2.2.1 Young's inequality

Lemma 2.2.1. ***(Young's inequality)*** 1. *Let a function f be real-valued increasing and continuous on $[0,\infty)$, and $f(0)=0$. Then for all $a, b \geqslant 0$*

$$ab \leqslant \int_0^a f\,\mathrm{d}x + \int_0^b f^{-1}\,\mathrm{d}x\,, \tag{2.2.1}$$

where f^{-1} is the inverse function. Moreover, the equality holds if and only if $b = f(a)$.

2. *Let a function f be real-valued increasing positive and continuous on $(0,\infty)$ satisfying*

$$\lim_{x\to 0+} f(x) = \infty\,, \qquad \lim_{x\to\infty} f(x) = 0\,, \qquad \int_0^1 f\,\mathrm{d}x < \infty\,.$$

Then for all $a \geqslant 0, b > 0$

$$ab \geqslant \int_0^a f\,\mathrm{d}x - \int_b^\infty f^{-1}\,\mathrm{d}x\,. \tag{2.2.2}$$

Moreover, the equality holds if and only if $a > 0$ and $b = f(a)$.

The geometric meaning of inequality (2.2.1) is the following.
Note that

$$ab = \operatorname{meas}(OACB) \leqslant \operatorname{meas}\Omega_1 + \operatorname{meas}\Omega_2 = \int_0^a f\,\mathrm{d}x + \int_0^b f^{-1}\,\mathrm{d}x\,.$$

(Equality holds only if $b = f(a)$.) Next we give an analytic proof of (2.2.1) based on these geometric considerations.

Proof If $b \leqslant f(a)$ ($\Longleftrightarrow f^{-1}(b) \leqslant a$), then

$$ab = \int_0^a \mathrm{d}x \int_0^b \mathrm{d}y = \int_0^{f^{-1}(b)} \mathrm{d}x \int_0^b \mathrm{d}y + \int_{f^{-1}(b)}^a \mathrm{d}x \int_0^b \mathrm{d}y\,.$$

Since $f(x) \leqslant b$ for all $0 \leqslant x \leqslant f^{-1}(b)$ and $f(x) \leqslant b$ for all $f^{-1}(b) \leqslant x \leqslant a$, we have

$$ab = \int_0^{f^{-1}(b)} \mathrm{d}x \int_0^{f(x)} \mathrm{d}y + \int_0^{f^{-1}(b)} \mathrm{d}x \int_{f(x)}^b \mathrm{d}y + \int_{f^{-1}(b)}^a \mathrm{d}x \int_0^{fx} \mathrm{d}y - \int_{f^{-1}(b)}^a \mathrm{d}x \int_b^{fx} \mathrm{d}y\,.$$

By combining the first and the third integrals in the right-hand side and interchanging the order of integration in the second one, we get

$$ab = \int_0^a \mathrm{d}x \int_0^{f(x)} \mathrm{d}y + \int_0^b \mathrm{d}y \int_0^{f^{-1}(y)} \mathrm{d}x - \int_{f^{-1}(b)}^a \mathrm{d}x \int_b^{f(x)} \mathrm{d}y$$

$$= \int_0^a f(x)\,\mathrm{d}x + \int_0^b f^{-1}(y)\,\mathrm{d}y - \int_{f^{-1}(b)}^a (f(x) - b)\,\mathrm{d}x\,.$$

Hence inequality (2.2.1) holds. Moreover, the equality holds if and only if

$$\int_{f^{-1}(b)}^a (f(x) - b)\,\mathrm{d}x = 0 \Longleftrightarrow f^{-1}(b) = a \Longleftrightarrow b = f(a)\,.$$

If $b > f(a)$, then by a similar argument

$$ab = \int_0^a f(x)\,\mathrm{d}x + \int_0^b f^{-1}(y)\,\mathrm{d}y - \int_{f(a)}^b (f(y) - a)\,\mathrm{d}x\,.$$

$$< \int_0^a f(x)\,\mathrm{d}x + \int_0^b f^{-1}(y)\,\mathrm{d}y\,.$$

□

Exercise 2.2.1. Give the geometric interpretation and an analytic proof of inequality (2.2.2).

For $-\infty < p \leqslant \infty$, the quantity p' satisfying

$$\frac{1}{p} + \frac{1}{p'} = 1$$

is called the *conjugate*. If $p = 1$, it is assumed that $p' := \infty$; if $p = \infty$, then $p' := 1$.

Corollary 2.2.1. 1. *If $1 < p < \infty$, then for all $a, b \geqslant 0$*

$$ab \leqslant \frac{a^p}{p} + \frac{b^{p'}}{p'}\,. \tag{2.2.3}$$

2. *If $0 < p < 1$, then for all $a \geqslant 0, b > 0$*

$$ab \geqslant \frac{a^p}{p} + \frac{b^{p'}}{p'}\,. \tag{2.2.4}$$

3. *In both inequalities the equality holds if and only if $a^p = b^{p'}$. (In the second case also $a > 0$.)*

Idea of the proof Set in Lemma 2.2.1 $f(x) = x^{p-1}$. □

Next we note some other equivalent forms of inequality (2.2.3). We assume here that $0^0 := 1$.

For all $0 \leqslant \gamma \leqslant 1$ and $a, b \geqslant 0$

$$a^\gamma b^{1-\gamma} \leqslant \gamma^\gamma (1-\gamma)^{1-\gamma}(a + b)\,. \tag{2.2.5}$$

The equality holds if an only if $(1-\gamma)a = \gamma b$.

For all $0 \leqslant \gamma \leqslant 1$ and $a, b \geqslant 0$

$$a^{\gamma} b^{1-\gamma} \leqslant \gamma a + (1-\gamma) b \,. \tag{2.2.6}$$

The equality holds if an only if $a = b$.

Inequality (2.2.3) implies inequalities (2.2.5) and (2.2.6) if to replace $\frac{a^p}{p}$ by a, $\frac{b'^p}{p'}$ by b and $\frac{1}{p}$ by γ, by γa, $(1-\gamma)b$ and γ respectively.

In its turn inequality (2.2.6) is equivalent to

$$\gamma \ln a + (1-\gamma) b \leqslant \ln(\gamma a + (1-\gamma) b) \,. \tag{2.2.7}$$

where $a, b > 0, 0 \leqslant \gamma \leqslant 1$, which is the property of concavity of the logarithmic function.

Exercise 2.2.2. As shown above inequality (2.2.3) can be proved as a corollary of inequality (2.2.1) and as a corollary of the concavity of the logarithmic function. Give one more proof by reducing it to the one-dimensional inequality $\frac{x^{1-p}}{p} + \frac{x}{p'} \geqslant 1$ for all $x > 0$.

Exercise 2.2.3. State and prove by induction the generalizations of inequalities (2.2.3), (2.2.5) and (2.2.6) for the case of the product of m multiples in the left-hand side. Find in which cases the equality holds.

2.2.2 Hölder's inequality for integrals

Theorem 2.2.1. ***(Hölder's inequality)*** *Let $\Omega \subset \mathbb{R}^n$ be a measurable set, functions f and g be measurable on Ω and $0 < p \leqslant \infty$.*

1. *If $1 \leqslant p \leqslant \infty$, then*

$$\int_{\Omega} |fg| \, \mathrm{d}x \leqslant \|f\|_{L_p(\Omega)} \, \|g\|_{L_{p'}(\Omega)} \,. \tag{2.2.8}$$

2. *If $0 < p < 1$, then*

$$\int_{\Omega} |fg| \, \mathrm{d}x \geqslant \|f\|_{L_p(\Omega)} \, \|g\|_{L_{p'}(\Omega)} \,. \tag{2.2.9}$$

If one of the factors in the right-hand sides of (2.2.8) or (2.2.9) is equal to zero, we assume that the product is also equal to 0 even if the second factor is infinite. See also Remark 2.1.4, explaining how to treat the factor $\|g\|_{L_{p'}(\Omega)}$ in (2.2.9) if g vanishes at some points $x \in \Omega$. If $1 \leqslant p \leqslant \infty$ and $f \in L_p(\Omega), g \in L_{p'}(\Omega)$ then inequality (2.2.8) implies, in particular, that $fg \in L_1(\Omega)$.

Idea of the proof Integrate over Ω the inequality obtained from Young's inequality (2.2.3) by setting

$$a := \frac{|f(x)|}{\|f\|_{L_p(\Omega)}}, \quad b = \frac{|g(x)|}{\|g\|_{L_{p'}(\Omega)}} \,. \tag{2.2.10}$$

□

Proof *Step* 1. If $p = 1$ hence $p' = \infty$, or $p = \infty$ hence $p' = 1$, inequality (2.2.8) is already proved in Lemma 2.1.9.

Step 2. Next let $1 < p < \infty$. If either $\|f\|_{L_p(\Omega)}$ or $\|g\|_{L_{p'}(\Omega)}$ is equal to 0 or ∞, then inequality (2.2.8) is trivial. Assume that $0 < \|f\|_{L_p(\Omega)}, \|g\|_{L_{p'}(\Omega)} < \infty$.

Inequality (2.2.3) with a and b defined by (2.2.10) implies that for all $x \in \Omega$

$$\frac{|f(x)g(x)|}{\|f\|_{L_p(\Omega)}\,\|g\|_{L_{p'}(\Omega)}} \leqslant \frac{1}{p}\,\frac{|f(x)|^p}{\|f\|^p_{L_p(\Omega)}} + \frac{1}{p'}\,\frac{|g(x)|^{p'}}{\|g\|^{p'}_{L_{p'}(\Omega)}}\,. \tag{2.2.11}$$

Since both parts of the inequality are measurable on Ω, by Theorem 1.3.4

$$\frac{\int\limits_\Omega |f(x)g(x)|\,\mathrm{d}x}{\|f\|_{L_p(\Omega)}\,\|g\|_{L_{p'}(\Omega)}} \leqslant \frac{1}{p}\,\frac{\int\limits_\Omega |f(x)|^p\,\mathrm{d}x}{\|f\|^p_{L_p(\Omega)}} + \frac{1}{p'}\,\frac{\int\limits_\Omega |g(x)|^{p'}\,\mathrm{d}x}{\|g\|^{p'}_{L_{p'}(\Omega)}} = \frac{1}{p} + \frac{1}{p'} = 1\,,$$

which implies the desired inequality (2.2.8).

Step 3. If $0 < p < 1$, then by a similar argument starting with inequality (2.2.4) instead of (2.2.3), we obtain inequality (2.2.9). □

Remark 2.2.1. The proof above is one of the implementations of the general idea: *each numerical inequality gives rise to an integral inequality if its entries are replaced by the values of some functions and integration is carried out.*

If $p = 2$, then $p' = 2$ as well and inequality (2.2.8) takes the form

$$\int\limits_\Omega |fg|\,\mathrm{d}x \leqslant \|f\|_{L_2(\Omega)}\,\|g\|_{L_2(\Omega)}\,.$$

This is the *Cauchy—Bunyakovskiĭ inequality.** It is a particular case of the general Cauchy—Bunyakovskiĭ inequality (1.1.4) for semi-inner-product spaces, since $\int\limits_\Omega f\bar{g}\,\mathrm{d}x$ is a semi-inner product on $L_2(\Omega)$, and hence can also be proved as in Exercise 1.1.4. See Section 3.1.1.

