Lemma 2.1.2. Consider, together with the subset N introduced above, the sets $N_1 := \{x \in \Omega\colon 0 < |f(x)| \leqslant 1\}$ and $N_2 := \{x \in \Omega\colon |f(x)| > 1\}$. Apply Theorem 1.3.10 and the Dominated Convergence Theorem 1.3.11 to justify passing to the limit under the integral sign over N_1, N_2 respectively. □

Proof First of all note that the existence of the limit in (2.1.61) follows by the monotonicity of $\|f\|^*_{L_p(\Omega)}$. (See (2.1.59).)

Step 1. Assume that $\operatorname{meas} N = 0$. Denote $\alpha(p) := \frac{1}{\operatorname{meas}\Omega}\int\limits_\Omega |f|^p\,dx$. Equality (2.1.57) implies that $\alpha(p) \to 1$ as $p \to 0+$. Hence $\ln\alpha(p) = \ln[1 + (\alpha(p) - 1)] \sim \alpha(p) - 1$ as $p \to 0+$ and

$$\lim_{p\to 0+} \|f\|^*_{L_p(\Omega)} = \exp\left(\lim_{p\to 0+} \frac{\ln\alpha(p)}{p}\right) = \exp\left(\lim_{p\to 0+} \frac{\alpha(p)-1}{p}\right)$$

$$= \exp\left(\frac{1}{\operatorname{meas}\Omega} \lim_{p\to 0+} \int\limits_\Omega \frac{|f|^p - 1}{p}\,dx\right). \tag{2.1.62}$$

The function $\frac{a^p-1}{p}$ is for $a > 1$ increasing and for $0 < a < 1$ decreasing on $(0,\infty)$ and $\lim\limits_{p\to 0} \frac{a^p-1}{p} = \ln a$. Therefore

$$\frac{a^p-1}{p} \leqslant \frac{a^{p_0}-1}{p_0}\,,\quad 0 < p \leqslant p_0,\ a > 1\,;\quad \frac{1-a^p}{p} \leqslant |\ln|f||\ ,\quad 0 < p < \infty,\ 0 < a \leqslant 1\,.$$

Consequently for all $x \in N_1 \cup N_2$

$$\lim_{p\to 0+} \frac{|f(x)|^p - 1}{p} = \ln|f(x)|\ , \tag{2.1.63}$$

for all $x \in N_2$ and $0 < p \leqslant p_0$

$$\frac{|f(x)|^p - 1}{p} \leqslant \frac{|f(x)|^{p_0} - 1}{p_0}\,, \tag{2.1.64}$$

and for all $x \in N_1$ and $0 < p < \infty$

$$\frac{1 - |f(x)|^p}{p} \leqslant |\ln|f(x)||\ . \tag{2.1.65}$$

Since the function $\frac{|f|^{p_0}-1}{p_0}$ is non-negative and integrable on Ω, equality (2.1.63) and inequality (2.1.64) imply, by Theorem 1.3.11, that

$$\lim_{p\to 0+} \int\limits_{N_2} \frac{|f(x)|^p - 1}{p}\,dx = \int\limits_{N_2} |\ln f(x)|\,dx < +\infty\,.$$

Moreover, equality (2.1.63) and inequality (2.1.65) imply, by Theorem 1.3.10, that

$$\lim_{p\to 0+} \int\limits_{N_1} \frac{1 - |f(x)|^p}{p}\,dx = \int\limits_{N_1} |\ln f(x)|\,dx = -\int\limits_{N_1} |\ln f(x)|\,dx\,,$$

the limit being finite of equal to $+\infty$. Thus,

$$\lim_{p\to 0+} \int\limits_\Omega \frac{|f(x)|^p - 1}{p}\,dx = \int\limits_\Omega |\ln f(x)|\,dx\,,$$

the limit being finite or equal to $-\infty$, which together with (2.1.62) implies the desired equality (2.1.61).

Step 2. If $\operatorname{meas} N = \operatorname{meas}\Omega$, then $f \sim 0$ on Ω and equality (2.1.61) is trivial: $0 = 0$. If $0 < \operatorname{meas} N < \operatorname{meas}\Omega$, then equality (2.1.61) again takes the form $0 = 0$ because

$$\lim_{p\to 0+} \|f\|^*_{L_p(\Omega)} = \lim_{p\to 0+} \left(\frac{\operatorname{meas}(\Omega\setminus N)}{\operatorname{meas}\Omega}\right)^{\frac{1}{p}} \|f\|^*_{L_p(\Omega\setminus N)} = 0\,.$$

This follows since the first factor is equal to 0 whilst the second is finite by the first part of the proof. □

Exercise 2.1.15. Prove the statement of Exercise 2.1.14 by applying Theorem 2.1.4 and Example 2.1.1.

Finally we note that Theorem 2.1.4 implies that inequality (2.1.59) holds for all $-\infty \leqslant p < q \leqslant \infty$. (See Corollary 2.2.3 below.) In the extended version it implies, in particular, that

$$\|f\|^*_{L_{-1}(\Omega)} \leqslant \|f\|^*_{L_0(\Omega)} \leqslant \|f\|^*_{L_1(\Omega)}\,,$$

which is an integral analogue of the numerical inequality (2.1.16).