Corollary 2.2.2. *Let $\Omega \subset \mathbb{R}^n$ be a measurable set, functions f and g be measurable on Ω, $0 < p \leqslant \infty$, $-\infty < p_1, p_2 \leqslant \infty$, $p_1 \neq 0, p_2 \neq 0$ and*

$$\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}\,. \tag{2.2.12}$$

(We assume that $\frac{1}{\infty} := 0$.)

1. *If $p \leqslant p_1 \leqslant \infty$, then*

$$\|fg\|_{L_p(\Omega)} \leqslant \|f\|_{L_{p_1}(\Omega)}\,\|g\|_{L_{p_2}(\Omega)}\,. \tag{2.2.13}$$

2. *If $-\infty < p_1 < p, p_1 \neq 0$, then*

$$\|fg\|_{L_p(\Omega)} \geqslant \|f\|_{L_{p_1}(\Omega)}\,\|g\|_{L_{p_2}(\Omega)}\,. \tag{2.2.14}$$

* Also known as the *Schwarz inequality* or the *Cauchy—Schwarz inequality.*

If $p = 1$, then the corollary coincides with Theorem 2.2.1. Thus, the corollary is, in fact, a generalization of Theorem 2.2.1. However, this is a generalization which is equivalent to the initial statement.

Idea of the proof If $p < \infty$, apply the following particular case of equality (2.1.48)

$$\|fg\|_{L_p(\Omega)} = \| \, |f|^p \, |g|^p \, \|_{L_1(\Omega)}^{\frac{1}{p}}$$

and Theorem 2.2.1. □

Proof *Step*1. If $p < \infty$ and $p \leqslant p_1$, then applying equality (2.1.48) and Hölder's inequality (2.2.8) with the exponent $\frac{p_1}{p} \geqslant 1$, and taking into account that, by (2.2.12), $\left(\frac{p_1}{p}\right)' = \frac{p_2}{p}$, we get

$$\| \, |f|^p \, |g|^p \, \|_{L_1(\Omega)} \leqslant \| \, |f|^p \, \|_{L_{\frac{p_1}{p}}(\Omega)} \, \| \, |g|^p \, \|_{L_{\frac{p_2}{p}}(\Omega)} = \|f\|_{L_{p_1}(\Omega)}^p \, \|g\|_{L_{p_2}(\Omega)}^p \, ,$$

which implies inequality (2.2.13). Inequality (2.2.14) is proved similarly.

Step 2. If $p = \infty$ and $p_1 < \infty$, then, by (2.2.12), $p_2 = -p_1$, and by inequality (2.2.8) with $p = \infty$ we get

$$\|f\|_{L_{p_1}(\Omega)} = \left\|(fg)g^{-1}\right\|_{L_{p_1}(\Omega)} \leqslant \|fg\|_{L_\infty(\Omega)} \left\|g^{-1}\right\|_{L_{p_1}(\Omega)} \, ,$$

which implies inequality (2.2.14) since, (2.1.48), $\left\|g^{-1}\right\|_{L_{p_1}(\Omega)} = \|g\|_{L_{p_2}(\Omega)}^{-1}$.

Step 3. Finally, if $p = \infty$ and $p_1 = \infty$, then, by (2.1.48), also $p_2 = \infty$. In this case inequality (2.2.13) takes the form

$$\|fg\|_{L_\infty(\Omega)} \leqslant \|f\|_{L_\infty(\Omega)} \, \|f\|_{L_\infty(\Omega)} \, , \tag{2.2.15}$$

which can be proved similarly to inequality (2.1.52) taking into account that $\Omega_{M_1 M_2}(|fg|) \subset \Omega_{M_1}(|f|) \cup \Omega_{M_2}(|g|)$ where $M_1 = \|f\|_{L_\infty(\Omega)}$ and $M_2 = \|g\|_{L_\infty(\Omega)}$. □

Exercise 2.2.4. Prove inequality (2.2.15) by passing to the limit in (2.2.8) and applying Theorem 2.1.2.

Exercise 2.2.5. Derive inequality (2.2.9) from inequality (2.2.8) by applying the argument used in the second part of the proof of Corollary 2.2.2.

Corollary 2.2.3. *Let $\Omega \subset \mathbb{R}^n$ be a measurable set of finite measure and let $0 < p < q \leqslant \infty$. Then*

$$L_q(\Omega) \subset L_p(\Omega) \tag{2.2.16}$$

and for all $f \in L_q(\Omega)$

$$\|f\|_{L_p(\Omega)} \leqslant (\operatorname{meas}\Omega)^{\frac{1}{p}-\frac{1}{q}} \, \|f\|_{L_q(\Omega)} \, . \tag{2.2.17}$$

(*In the notation of Section 2.1.6 this is inequality* (2.1.59).)

Idea of the proof Apply Hölder's inequality to $\int\limits_\Omega |f|^p \, dx = \int\limits_\Omega |f|^p \cdot 1 \, dx$. □

Proof If $q = \infty$, the statement was proved in Corollary 2.1.3. Let $0 < p < q < \infty$. By applying Hölder's inequality with the exponents $\frac{q}{p} > 1$ and $\left(\frac{q}{p} > 1\right)' = \frac{q}{q-p}$ we get

$$\int_\Omega |f|^p \, dx \leqslant \| \, |f|^p \, \|_{L_{\frac{q}{p}}(\Omega)} \, \| 1 \|_{L_{\frac{q}{q-p}}(\Omega)} = \|f\|^p_{L_q(\Omega)} \, (\text{meas}\, \Omega)^{\frac{q-p}{q}} ,$$

and inequality (2.2.17) follows. It also implies inclusion (2.2.16). Indeed, if $f \in L_q(\Omega)$, then $\|f\|_{L_q(\Omega)} < \infty$. Hence by (2.2.17) $\|f\|_{L_p(\Omega)} < \infty$ which means that $f \in L_p(\Omega)$. □

Remark 2.2.2. If $f \equiv 1$ on Ω, then inequality (2.2.17) turns into an equality. Hence the factor $(\text{meas}\, \Omega)^{\frac{1}{p} - \frac{1}{q}}$ is not replaceable by a smaller one. Following the tradition we call it a *sharp constant* in inequality (2.2.17), thus emphasizing that it is independent of $f \in L_q(\Omega)$, though clearly it is a function of Ω, p and q. This fact may be stated also in the following way. Consider the identity operator I as an operator acting from $L_q(\Omega)$ to $L_p(\Omega)$, which is possible due to inclusion (2.2.16). Then the operator $I\colon L_q(\Omega) \to L_p(\Omega)$ is bounded and, moreover,

$$\|I\|_{L_q(\Omega) \to L_p(\Omega)} \equiv \sup_{f \in L_q(\Omega), f \not\sim 0} \frac{\|f\|_{L_p(\Omega)}}{\|f\|_{L_q(\Omega)}} = (\text{meas}\, \Omega)^{\frac{1}{p} - \frac{1}{q}} . \tag{2.2.18}$$

Inclusion (2.2.16) and, in general, any inclusion $Z_1 \subset Z_2$ of function spaces Z_1 and Z_2 is often called an *embedding*. Also the identity operator $I\colon L_q(\Omega) \to L_p(\Omega)$, in general case $I\colon Z_1 \to Z_2$, is called an *embedding operator*. If this operator is bounded, as in the case under consideration, then the corresponding embedding is called a *continuous embedding*.

Exercise 2.2.6. Prove that for $\Omega := \mathbb{R}^n$ embedding (2.2.16) does not hold for $q > p$ and for $q < p$. Prove also that if $\Omega := B_1$ and $q < p$, then again embedding (2.2.16) does not hold.

In the notation of Section 2.1.6 inequalities (2.2.8) and (2.2.9) in the case $0 < \text{meas}\, \Omega < \infty$ take the form

$$\|fg\|_{L_1(\Omega)} \leqslant \|f\|^*_{L_p(\Omega)} \, \|g\|^*_{L_{p'}(\Omega)} \, , \quad 1 \leqslant p \leqslant \infty , \tag{2.2.19}$$

and

$$\|fg\|_{L_1(\Omega)} \geqslant \|f\|^*_{L_p(\Omega)} \, \|g\|^*_{L_{p'}(\Omega)} \, , \quad 1 \leqslant p \leqslant \infty , \quad 0 < p < 1 . \tag{2.2.20}$$

Under additional assumptions of monotonicity type these inequalities can be improved as the following statement shows.

Lemma 2.2.2. (Chebyshev's inequality) *Let $-\infty < a < b < \infty$ and let functions f, g be non-negative on (a, b).*

1. *If f is non-decreasing and g is non-increasing on (a, b), then $fg \in L_1((a, b))$ and*

$$\|fg\|_{L_1(\Omega)} \leqslant \|f\|^*_{L_1(\Omega)} \, \|g\|^*_{L_1(\Omega)} \, . \tag{2.2.21}$$

2. *If both f and g are non-decreasing or both are non-increasing on (a, b), then*

$$\|fg\|_{L_1(\Omega)} \geqslant \|f\|^*_{L_1(\Omega)} \, \|g\|^*_{L_1(\Omega)} \, . \tag{2.2.22}$$

Idea of the proof Prove that $\int\limits_a^b \varphi g \, dx \leqslant g(c) \int\limits_a^b \varphi \, dx = 0$, where for all $x \in (a,b)$ $\varphi(x) := f(x) - (b-a)^{-1} \int\limits_a^b f \, dy$, and $c \in (a,b)$ is such that $\varphi(x) \leqslant 0$ if $x \in (a,c)$ and $\varphi(x) \geqslant 0$ if $x \in (c,b)$. □

Exercise 2.2.7. Prove the lemma by applying the above hint.

Next we continue to derive corollaries of Hölder's inequality.

Corollary 2.2.4. (Multiplicative inequality) *Let $\Omega \subset \mathbb{R}^n$ be a measurable set and $0 < p_1 < p < p_2 \leqslant \infty$. Then*

$$L_{p_1}(\Omega) \cap L_{p_2}(\Omega) \subset L_p(\Omega) \tag{2.2.23}$$

and

$$\|f\|_{L_p(\Omega)} \leqslant \|f\|^{\alpha}_{L_{p_1}(\Omega)} \, \|f\|^{1-\alpha}_{L_{p_1}(\Omega)} \tag{2.2.24}$$

for all $f \in L_{p_1}(\Omega) \cap L_{p_2}(\Omega)$, where $\alpha \in (0,1)$ is defined by

$$\frac{1}{p} = \frac{\alpha}{p_1} + \frac{1-\alpha}{p_2}. \tag{2.2.25}$$

Idea of the proof Apply inequality (2.2.13) with exponents $p, \frac{p_1}{\alpha p}$ and $\frac{p_2}{(1-\alpha)p}$ to $|f| = |f|^{\alpha} \cdot |f|^{(1-\alpha)}$. □

Proof Since by (2.2.25) $\left(\frac{p_1}{\alpha}\right)^{-1} + \left(\frac{p_2}{1-\alpha}\right)^{-1} = p^{-1}$, inequality (2.2.13) and equality (2.1.1) imply that

$$\|f\|_{L_p(\Omega)} \leqslant \|\, |f|^{\alpha} \,\|_{L_{\frac{p_1}{\alpha}}(\Omega)} \, \|\, |f|^{1-\alpha} \,\|_{L_{\frac{p_2}{1-\alpha}}(\Omega)} = \|f\|^{\alpha}_{L_{p_1}(\Omega)} \, \|f\|^{1-\alpha}_{L_{p_2}(\Omega)} \, .$$

□

Remark 2.2.3. Inequality (2.2.24) is the simplest of the so-called *interpolation inequalities.*

Corollary 2.2.5. (Inequality with a parameter) *Let $\Omega \subset \mathbb{R}^n$ be a measurable set and $0 < p_1 < p < p_2 \leqslant \infty$. Then for all $\varepsilon > 0$*

$$\|f\|_{L_p(\Omega)} \leqslant \alpha^{\alpha}(1-\alpha)^{1-\alpha}\big(\varepsilon^{1-\alpha}\, \|f\|_{L_{p_1}(\Omega)} + \varepsilon^{-\alpha}\, \|f\|_{L_{p_2}(\Omega)}\big) \tag{2.2.26}$$

for all $f \in L_{p_1}(\Omega) \cap L_{p_2}(\Omega)$, where $\alpha \in (0,1)$ is defined by (2.2.25).

Idea of the proof Apply inequalities (2.2.24) and (2.2.5) taking into account that

$$\|f\|^{\alpha}_{L_{p_1}(\Omega)} \, \|f\|^{1-\alpha}_{L_{p_1}(\Omega)} = \big(\varepsilon^{1-\alpha}\, \|f\|_{L_{p_1}(\Omega)}\big)^{\alpha} \big(\varepsilon^{-\alpha}\, \|f\|_{L_{p_2}(\Omega)}\big)^{1-\alpha}.$$

□

Exercise 2.2.8. For a fixed $f \in L_{p_1}(\Omega) \cap L_{p_2}(\Omega)$ inequality (2.2.26) holds for all $\varepsilon > 0$. So the right-hand side as a function of ε is greater than or equal to $\|f\|_{L_p(\Omega)}$. Hence, the minimum of the right-hand side with respect to $\varepsilon > 0$ is also greater than or equal to $\|f\|_{L_p(\Omega)}$. Taking into account this observation prove, by minimizing the right-hand side, that inequality (2.2.26) implies inequality (2.2.24). (Thus, inequalities (2.2.26) and (2.2.24) are equivalent.)

Inequality (2.2.26) implies, by taking $\varepsilon = 1$, the so-called *additive inequality*:

$$\|f\|_{L_p(\Omega)} \leqslant \alpha^\alpha(1-\alpha)^{1-\alpha}\big(\|f\|_{L_{p_1}(\Omega)} + \|f\|_{L_{p_2}(\Omega)}\big) \qquad (2.2.27)$$

Exercise 2.2.9. Inequality (2.2.27) holds for all measurable sets Ω and for all functions $f \in L_{p_1}(\Omega) \cap L_{p_2}(\Omega)$. Given Ω, f and $\delta > 0$, apply it, taking into account formulae (2.1.53) and (2.1.55), to the set $\delta\Omega$ and the function f_δ defined for $x \in \delta\Omega$ by: $f_\delta(x) := f(\delta x)$. Put $\varepsilon := \delta^{n(\frac{1}{p_1}-\frac{1}{p_2})}$ to obtain inequality (2.2.26).

Remark 2.2.4. Let $0 < p_1 < p_2 \leqslant \infty$ be fixed. The above argument shows that 1) inequality (2.2.24) for all measurable Ω and f, 2) inequality (2.2.26) for all measurable Ω and f, and for all $\varepsilon > 0$, and 3) inequality (2.2.27) for all measurable Ω and f, are equivalent. Next, assume that a measurable set Ω is such that $\varepsilon\Omega = \Omega$ for all $\varepsilon > 0$ be fixed. (For example, $\Omega = \mathbb{R}^n$.) A similar argument shows that 1) inequality (2.2.24) for all measurable f, 2) inequality (2.2.26) for all measurable f and for all $\varepsilon > 0$, and 3) inequality (2.2.27) for all measurable f, are equivalent.

Corollary 2.2.6. *Let $\Omega \subset \mathbb{R}^n$ be a measurable set, $m \in \mathbb{N}$, functions $f_1, \dots, f_m$ be measurable on Ω, $1 \leqslant p \leqslant p_1, \dots, p_m \leqslant \infty$, and*

$$\sum_{k=1}^{m} \frac{1}{p_k} = \frac{1}{p}. \qquad (2.2.28)$$

Then

$$\Big\|\prod_{k=1}^{m} f_k\Big\|_{L_p(\Omega)} \leqslant \prod_{k=1}^{m} \|f_k\|_{L_{p_1}(\Omega)}. \qquad (2.2.29)$$

Idea of the proof Apply Corollary 2.2.13 and induction. □

Exercise 2.2.10. Prove the corollary.

Sometimes it is convenient to apply the variant of inequality (2.2.29) with $p = 1$ obtained by replacing p_k by α_k and $|f_k|$ by $|f_k|^{\alpha_k}$: if $\alpha_k \geqslant 0$ and $\alpha_1 + \dots + \alpha_k = 1$, then

$$\int_\Omega \Big(\prod_{k=1}^{m} |f_k|^{\alpha_k}\Big)\,dx \leqslant \prod_{k=1}^{m} \Big(\int_\Omega |f_k|\,dx\Big)^{\alpha_k}. \qquad (2.2.30)$$

Finally we discuss the cases in which equality is attained in Hölder's inequality, its variants and corollaries. We say that complex-valued functions f and g are *almost proportional* on $\Omega \subset \mathbb{R}^n$ if there exist $A, B \in \mathbb{C}$ satisfying $|A| + |B| > 0$ such that $A\,|f(x)|^p = B\,|g(x)|^{p'}$ for almost all $x \in \Omega$.

Lemma 2.2.3. *Let $\Omega \subset \mathbb{R}^n$ be a measurable set, $0 < p \leqslant \infty$, $f \in L_p(\Omega)$ and $g \in L_{p'}(\Omega)$.*

1. *For $1 < p < \infty$ equality is attained in inequality (2.2.8) if and only if $|f|^p$ and $|g|^{p'}$ are almost proportional on Ω.*

2. *For $0 < p < 1$ equality is attained in inequality (2.2.9) if and only if a) $\|f\|_{L_p(\Omega)} = 0$, b) $\|g\|_{L_{p'}(\Omega)} = 0$ and $f \sim 0$ on the set $\{x \in \Omega\colon g(x) \neq 0\}$ or c) both $\|f\|_{L_p(\Omega)}, \|g\|_{L_{p'}(\Omega)} > 0$ and $|f|^p$ and $|g|^{p'}$ are almost proportional on Ω.*

3. *For $p = 1$ equality is attained in inequality (2.2.8) if and only if a) $f \sim 0$ on Ω or b) $f \not\sim 0$ on Ω and $|g| = \|g\|_{L_\infty(\Omega)}$ almost everywhere on the set $\{x \in \Omega\colon f(x) \neq 0\}$.*

4. *For $p = \infty$ in the previous statement f and g should be swapped.*

Idea of the proof If $p \neq 1$ and $p \neq \infty$, the proof is based on part 3 of Corollary 2.2.1. If, say $1 < p < \infty$, and $|f|^p$ and $|g|^{p'}$ are not almost proportional, verify that in the proof of Hölder's inequality in inequality (2.2.11) strict inequality holds on a subset of positive measure and apply part 2 of Theorem 1.3.4. If $p = 1$ apply inequality 2.1.23 and again part 2 of Theorem 1.3.4. □

Proof *Step* 1. Let $1 < p < \infty$. If $f \sim 0$ or $g \sim 0$ on Ω, inequality (2.2.8) takes the form 0=0. Assume that $f \not\sim 0$ and $g \not\sim 0$ on Ω. Then the functions $|f|^p$ and $|g|^{p'}$ are almost proportional if and only if, for $F(x) := \frac{f(x)}{\|f\|_{L_p(\Omega)}}$ and $G(x) := \frac{g(x)}{\|g\|_{L_{p'}(\Omega)}}$, the equality $|F(x)|^p = |G(x)|^{p'}$ holds for almost all $x \in \Omega$.

Indeed, if there exist $A, B \in \mathbb{C}$ satisfying $|A| + |B| > 0$ such that $A\,|f(x)|^p = B\,|g(x)|^{p'}$ for almost all $x \in \Omega$, then $A, B \neq 0$ and $A \int\limits_\Omega |f(x)|^p \,dx = B \int\limits_\Omega |g(x)|^{p'} \,dx$. Hence

$$|F(x)|^p = \frac{|f(x)|^p}{\int\limits_\Omega |f(y)|^p \,dy} = \frac{B}{A} \frac{|g(x)|^{p'}}{\int\limits_\Omega |g(y)|^{p'} \,dy} = |G(x)|^{p'}$$

for almost all $x \in \Omega$. Conversely, if $|F(x)|^p = |G(x)|^{p'}$ for almost all $x \in \Omega$, then $A\,|f(x)|^p = B\,|g(x)|^{p'}$ for almost all $x \in \Omega$, where $A = \int\limits_\Omega |g(y)|^{p'} \,dy$ and $B = \int\limits_\Omega |f(y)|^p \,dy$.

So, first assume that $|F(x)|^p = |G(x)|^{p'}$ for almost all $x \in \Omega$. Then $|g(x)| = C\,|f(x)|^{p-1}$ for almost all $x \in \Omega$, where $C = \left(\frac{\|g\|_{L_{p'}(\Omega)}}{\|f\|_{L_p(\Omega)}}\right)^{\frac{1}{p'}}$, and both parts of inequality (2.2.8) are equal to $C\,\|f\|^p_{L_p(\Omega)}$.