2.1.7 Further exercises

Exercise 2.1.16. Let $0 < p < \infty$ and let $\{r_k\}_{k\in\mathbb{N}}$ be an ordered sequence of all rational points in $\mathbb{R}^n$. Moreover, let a function $\mu \in L_1(\mathbb{R}^n) \cap C(\mathbb{R}^n \setminus \{0\})$ be non-negative and such that $\lim\limits_{x\to 0} \mu(x) = \infty$. Prove that the function f, defined by

$$f(x) := \left(\sum_{k=1}^{\infty} 2^{-k}\mu(x - r_k)\right)^{\frac{1}{p}}$$

for all $x \in \mathbb{R}^n$ for which the series converges and $f(x) := 0$ otherwise, belongs to $L_p(\mathbb{R}^n)$ and is essentially unbounded on all balls in $\mathbb{R}^n$, i.e. $\|f\|_{L_\infty(B(x,r))} = \infty$ for all $x \in \mathbb{R}^n$ and $r > 0$.

Exercise 2.1.17. Prove that in inequality (2.1.5) the equality is attained if and only if all a_k are equal to 0 or only one of them is non-zero.

Exercise 2.1.18. Let $\Omega \subset \mathbb{R}^n$. Prove that equality (2.1.31) is satisfied for all real-valued functions $f \in C(\Omega)$ if and only if $\operatorname{meas}\Omega \cap B(x,r) \neq 0$ for all $x \in \Omega$ and $r > 0$.

Exercise 2.1.19. Prove that for $-\infty < M < \infty$ Definitions 2.1.4 and 2.1.5 are also equivalent to the following one: $\sup\operatorname{vrai}\limits_{\Omega} f = M$ if and only if

$$\text{for almost all } x \in \Omega \quad f(x) \leqslant M\,,$$

and for all $\varepsilon > 0$ there exists a subset $G \subset \Omega$ such that $\operatorname{meas} G \neq 0$ and

$$\text{for almost all } x \in G \quad f(x) > M - \varepsilon\,.$$

Also $\sup\limits_{\Omega}\operatorname{vrai} f = \infty$ if and only if for all $N > 0$ there exists a subset $G \subset \Omega$ such that $\operatorname{meas} G \neq 0$ and

$$\text{for almost all } x \in G \quad f(x) > N .$$

This definition is closer to the definition of $\sup\limits_{\Omega} f$. However, Definitions 2.1.4 and 2.1.5 are more convenient for applications.

Exercise 2.1.20. By applying formula (2.1.42) with $p \in \mathbb{N}$ to $f(x) = \sin x$ on $\Omega = [0, \frac{\pi}{2}]$ prove the *Wallis* formula

$$\frac{\pi}{2} = \prod_{k=1}^{\infty} \frac{(2k)^2}{(2k-1)(2k+1)} .$$

Exercise 2.1.21. Let $\Omega \subset \mathbb{R}^n$ be a measurable set, a function f and a non-negative function g be measurable on Ω. Prove that if $g \in L_1(\Omega)$, then

$$\lim_{p \to \infty} \left(\int_{\Omega} |f|^p \, g \, dx \right)^{\frac{1}{p}} = \|f\|_{L_\infty(\Omega)} .$$

Next assume that $g \notin L_1(\Omega)$. If $\|f\|_{L_\infty(\Omega)} = \infty$, prove that this equality holds. If $\|f\|_{L_\infty(\Omega)} = \infty$, prove that it holds if and only if $\limsup\limits_{p \to \infty} \left(\int\limits_{\Omega} |f|^p \, g \, dx \right)^{\frac{1}{p}} < \infty$.

Exercise 2.1.22. Let $-\infty < a, b < \infty$, a function $f \in C([a,b])$ be non-negative, and let $x_0 \in [a,b]$ be a point of strict absolute maximum of f on $[a,b]$. Prove that

$$\lim_{p \to \infty} \frac{\int\limits_a^b x f(x)^p \, dx}{\int\limits_a^b f(x)^p \, dx} = x_0 .$$

Note that the above ratio is the x-coordinate x_p of the centre of mass of the curvilinear trapezium $T_p = \{(x, y) \in \mathbb{R}^2 \colon a \leqslant x \leqslant b, 0 \leqslant y \leqslant |f(x)|^p\}$. Therefore, the statement means that $\lim\limits_{p \to \infty} x_p = x_0$, which is geometrically understandable, because the assumption that f has only one point of strict absolute maximum x_0 implies that for large p the major part of T_p is concentrated around the vertical line $x = x_0$. For similar reasons one can expect that the statement will remain valid if x and x_0 are replaced by $g(x)$, $g(x_0)$ respectively, where $g \in C([a,b])$.

2.2 Hölder's inequality

Hölder's inequality is the main inequality in the theory of L_p-spaces. One may say, without exaggeration, that it, or its corollaries, are in that way or other used in proofs of all main inequalities related to L_p-spaces.