Next assume that $|F(x)|^p \neq |G(x)|^{p'}$ on a subset of positive measure Ω_1. Then by parts 1 and 3 of Corollary 2.2.1 inequality (2.2.11) holds and is strict on Ω_1. Hence by Theorem 1.3.4

$$\frac{\int\limits_\Omega |f(x)g(x)| \,dx}{\|f\|_{L_p(\Omega)}\,\|g\|_{L_{p'}(\Omega)}} < \frac{1}{p} \frac{\int\limits_\Omega |f(x)|^p \,dx}{\|f\|^p_{L_p(\Omega)}} + \frac{1}{p'} \frac{\int\limits_\Omega |g(x)|^{p'} \,dx}{\|g\|^{p'}_{L_{p'}(\Omega)}} = \frac{1}{p} + \frac{1}{p'} = 1\,.$$

Therefore

$$\int\limits_\Omega |fg| \,dx < \|f\|_{L_p(\Omega)}\,\|g\|_{L_{p'}(\Omega)}\,.$$

Step 2. The case $0 < p < \infty$ is similar to the case $1 < p < \infty$.

Step 3. Let $p = 1$. If $f \sim 0$ on Ω, or $f \not\sim 0$ on Ω and $|g| = \|g\|_{L_\infty(\Omega)}$ almost everywhere on the set $\{x \in \Omega\colon f(x) \neq 0\}$, then clearly inequality (2.2.8) turns to an equality. If $f \not\sim 0$ on Ω and $|g| < \|g\|_{L_\infty(\Omega)}$ on a subset Ω_1 of $\{x \in \Omega\colon f(x) \neq 0\}$ of positive measure, then the inequality

$$|f(x)g(x)| \leqslant |f(x)| \cdot \|g\|_{L_\infty(\Omega)}\,,$$

which by inequality (2.1.23) holds for almost all $x \in \Omega$, is strict on Ω_1. Hence by part 2 of Theorem 1.3.4

$$\int_\Omega |fg|\, dx < \int_\Omega |f|x \cdot \|g\|_{L_\infty(\Omega)}\, dx = \|f\|_{L_1(\Omega)} \|g\|_{L_\infty(\Omega)} \cdot$$

Step 4. If $p = \infty$, then $p' = 1$ and the statement follows by swapping f and g in Step 3. □

Exercise 2.2.11. By applying this lemma prove, assuming that $\operatorname{meas}\Omega > 0$, that in inequality (2.2.24) equality is attained if and only if $|f| \sim A\chi$ on Ω, where $A \geqslant 0$ and χ is the characteristic function of a measurable subset of Ω of positive measure.

2.2.3 Hölder's inequality for sequences

By considering step-functions one can obtain inequalities for sequences analogous to the appropriate inequalities for integrals.

Corollary 2.2.7. (Hölder's inequality for sequences) *Let $0 < p \leqslant \infty$, $a \in l_p$ and $g \in l_{p'}$.*

1. *If $1 \leqslant p \leqslant \infty$, then* $ab \in l_1$ and*

$$\sum_{k=1}^{\infty} |a_k b_k| \leqslant \|a\|_{l_p} \|b\|_{l_{p'}} \cdot \tag{2.2.31}$$

2. *If $0 < p < 1$, then*

$$\sum_{k=1}^{\infty} |a_k b_k| \geqslant \|a\|_{l_p} \|b\|_{l_{p'}} \cdot \tag{2.2.32}$$

(If at least one of $b_k = 0$, we assume that $\|b\|_{l_{p'}} = 0$.)

Idea of the proof In Theorem 2.2.1 consider $\Omega = (0, \infty)$ and step-functions f and g defined by $f(x) = a_k, g(x) = b_k$ for $x \in (k-1, k], k \in \mathbb{N}$. □

Proof Since

$$\int_0^\infty |fg|\, dx = \sum_{k=1}^{\infty} |a_k b_k|, \ \|f\|_{L_p((0,\infty))} = \|a\|_{l_p}, \ \|g\|_{L_{p'}((0,\infty))} = \|b\|_{l_{p'}}, \tag{2.2.33}$$

inequalities (2.2.31) and (2.2.32) follow from inequalities (2.2.8), (2.2.9) respectively. □

Remark 2.2.5. Note that one might have started with an arbitrary measurable set $\Omega \subset \mathbb{R}^n$ of positive measure. Indeed, by Exercise 1.5.5 Ω can be represented as $\Omega = \bigcup_{k=1}^{\infty} \Omega_k$ where $\Omega_k, k \in \mathbb{N}$, are disjoint subsets such that $\operatorname{meas}\Omega_k = 1$ for all $k \in \mathbb{N}$ if $\operatorname{meas}\Omega = \infty$, and $\operatorname{meas}\Omega_k = 2^{-k}\operatorname{meas}\Omega$ for all $k \in \mathbb{N}$ if $\operatorname{meas}\Omega < \infty$. If, in the first case, $f(x) = a_k$ on Ω_k, and, in the second case, $f(x) = 2^{\frac{k}{p}}(\operatorname{meas}\Omega)^{-\frac{1}{p}} a_k$ on Ω_k, then $\|f\|_{L_p(\Omega)} = \|a\|_{l_p}$.

* Given $a = \{a_k\}_{k\in\mathbb{N}}$ and $b = \{b_k\}_{k\in\mathbb{N}}$, here we assume that $ab := \{a_k b_k\}_{k\in\mathbb{N}}$.

If $m \in \mathbb{N}$ and sequences a and b are such that $a_k = b_k = 0$ for $k > m$, then inequalities (2.2.8) and (2.2.9) imply corresponding inequalities for finite sums, for example for $1 \leqslant p \leqslant \infty$

$$\sum_{k=1}^{m} |a_k b_k| \leqslant \left(\sum_{k=1}^{m} |a_k|^p\right)^{\frac{1}{p}} \left(\sum_{k=1}^{m} |b_k|^{p'}\right)^{\frac{1}{p'}}. \tag{2.2.34}$$

In its turn, inequality (2.2.31) can be obtained from (2.2.34) by passing to the limit.

Exercise 2.2.12. Give a direct proof of inequality (2.2.34) similar to the proof of inequality (2.2.8) given in Section 2.2.2.

Remark 2.2.6. If $m = 2$, it is also possible to give a simple proof of inequality (2.2.34), which takes the form

$$|a_1| \cdot |a_2| + |b_1| \cdot |b_2| \leqslant (|a_1|^p + |a_2|^p)^{\frac{1}{p}} (|b_1|^{p'} + |b_2|^{p'})^{\frac{1}{p'}}, \tag{2.2.35}$$

by reducing it to the one-dimensional inequality. Indeed, if $a_2 b_1 = 0$ it is trivial. Let $a_2 b_1 \neq 0$. Dividing (2.2.35) by $|a_2| \cdot |b_1|$ and setting $x := \frac{|a_1|}{|a_2|}, y := \frac{|b_2|}{|b_1|}$ we see that (2.2.35) is equivalent to the inequality $(x+y)(x^p+1)^{-\frac{1}{p}} \leqslant (y^{p'}+1)^{\frac{1}{p'}}$, which is easily proved by finding, for a fixed $y \geqslant 0$, the maximum of the function $(x+y)(x^p+1)^{-\frac{1}{p}}$ on $[0, \infty)$.

Exercise 2.2.13. Prove inequality (2.2.34) by induction starting with inequality (2.2.35).

Exercise 2.2.14. By applying Definition 1.3.18 deduce inequality (2.2.8) from inequality (2.2.34).

Corollary 2.2.8. *Let $m \in \mathbb{N}$ and $a_1, \dots, a_m \geqslant 0$.*

1. *If $1 \leqslant p < \infty$, then*

$$a_1^p + \cdots + a_m^p \leqslant (a_1 + \cdots + a_m)^p \leqslant m^{p-1}(a_1^p + \cdots + a_m^p). \tag{2.2.36}$$

2. *If $0 < p < 1$, then*

$$m^{p-1}(a_1^p + \cdots + a_m^p) \leqslant (a_1 + \cdots + a_m)^p \leqslant a_1^p + \cdots + a_m^p. \tag{2.2.37}$$

Idea of the proof Apply inequality (2.2.34) and its variant for $0 < p < 1$ with $b_1 = \cdots = b_m = 1$, and inequality (2.1.5). □

Proof The left of inequalities (2.2.36) is inequality (2.1.5). The right one follows by (2.2.34) since

$$a_1 + \cdots + a_m = a_1 \cdot 1 + \cdots + a_m \cdot 1 \leqslant \left(\sum_{k=1}^{m} a_k^p\right)^{\frac{1}{p}} \left(\sum_{k=1}^{m} 1\right)^{\frac{1}{p'}} = m^{1-\frac{1}{p}} \left(\sum_{k=1}^{m} a_k^p\right)^{\frac{1}{p}}.$$

The case $0 < p < 1$ is treated similarly. □

Let $0 < p < q < \infty$. Replacing in the second of inequalities (2.2.36) p by $\frac{q}{p}$ and a_k by a_k^p yields

$$(a_1^p + \cdots + a_m^p)^{\frac{1}{p}} \leqslant m^{\frac{1}{p}-\frac{1}{q}}(a_1^q + \cdots + a_m^q)^{\frac{1}{q}}\,, \tag{2.2.38}$$

which in the notation of Section 2.1.2 is inequality (2.1.15). Next we give a useful generalization of inequality (2.1.49), the simplest particular case of the right of inequalities (2.2.36).

Lemma 2.2.4. *Let $1 < p < \infty$. Then for all $a, b \geqslant 0$ and $\gamma > 1$*

$$(a+b)^p \leqslant \gamma a^p + C_p(\gamma) b^p \tag{2.2.39}$$

where

$$C_p(\gamma) = \left(1 - \gamma^{\frac{1}{1-p}}\right)^{1-p}. \tag{2.2.40}$$

Moreover, the equality holds if and only if $b = \left(\gamma^{\frac{1}{p-1}} - 1\right)a$.

The important point about this inequality is that the coefficient at a^p can be arbitrarily close to 1 at the expense of the coefficient at b^p which tends to ∞ as $\gamma \to 1+$.

Idea of the proof Reduce inequality (2.2.39) to the equivalent one-dimensional inequality. □

Proof If $b = 0$, the inequality is trivial. Let $b > 0$. Then, by dividing by b and putting $x = \frac{a}{b}$, we see that inequality (2.2.39) is equivalent to the inequality $(1+x)^p - \gamma x^p \leqslant C_p(\gamma)$ for all $x > 0$. The first statement follows since

$$\max_{x>0} \left((1+x)^p - \gamma x^p\right) = \left(1 - \gamma^{\frac{1}{1-p}}\right)^{1-p},$$

and the second one since $x = \left(\gamma^{\frac{1}{p-1}} - 1\right)^{-1}$ is the only point of maximum. □

Remark 2.2.7. Note that if $\gamma = 2^{p-1}$, then $C_p(\gamma) = 2^{p-1}$ and (2.2.40) coincides with (2.1.49). Note also that if $a > 0$, then

$$\min_{\gamma>1} \left(\gamma a^p + C_p(\gamma) b^p\right) = (a+b)^p \tag{2.2.41}$$

and $\gamma = \left(\frac{a+b}{a}\right)^{p-1}$ is the only point of minimum.

2.2.4 Converse Hölder's inequality

Assume that $\Omega \subset \mathbb{R}^n$ be a measurable set, $1 \leqslant p \leqslant \infty$, $f \in L_p(\Omega)$ and $\|f\|_{L_p(\Omega)} \leqslant M$. Then by Hölder's inequality

$$\left| \int_\Omega fg \, \mathrm{d}x \right| \leqslant M \, \|g\|_{L_{p'}(\Omega)} \tag{2.2.42}$$

for all $g \in L_{p'}(\Omega)$. This statement may be inverted as the following theorem shows.

Theorem 2.2.2. (Converse Hölder's inequality) *Let $\Omega \subset \mathbb{R}^n$ be a measurable set, a function f be measurable on Ω and $M \geqslant 0$.*

1. *If $1 \leqslant p \leqslant \infty$ and for all functions $g \in L_{p'}(\Omega)$ the functions fg are integrable on Ω and inequality (2.2.42) holds, then $f \in L_p(\Omega)$ and $\|f\|_{L_p(\Omega)} \leqslant M$.*

2. *If $0 < p < 1$ and for all functions $g \in L_{p'}(\Omega)$ the functions fg are integrable on Ω and the inequality*

$$\left| \int\limits_\Omega fg \, dx \right| \geqslant M \, \|g\|_{L_{p'}(\Omega)} \tag{2.2.43}$$

holds, then $\|f\|_{L_p(\Omega)} \geqslant M$.

Idea of the proof Consider the functions h_p, defined by* $h_p(x) := \frac{|f(x)|^p}{f(x)}$ if $f(x) \neq 0$ and $h_p(x) := 0$ if $f(x) = 0$, and the characteristic functions χ_k of the sets $\Omega_k := \{x \in \Omega\colon |x| < k, \frac{1}{k} < |f(x)| < k\}$, $k \in \mathbb{N}$. If $p = 1$ take in (2.2.42) $g = h_1$, otherwise take in (2.2.42) or (2.2.43) $g = h_p \chi_k$ and pass to the limit as $k \to \infty$. □

Proof The case $f \sim 0$ being trivial, assume that $f \not\sim$ on Ω. Hence $\|f\|_{L_p(\Omega)} > 0$.

1. If $p = 1$, then $h_1 \in L_\infty(\Omega)$ and $\|h_1\|_{L_\infty(\Omega)} = 1$. Hence, one may take $g = h_1$ in (2.2.42), and the statement follows because $\int\limits_\Omega f h_1 \, dx = \|f\|_{L_1(\Omega)}$.

2. Next let $1 < p < \infty$. If it were known that $f \in L_p(\Omega)$, then inequality (2.2.42) with $g = h_p$ would imply the statement because

$$\|h_p\|_{L_{p'}(\Omega)} = \|f\|^{p-1}_{L_p(\Omega)}, \qquad \int\limits_\Omega f h_p \, dx = \|f\|^p_{L_p(\Omega)}.$$

In order to prove that $f \in L_p(\Omega)$ and simultaneously obtain the desired inequality we take $g = h_p \chi_k$ with any $k \in \mathbb{N}$. Note that $\operatorname{meas} \Omega_k > 0$ for some $k_0 \in \mathbb{N}$. Otherwise, $\operatorname{meas}\{x \in \Omega\colon f(x) \neq 0\} = \operatorname{meas} \bigcup\limits_{k=1}^{\infty} \Omega_k = 0$, hence $f \sim 0$. Note that the functions $h_p \chi_k$ are measurable on Ω and for $k \geqslant k_0$

$$0 < k^{1-p} (\operatorname{meas} \Omega_k)^{1-\frac{1}{p}} \leqslant \|h_p \chi_k\|_{L_{p'}(\Omega)} = \|f \chi_k\|^{p-1}_{L_p(\Omega)} \leqslant k^{p-1} (\operatorname{meas} \Omega_k)^{1-\frac{1}{p}} < \infty.$$

Therefore inequality (2.2.42) with $g = h_p \chi_k$ implies that for all $k \in \mathbb{N}$

$$\|f \chi_k\|^p_{L_p(\Omega)} \leqslant M \, \|f \chi_k\|^{p-1}_{L_p(\Omega)}$$

and, hence, $\|f \chi_k\|_{L_p(\Omega)} \leqslant M$ for $k \geqslant k_0$. Since $|f(x)|^p \chi_k(x) \leqslant |f(x)|^p \chi_{k+1}(x)$ for all $x \in \Omega$ and $k \in \mathbb{N}$ and $\lim k \to \infty |f(x)|^p \chi_k(x) = |f(x)|^p$ for all $x \in \Omega$ by Monotone Convergence Theorem (see Corollary 1.3.7)

$$\|f\|_{L_p(\Omega)} = \lim_{k \to \infty} \|f \chi_k\|_{L_p(\Omega)} \leqslant M.$$

3. If $0 < p < 1$, the argument is similar.

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* If f is real-valued, then $h_p(x) := |f(x)|^{p-1} \operatorname{sgn} f(x)$.

4. The case $p = \infty$ can be reduced to the case $1 < p < \infty$. Indeed, inequality (2.2.42) with $p = \infty$ and inequality (2.2.17) imply that for all $1 < p < \infty$, for all $k \in \mathbb{N}$ and for all $g \in L_{p'}(\Omega \cap B_k)$

$$\left| \int_\Omega fg \, \mathrm{d}x \right| \leqslant M \, \|g\|_{L_1(\Omega)} \leqslant M \, (\mathrm{meas}(\Omega \cap B_k))^{\frac{1}{p}} \, \|g\|_{L_{p'}(\Omega)} \, .$$

Hence, by Step 3

$$\|f\|_{L_p(\Omega \cap B_k)} \leqslant M (\mathrm{meas}(\Omega \cap B_k))^{\frac{1}{p}} \, .$$

Passing to the limit as $p \to \infty$ we get, by Theorem 2.1.2, that $\|f\|_{L_\infty(\Omega \cap B_k)} \leqslant M$. Finally, passing to the limit as $k \to \infty$ and taking into account Lemma 2.1.12, we obtain the desired inequality. □

Corollary 2.2.9. *Let $\Omega \subset \mathbb{R}^n$ be a measurable set and a function f be measurable on Ω. If $\int_\Omega fg \, \mathrm{d}x = 0$ for all functions $g \in L_\infty(\Omega)$, then $f \sim 0$ on Ω.*

This is a particular case of the theorem corresponding to $M = 0$ and $p = \infty$.

Corollary 2.2.10. (Duality formula) *If $\Omega \subset \mathbb{R}^n$ is a measurable set, a function f is measurable on Ω and $1 \leqslant p \leqslant \infty$, then*

$$\|f\|_{L_p(\Omega)} = \sup_{\|g\|_{L_{p'}(\Omega)}=1} \left| \int_\Omega fg \, \mathrm{d}x \right| . \tag{2.2.44}$$

Idea of the proof Denote the right-hand side of this formula by M. Apply Hölder's inequality to prove that $M \leqslant \|f\|_{L_p(\Omega)}$, and the definition of a supremum and the theorem to prove that $M \geqslant \|f\|_{L_p(\Omega)}$. □

Proof By inequality (2.2.8)

$$M \leqslant \sup_{\|g\|_{L_{p'}(\Omega)}=1} \|f\|_{L_p(\Omega)} \, \|g\|_{L_{p'}(\Omega)} = \|f\|_{L_p(\Omega)} \, .$$

If $M = \infty$, this implies that $\|f\|_{L_p(\Omega)} = \infty$, hence formula (2.2.44) holds. Assume that $M < \infty$. Then, on the other hand, by the definition of a supremum, $\left| \int_\Omega fg \, \mathrm{d}x \right| \leqslant M$ for all $g \in L_{p'}(\Omega)$ satisfying $\|g\|_{L_{p'}(\Omega)} = 1$. Given an arbitrary $h \in L_{p'}(\Omega)$ which is not equivalent to 0, take here $g := \frac{h}{\|h\|_{L_{p'}(\Omega)}}$. Since $\|g\|_{L_{p'}(\Omega)} = 1$, this implies that

$$\left| \int_\Omega fh \, \mathrm{d}x \right| \leqslant M \, \|h\|_{L_{p'}(\Omega)} \tag{2.2.45}$$

for all $h \in L_{p'}(\Omega)$. Hence by the first part of Theorem 2.2.2 $\|f\|_{L_p(\Omega)} \leqslant M$, and formula (2.2.44) follows. □

Remark 2.2.8. At a first glance formula (2.2.44) does not seem to be useful, because $\|f\|_{L_p(\Omega)}$ is represented via an expression, more complicated than the initial one, which involves integrals $\int\limits_\Omega fg\,\mathrm{d}x$ for all functions $g \in L_{p'}(\Omega)$ satisfying $\|g\|_{L_{p'}(\Omega)} = 1$ and, moreover, taking supremum with respect to all such functions. However, the advantage, which can be exploited, is that the integral $\int\limits_\Omega fg\,\mathrm{d}x$ is, in contrast to $\|f\|_{L_p(\Omega)}$, linear with respect to f.

Remark 2.2.9. Let us consider the linear functional on $L_p(\Omega)$ where $1 \leqslant p \leqslant \infty$ associated with a measurable function f, defined for all functions $g \in L_p(\Omega)$ by

$$l_f(g) := \int\limits_\Omega fg\,\mathrm{d}x\,. \tag{2.2.46}$$

The right-hand side of formula (2.2.44) is the norm of this functional if p' is replaced by p. (See formula (1.1.5).) Hence

$$\|l_f\| = \|f\|_{L_{p'}(\Omega)}\,\cdot \tag{2.2.47}$$

Exercise 2.2.15. Prove the following variant of formula (2.2.44)

$$\|f\|_{L_p(\Omega)} = \sup_{\|g\|_{L_{p'}(\Omega)}=1} \int\limits_\Omega |fg|\,\mathrm{d}x\,. \tag{2.2.48}$$

2.2.5 Further exercises

Exercise 2.2.16. Let $1 \leqslant p \leqslant \infty, f \in L_p(0,\infty)$ and $F(x) := \int\limits_0^x f\,\mathrm{d}y$ for $x \in (0,\infty)$. Prove that $F(x) = o(x^{\frac{1}{p'}})$ as $x \to 0+$ and $F(x) = O(x^{\frac{1}{p'}})$ as $x \to \infty$.

Exercise 2.2.17. Let $\Omega \subset \mathbb{R}^n$ be a measurable set, and let functions f, g, w be measurable on Ω and $w \geqslant 0$. Prove that if $1 < p < \infty$, then

$$\int\limits_\Omega |fg|\,w\,\mathrm{d}x \leqslant \Big(\int\limits_\Omega |f|^p\,w\,\mathrm{d}x\Big)^{\frac{1}{p}} \Big(\int\limits_\Omega |g|^{p'}\,w\,\mathrm{d}x\Big)^{\frac{1}{p'}}. \tag{2.2.49}$$

State and prove a similar inequality for $0 < p < 1$.

Exercise 2.2.18. In development of Exercise 2.2.6 prove, by applying Exercise 1.5.5, that 1) for *any* measurable set $\Omega \subset \mathbb{R}^n$ of infinite measure embedding (2.2.16) does not hold for $q > p$ and for $q < p$, 2) for *any* measurable set $\Omega \subset \mathbb{R}^n$ of non-zero finite measure embedding (2.2.16) does not hold for $q < p$.

Exercise 2.2.19. Prove that if

$$\operatorname{meas}\Omega < \left(\frac{p_1(p_2-p)}{\varepsilon p_2(p-p_1)}\right)^{\frac{p_1p_2}{p_2-p_1}},$$

then inequality (2.2.26) is strict. Otherwise the equality in (2.2.26) is attained if and only if $|f| \sim A\chi$ on Ω, where $A \geqslant 0$ and χ is the characteristic function of a measurable subset Ω_1 of Ω satisfying

$$\operatorname{meas}\Omega_1 = \left(\frac{p_1(p_2-p)}{\varepsilon p_2(p-p_1)}\right)^{\frac{p_1p_2}{p_2-p_1}}.$$

(If $p_2 = \infty$, then $\left(\frac{p_1(p_2-p)}{\varepsilon p_2(p-p_1)}\right)^{\frac{p_1p_2}{p_2-p_1}}$ should be replaced by the limit of this expression as $p_2 \to \infty$ equal to $\left(\frac{p_1}{\varepsilon p}\right)^{p_1}$.)

Exercise 2.2.20. Prove that if the entries of a determinant Δ of order m are functions in $L_m(\Omega)$ where $\Omega \subset \mathbb{R}^n$ is a measurable set, then $\Delta \in L_1(\Omega)$.

Exercise 2.2.21. Let $\Omega \subset \mathbb{R}^n$ be a measurable set, $m \in \mathbb{N}$, functions $f_1, \dots, f_m$ be measurable on Ω, $0 < p \leqslant p_1, \dots, p_m \leqslant \infty$, and let equality (2.2.28) be satisfied. If all $p_k \geqslant 1$, then by Corollary 2.2.6 inequality (2.2.29) holds. What happens if one of p_k is less that 1? two or more of p_k are less that 1?

Exercise 2.2.22. Let $\Omega \subset \mathbb{R}^n$ be a measurable set, $1 \leqslant p_1, p_2, p_3 \leqslant \infty$, $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = 1$ and let $f_k \in L_{p_k}(\Omega)$ for $k \in \{1, 2, 3\}$. Prove that

$$\int_\Omega |f_1 f_2 f_3|\, dx \leqslant \left(\|f_1 f_2\|_{L_{p_{12}}(\Omega)} \|f_2 f_3\|_{L_{p_{23}}(\Omega)} \|f_1 f_3\|_{L_{p_{13}}(\Omega)} \right)^{\frac{1}{2}}$$

$$\leqslant \|f_1\|_{L_{p_1}(\Omega)} \|f_2\|_{L_{p_2}(\Omega)} \|f_3\|_{L_{p_3}(\Omega)},$$

where $\frac{1}{p_{kl}} = \frac{1}{p_k} + \frac{1}{p_l}$. State and prove the generalization of this inequality for the product of m factors where $m > 3$.

2.3 Minkowski's inequality

2.3.1 Case $1 \leqslant p \leqslant \infty$

Inequality (2.1.51) proved in Section 2.1.5 and (2.1.6) imply that for $1 < p < \infty$

$$\|f+g\|_{L_p(\Omega)} \leqslant 2^{1-\frac{1}{p}} \left(\|f\|_{L_p(\Omega)} + \|g\|_{L_p(\Omega)} \right). \tag{2.3.1}$$

If $p = 1$, then the numerical factor in the right-hand side of (2.3.1) is equal to 1. In Section 2.1.5 it is proved that for $p = \infty$ this inequality also holds with the factor 1. (See inequality (2.1.52).) So is the case $1 < p < \infty$: one can obtain a better estimate in which the factor $2^{1-\frac{1}{p}}$ is replaced by 1.

Theorem 2.3.1. (Minkowski's inequality) *Let $\Omega \subset \mathbb{R}^n$ be a measurable set and $1 \leqslant p \leqslant \infty$. If $f, g \in L_p(\Omega)$, then $f + g \in L_p(\Omega)$ and*

$$\|f+g\|_{L_p(\Omega)} \leqslant \|f\|_{L_p(\Omega)} + \|g\|_{L_p(\Omega)}. \tag{2.3.2}$$

Idea of the first proof Apply inequality (2.2.39) with an arbitrary $\gamma > 1$ to $|f(x) + g(x)|^p$, integrate over Ω and next minimize the right-hand side over γ. □

First proof By inequality (2.2.39) we get that for all $x \in \Omega$ and for all $\gamma > 1$

$$|f(x) + g(x)|^p \leqslant \big(|f(x)| + |g(x)| \big)^p \leqslant \gamma \, |f(x)|^p + C_p(\gamma) \, |g(x)|^p \, . \tag{2.3.3}$$

Since the function in the right-hand side of this inequality is integrable on Ω and the function in the left-hand side is measurable on Ω, by Theorem 1.3.4 the function in the left-hand side is also integrable on Ω, hence $f + g \in L_p(\Omega)$, and

$$\int_\Omega |f+g|^p \, \mathrm{d}x \leqslant \gamma \int_\Omega |f(x)|^p \, \mathrm{d}x + C_p(\gamma) \int_\Omega |g(x)|^p \, \mathrm{d}x \, . \tag{2.3.4}$$

Therefore by formula (2.2.41)

$$\|f+g\|_{L_p(\Omega)} \leqslant \big(\min_{\gamma>1} \big(\gamma \, \|f\|^p_{L_p(\Omega)} + C_p(\gamma) \, \|g\|^p_{L_p(\Omega)} \big) \big)^{\frac{1}{p}} = \|f\|_{L_p(\Omega)} + \|g\|_{L_p(\Omega)} \, . \; \square$$

Next we give a standard proof of Minkowski's inequality, based on application of Hölder's inequality.

Idea of the second proof Note that

$$|f+g|^p = |f+g| \cdot |f+g|^{p-1} \leqslant (|f|+|g|) \, |f+g|^{p-1} = |f| \cdot |f+g|^{p-1} + |g| \cdot |f+g|^{p-1} \tag{2.3.5}$$

and apply inequality (2.2.8). □

Second proof Since the entries of inequality (2.3.5) are measurable on Ω, by Theorem 1.3.4 and Hölder's inequality

$$\int_\Omega |f+g| \, \mathrm{d}x \leqslant \int_\Omega |f| \cdot |f+g|^{p-1} \, \mathrm{d}x + \int_\Omega |g| \cdot |f+g|^{p-1} \, \mathrm{d}x \tag{2.3.6}$$

$$\leqslant \|f\|_{L_p(\Omega)} \, \big\| \, |f+g|^{p-1} \big\|_{L_{p'}(\Omega)} + \|g\|_{L_p(\Omega)} \, \big\| \, |f+g|^{p-1} \big\|_{L_{p'}(\Omega)} \, .$$

So,

$$\|f+g\|^p_{L_p(\Omega)} \leqslant \|f+g\|^{p-1}_{L_p(\Omega)} \big(\|f\|_{L_p(\Omega)} + \|g\|_{L_p(\Omega)} \big) \, . \tag{2.3.7}$$

If $\|f+g\|_{L_p(\Omega)} = 0$, inequality (2.3.2) is trivial. Assume that $\|f+g\|_{L_p(\Omega)} > 0$. By inequality (2.3.1) $\|f+g\|_{L_p(\Omega)} < \infty$. Therefore (2.3.7) implies (2.3.2). □

Remark 2.3.1. One can complete the second proof without referring to the separately proved inequality (2.3.1). The important point is that inequality (2.3.6) implies inequality (2.3.2), but under the assumption that $\|f+g\|_{L_p(\Omega)} < \infty$. Let χ_k be the characteristic function of the set $\{x \in \Omega \colon |x| < k, |f(x) + g(x)| < k\}$, $k \in \mathbb{N}$. Then $\|f\chi_k + g\chi_k\|_{L_p(\Omega)} < \infty$. Since $|f(x)\chi_k(x) + g(x)\chi_k(x)| \to |f(x) + g(x)|$ as $k \to \infty$ for all $x \in \Omega$, by the Fatou theorem 1.3.9

$$\|f+g\|_{L_p(\Omega)} \leqslant \sup_{k \in \mathbb{N}} \|f\chi_k + g\chi_k\|_{L_p(\Omega)} \leqslant \sup_{k \in \mathbb{N}} \big(\|f\chi_k\|_{L_p(\Omega)} + \|g\chi_k\|_{L_p(\Omega)} \big)$$

$$\leqslant \|f\|_{L_p(\Omega)} + \|g\|_{L_p(\Omega)} \, .$$

Corollary 2.3.1. *Let* $1 \leqslant p \leqslant \infty$ *and* $a, b \in l_p$. *Then* $a + b \in l_p$ *and*

$$\|a+b\|_{l_p} \leqslant \|a\|_{l_p} + \|b\|_{l_p} \, . \tag{2.3.8}$$

Idea of the proof The same as the idea of the proof of Corollary 2.2.7. □

Corollary 2.3.2. *Let $\Omega \subset \mathbb{R}^n$ be a measurable set, $1 \leqslant p \leqslant \infty$ and $m \in \mathbb{N}$. If $f_1, \dots, f_m \in L_p(\Omega)$, then $\sum\limits_{k=1}^{m} f_k \in L_p(\Omega)$ and*

$$\Big\| \sum_{k=1}^{m} f_k \Big\|_{L_p(\Omega)} \leqslant \sum_{k=1}^{m} \|f_k\|_{L_p(\Omega)} \,. \tag{2.3.9}$$

Idea of the proof Apply induction. □

Corollary 2.3.3. *Let $1 \leqslant p \leqslant \infty$ and $m \in \mathbb{N}$. If $a_1, \dots, a_m \in l_p$, then $\sum\limits_{k=1}^{m} a_k \in l_p$ and*

$$\Big\| \sum_{k=1}^{m} a_k \Big\|_{l_p} \leqslant \sum_{k=1}^{m} \|a_k\|_{l_p} \,. \tag{2.3.10}$$

Idea of the proof Apply induction starting with inequality (2.3.8) or the idea of the proof of Corollary 2.2.7. □

Next we investigate the cases in which Minkowski's inequality turns into an equality. We say that two complex numbers z and w are *positively proportional* if there exist $A, B \geqslant 0$ satisfying $|A| + |B| > 0$ such that $Az = Bw$, which is equivalent to $zw \neq 0$ and $\arg z = \arg w$, or $zw = 0$. (Geometrically, the position vectors corresponding to z and w have the same direction.) If z and w are real, this means that they have the same sign. Note that

$$|z + w| = |z| + |w| \iff z \text{ and } w \text{ are positively proportional}. \tag{2.3.11}$$

Moreover, we say that two complex-valued functions f and g are *positively almost proportional* on $\Omega \subset \mathbb{R}^n$ if there exist $A, B \geqslant 0$ satisfying $|A| + |B| > 0$ such that $Af(x) = Bg(x)$ for almost all $x \in \Omega$. Note that f and g are *positively almost proportional* on Ω if and only if $|f|$ and $|g|$ are almost proportional on Ω and $f(x)$ and $g(x)$ are positively proportional for almost all $x \in \Omega$. If f and g are real valued, the second part of the statement means that they have the same sign almost everywhere on Ω.

Pay attention to the distinction in statements 'f and g are positively almost proportional on Ω' and 'f and g are positively proportional almost everywhere on Ω'. The second one is much weaker, because it requires that for almost all $x \in \Omega$ there exist $A(x)$ and $B(x)$ satisfying $|A(x)| + |B(x)| > 0$ such that $A(x)f(x) = B(x)g(x)$, whilst the first one requires that $A(x) = A$ and $B(x) = B$ are independent of x.

Lemma 2.3.1. *Let $\Omega \subset \mathbb{R}^n$ be a measurable set, $1 \leqslant p < \infty$ and $f, g \in L_p(\Omega)$. The equality is attained in Minkowski's inequality* (2.3.2) *if and only if*

1) *for $p = 1$, the functions f and g are positively proportional almost everywhere on Ω,*

2) *for $1 < p < \infty$, the functions f and g are positively almost proportional on Ω.*

Idea of the proof If $p = 1$, apply (2.3.11). If $1 < p < \infty$, consider inequality (2.3.3) with

$$\gamma = \left(\frac{\|f\|_{L_p(\Omega)} + \|g\|_{L_p(\Omega)}}{\|f\|_{L_p(\Omega)}} \right)^{p-1}, \tag{2.3.12}$$

the choice being explained by Remark 2.2.7. Apply the second statement of Lemma 2.2.4 and 2.3.11 to find in which cases it turns into an inequality. □

Proof *Step* 1. Let $p = 1$. If f and g are positively proportional for almost everywhere on Ω, then by (2.3.11) $|f + g| = |f| + |g|$ almost everywhere on Ω. Hence, integration over Ω implies (2.3.2) with the equality. Otherwise $|f + g| < |f| + |g|$ on a subset of Ω of positive measure. Hence, by Theorem 1.3.4 integration over Ω implies (2.3.2) with the strict inequality.

Step 2. Let $1 < p < \infty$. By the second statement of Lemma 2.2.4 the second inequality in (2.3.3) with γ defined above turns into an equality for almost all $x \in \Omega$ if and only if $|f(x)| = \left(\gamma^{\frac{1}{p-1}} - 1\right)^{-1}|g(x)| = \frac{\|f\|_{L_p(\Omega)}}{\|g\|_{L_p(\Omega)}} |g(x)|$ for almost all $x \in \Omega$, i.e. if and only if $|f|$ and $|g|$ are almost proportional on Ω. By 2.3.11 the first inequality in (2.3.3) turns into an equality for almost all $x \in \Omega$ if and only if $f(x)$ and $g(x)$ are positively proportional for almost all $x \in \Omega$. So, both of inequalities in (2.3.3) turn into equalities if and only if f and g are positively almost proportional on Ω.

If this holds, then integrating (2.3.3) we obtain (2.3.2) with the equality. If it does not, then by Theorem 1.3.4 we obtain (2.3.2) with the strict inequality. □

Exercise 2.3.1. State and prove an analogue of Lemma 2.3.1 for sequences.

Remark 2.3.2. Inequality (2.3.3) with γ defined by (2.3.12) takes the form

$$|f(x) + g(x)|^p \leqslant \left(\frac{|f(x)|^p}{\|f\|^{p-1}_{L_p(\Omega)}} + \frac{|g(x)|^p}{\|g\|^{p-1}_{L_p(\Omega)}} \right) \left(\|f\|_{L_p(\Omega)} + \|g\|_{L_p(\Omega)} \right)^{p-1}. \tag{2.3.13}$$

It can also be proved directly by applying inequality (2.2.34) with $m = 2$. Indeed,

$$|f(x)| + |g(x)| = \left(|f(x)| \, \|f\|^{-\frac{1}{p'}}_{L_p(\Omega)} \right) \|f\|^{\frac{1}{p'}}_{L_p(\Omega)} + \left(|g(x)| \, \|g\|^{-\frac{1}{p'}}_{L_p(\Omega)} \right) \|g\|^{\frac{1}{p'}}_{L_p(\Omega)}$$

$$\leqslant \left(\left(|f(x)| \, \|f\|^{-\frac{1}{p'}}_{L_p(\Omega)} \right)^p + \left(|g(x)| \, \|g\|^{-\frac{1}{p'}}_{L_p(\Omega)} \right)^p \right)^{\frac{1}{p}} \left(\|f\|_{L_p(\Omega)} + \|g\|_{L_p(\Omega)} \right)^{\frac{1}{p'}}.$$

Raising this inequality to the power p we arrive at (2.3.13). This gives one more proof of Theorem 2.3.1 since integration of (2.3.13) over Ω yields inequality (2.3.2). This proof is shorter than the first proof of Theorem 2.3.1. However, it contains the trick of dividing and multiplying by $\|\cdot\|^{\frac{1}{p'}}_{L_p(\Omega)}$ suggested by that proof. Compared with the second proof of Theorem 2.3.1, the advantage is that it does not require proving separately that $\|f + g\|_{L_p(\Omega)} < \infty$ or applying the approximation procedure like the one described in Remark 2.3.1.

Minkowski's inequality is the triangle inequality for the spaces $L_p(\Omega)$. As in the case of general normed spaces (see Section 1.1) it implies the reverse Minkowski's inequality

$$\|f - g\|_{L_p(\Omega)} \geqslant \left| \|f\|_{L_p(\Omega)} - \|g\|_{L_p(\Omega)} \right| \tag{2.3.14}$$

for $f, g \in L_p(\Omega)$.

2.3.2 Case $0 < p < 1$

Theorem 2.3.2. *Let $\Omega \subset \mathbb{R}^n$ be a measurable set and $0 < p < 1$. If $f, g \in L_p(\Omega)$, then $f + g \in L_p(\Omega)$ and*

$$\|f + g\|_{L_p(\Omega)} \leqslant 2^{\frac{1}{p}-1}\big(\|f\|_{L_p(\Omega)} + \|g\|_{L_p(\Omega)} \big) . \tag{2.3.15}$$

Idea of the proof Apply first inequality (2.1.6) and then inequality (2.1.49). □

Proof By inequality (2.1.6) for all $x \in \Omega$

$$|f(x) + g(x)|^p \leqslant (|f(x)| + |g(x)|)^p \leqslant |f(x)|^p + |g(x)|^p . \tag{2.3.16}$$

Since both parts of the inequality are measurable on Ω by Theorem 1.3.4 and inequality (2.1.49)

$$\|f + g\|_{L_p(\Omega)} = \Big(\int\limits_\Omega |f + g|^p \, dx \Big)^{\frac{1}{p}} \leqslant \Big(\int\limits_\Omega |f|^p \, dx + \int\limits_\Omega |g|^p \, dx \Big)^{\frac{1}{p}} \tag{2.3.17}$$

$$\leqslant 2^{\frac{1}{p}-1}\Big(\Big(\int\limits_\Omega |f|^p \, dx \Big)^{\frac{1}{p}} + \Big(\int\limits_\Omega |g|^p \, dx \Big)^{\frac{1}{p}} \Big) = 2^{\frac{1}{p}-1}\big(\|f\|_{L_p(\Omega)} + \|g\|_{L_p(\Omega)} \big) . \tag{2.3.18}$$

□

Corollary 2.3.4. *Let $0 < p < 1$ and $a, b \in l_p$. Then $a + b \in l_p$ and*

$$\|a + b\|_{l_p} \leqslant 2^{\frac{1}{p}-1}(\|a\|_{l_p} + \|b\|_{l_p}) . \tag{2.3.19}$$

Idea of the proof The same as the idea of the proof of Corollary 2.2.7. □

Corollary 2.3.5. *Let $\Omega \subset \mathbb{R}^n$ be a measurable set, $0 < p < 1$ and $m \in \mathbb{N}$. If $f_1, \dots, f_m \in L_p(\Omega)$, then $\sum\limits_{k=1}^{m} f_k \in L_p(\Omega)$ and*

$$\Big\| \sum_{k=1}^{m} f_k \Big\|_{L_p(\Omega)} \leqslant m^{\frac{1}{p}-1} \sum_{k=1}^{m} \|f_k\|_{L_p(\Omega)} . \tag{2.3.20}$$

Idea of the proof Apply inequalities (2.2.36) and (2.2.37). □

Exercise 2.3.2. Prove the corollary.

Corollary 2.3.6. *Let $0 < p < 1$ and $m \in \mathbb{N}$. If $a_1, \dots, a_m \in l_p$, then $\sum\limits_{k=1}^{m} a_k \in l_p$ and*

$$\Big\| \sum_{k=1}^{m} a_k \Big\|_{l_p} \leqslant m^{\frac{1}{p}-1} \sum_{k=1}^{m} \|a_k\|_{l_p} . \tag{2.3.21}$$

Idea of the proof The same as the idea of the proof of Corollary 2.2.7. □

Lemma 2.3.2. *Let $\Omega \subset \mathbb{R}^n$ be a measurable set, $0 < p < 1$ and $f, g \in L_p(\Omega)$. The equality is attained in inequality (2.3.15) if and only if $fg \sim 0$ on Ω and $\|f\|_{L_p(\Omega)} = \|g\|_{L_p(\Omega)}$.*

Idea of the proof By applying (2.3.11), the description of the cases of equality in (2.1.6) and (2.1.49), and Theorem 1.3.4, investigate in which cases all inequalities in (2.3.16) – (2.3.18) turn into equalities. □

Exercise 2.3.3. Prove the lemma.

Similarly, the equality is attained in inequality (2.3.19) if and only if $a_k b_k = 0$ for all $k \in \mathbb{N}$ and $\|a\|_{l_p} = \|b\|_{l_p}$.

The above statements imply, in particular, that the factor $2^{\frac{1}{p}-1} > 1$ in inequalities (2.3.15) and (2.3.19) is sharp, i.e. the smallest possible one.

Remark 2.3.3. Each integral inequality involving arbitrary measurable functions implies an analogue for sequences, which can be obtained in the spirit of the proof of Corollary 2.2.7. The converse implication is not always true. In some cases it is possible to derive an analogues integral inequality. See, for example, Exercise 2.2.14. In some other cases it is not possible. For example, Jensen's inequality does not have a reasonable integral analogue, which follows by Corollary 2.2.3 and Exercise 2.2.6.

Finally we note the following useful generalization of inequality (2.3.15).

Lemma 2.3.3. *Let $\Omega \subset \mathbb{R}^n$ be a measurable set and $0 < p < 1$. For all $\gamma > 1$*

$$\|f+g\|_{L_p(\Omega)} \leqslant \gamma \|f\|_{L_p(\Omega)} + (1-\gamma^{p'})^{\frac{1}{p'}} \|g\|_{L_p(\Omega)} \tag{2.3.22}$$

for all $f, g \in L_p(\Omega)$.

The important point about this inequality is that, similarly to inequality (2.2.39), the coefficient at $\|f\|_{L_p(\Omega)}$ can be arbitrarily close to 1 at the expense of the coefficient at $\|g\|_{L_p(\Omega)}$ which tends to ∞ as $\gamma \to 1+$.

Idea of the proof Apply first inequality (2.1.6) and then inequality (2.2.39). □

If $\gamma = 2^{p-1}$, then inequality (2.3.22) coincides with (2.3.15).

Exercise 2.3.4. Prove the lemma.

Corollary 2.3.7. *Let $\Omega \subset \mathbb{R}^n$ be a measurable set and $0 < p < 1$. Provided $0 < \gamma < 1$,*

$$\|f-g\|_{L_p(\Omega)} \geqslant \gamma \|f\|_{L_p(\Omega)} - \left(\gamma^{p'}-1\right)^{\frac{1}{p'}} \|g\|_{L_p(\Omega)}$$

for all $f, g \in L_p(\Omega)$.

2.3.3 Further exercises

Exercise 2.3.5. Let $\Omega \subset \mathbb{R}^n$ be a measurable set, $0 < p < 1$ and $f, g \in L_p(\Omega)$. The equality is attained in inequality (2.3.22) if and only if $fg \sim 0$ on Ω and $\|g\|_{L_p(\Omega)} = (\gamma^{p'}-1)^{\frac{1}{p}} \|f\|_{L_p(\Omega)}$.

Exercise 2.3.6. Let $\Omega \subset \mathbb{R}^n$ be a measurable set, $0 < p < 1$ and $m \in \mathbb{N}$. If $f_1, \dots, f_m \in L_p(\Omega)$, then for all $\gamma > 1$

$$\|\sum_{k=1}^{m} f_k\|_{L_p(\Omega)} \leqslant \gamma \|f_1\|_{L_p(\Omega)} + (1-\gamma^{p'})^{\frac{1}{p'}}(m-1)^{\frac{1}{p}-1}\sum_{k=1}^{m}\|f_k\|_{L_p(\Omega)} ,$$

and for all $0 < \gamma < 1$

$$\|\sum_{k=1}^{m} f_k\|_{L_p(\Omega)} \geqslant \gamma \|f_1\|_{L_p(\Omega)} - (\gamma^{p'}-1)^{\frac{1}{p'}}(m-1)^{\frac{1}{p}-1}\sum_{k=1}^{m}\|f_k\|_{L_p(\Omega)} .$$

Exercise 2.3.7. Let $0 < p \leqslant \infty$ and $m \in \mathbb{N}$. Moreover, let $\Omega_1, \dots, \Omega_m \subset \mathbb{R}^n$ be measurable sets and $f \in \bigcap\limits_{k=1}^{m} L_p(\Omega_k)$. Prove that $f \in L_p(\bigcup\limits_{k=1}^{m} \Omega_k)$ and

$$\|f\|_{L_p(\bigcup\limits_{k=1}^{m} \Omega_k)} \leqslant \max\{1, m^{\frac{1}{p}-1}\} \sum_{k=1}^{m} \|f\|_{L_p(\Omega_k)} .$$

Exercise 2.3.8. Let $\Omega \subset \mathbb{R}^n$ be a measurable set and $0 < p \leqslant \infty$. Prove that for non-negative functions $f, g \in L_p(\Omega)$

$$\min\{1, 2^{\frac{1}{p}-1}\}\left(\|f\|_{L_p(\Omega)} + \|g\|_{L_p(\Omega)}\right) \leqslant \|f+g\|_{L_p(\Omega)} . \qquad (2.3.23)$$

Exercise 2.3.9. For $1 < p < \infty$ give another proof of Lemma 2.3.1 based on the second proof of inequality (2.3.2) and Lemma 2.2.3.

Exercise 2.3.10. Let $\Omega \subset \mathbb{R}^n$ and functions f, g be bounded on Ω. Prove that equality $\|f+g\|_{C(\Omega)} = \|f\|_{C(\Omega)} + \|g\|_{C(\Omega)}$ holds if and only if there exists a sequence $\{x_k\}_{k\in\mathbb{N}}$ of points $x_k \in \Omega$ such that the limits $\lim\limits_{k\to\infty} f(x_k) = \lim\limits_{k\to\infty} g(x_k)$ exist, are positively proportional and

$$\big|\lim_{k\to\infty} f(x_k)\big| = \|f\|_{C(\Omega)} , \quad \big|\lim_{k\to\infty} g(x_k)\big| = \|g\|_{C(\Omega)} .$$

Exercise 2.3.11. Let $\Omega \subset \mathbb{R}^n$ be a measurable set and $f, g \in L_\infty(\Omega)$. Prove that equality (2.1.52) holds if and only if there exist functions F and G equivalent to f, g respectively, such that $\|F\|_{C(\Omega)} = \|f\|_{L_\infty(\Omega)}$ and $\|G\|_{C(\Omega)} = \|g\|_{L_\infty(\Omega)}$, and a sequence $\{x_k\}_{k\in\mathbb{N}}$ of points $x_k \in \Omega$ such that the limits $\lim\limits_{k\to\infty} F(x_k) = \lim\limits_{k\to\infty} G(x_k)$ exist, are positively proportional and

$$\big|\lim_{k\to\infty} F(x_k)\big| = \|f\|_{L_\infty(\Omega)} , \quad \big|\lim_{k\to\infty} G(x_k)\big| = \|g\|_{L_\infty(\Omega)} .$$

(*Hint* Consider the set $\omega := \Omega_M(|f|) \cup \Omega_N(|g|) \cup \Omega_L(|f+g|)$ where $M := \|f\|_{L_\infty(\Omega)}$, $N := \|g\|_{L_\infty(\Omega)}$ and $L := \|f+g\|_{L_\infty(\Omega)}$, and the functions F, G defined by $F(x) := f(x), G(x) := g(x)$ for $x \in \Omega \setminus \omega$ and $F(x) := G(x) := 0$ for $x \in \omega$.)

Exercise 2.3.12. Given sequences $a, b \in l_\infty$, state and prove the conditions on a and b ensuring that $\|a+b\|_{l_\infty} = \|a\|_{l_\infty} + \|b\|_{l_\infty}$.

Exercise 2.3.13. Let $m \in \mathbb{N}$, $a = \{a_k\}_{k=1}^m$, $b = \{b_k\}_{k=1}^m$ and $a_1, b_1, \dots, a_m, b_m \geqslant 0$. Prove that

$$\|a+b\|_{l_0}^* \geqslant \|a\|_{l_0}^* + \|b\|_{l_0}^* . \tag{2.3.24}$$

Moreover, prove that the equality is attained if and only if the sequences a and b are proportional. (*Hint* Apply the convexity of the function φ defined for all $0 \leqslant t \leqslant 1$ by $\varphi(t) := \prod\limits_{k=1}^{m} (ta_k + (1-t)b_k)^{\frac{1}{m}}$.)

Exercise 2.3.14. Let Ω be a measurable set, $0 < \operatorname{meas}\Omega < \infty$ and let functions f and g be non-negative and measurable on Ω. Prove that

$$\|f+g\|_{L_0(\Omega)}^* \geqslant \|f\|_{L_0(\Omega)}^* + \|g\|_{L_0(\Omega)}^* . \tag{2.3.25}$$

2.4 Convergence in $L_p(\Omega)$. Completeness of the spaces $L_p(\Omega)$

In this section we discuss the notion of convergence of a sequence of functions in $L_p(\Omega)$ and compare it with other types of convergence. The main result of the section is the proof of the completeness of the spaces $L_p(\Omega)$.

Lemma 2.4.1. (Uniqueness of a limit) *Let $\Omega \subset \mathbb{R}^n$ be a measurable set and $0 < p \leqslant \infty$. Moreover, let $f, g \in L_p(\Omega)$, $f_k \in L_p(\Omega), k \in \mathbb{N}$ and $f_k \to f$ in $L_p(\Omega)$ as $k \to \infty$, i.e.*

$$\lim_{k\to\infty} \|f_k - f\|_{L_p(\Omega)} = 0 . \tag{2.4.1}$$

Then $f_k \to g$ in $L_p(\Omega)$ as $k \to \infty$ if and only if g is equivalent to f on Ω.

Idea of the proof Apply inequalities (2.3.2) and (2.3.15) to $\|f-g\|_{L_p(\Omega)}$. □

Proof By (2.3.2) and (2.3.15)

$$\|f-g\|_{L_p(\Omega)} = \|f - f_k + f_k - g\|_{L_p(\Omega)} \leqslant A_p \left(\|f_k - f\|_{L_p(\Omega)} + \|f_k - g\|_{L_p(\Omega)} \right)$$

where $A_p := \max\{1, 2^{\frac{1}{p}-1}\}$. Passing to the limit as $k \to \infty$ we obtain that $\|f-g\|_{L_p(\Omega)} = 0$. Hence by Lemma 2.1.10 $f \sim g$ on Ω. □

Lemma 2.4.2. (Boundedness of a convergent sequence) *Let $\Omega \subset \mathbb{R}^n$ be a measurable set and $0 < p \leqslant \infty$. Moreover, let $\{f_k\}_{k\in\mathbb{N}}$ be a convergent sequence of functions $f_k \in L_p(\Omega)$, i.e. there exists a function $f \in L_p(\Omega)$ such that $f_k \to f$ in $L_p(\Omega)$ as $k \to \infty$. Then the sequence $\{f_k\}_{k\in\mathbb{N}}$ is bounded.*

The proof is similar to the standard proof for a sequence of complex numbers.

Lemma 2.4.3. *Let $\Omega \subset \mathbb{R}^n$ be a measurable set of finite and $0 < q < p \leqslant \infty$. Moreover, let $f \in L_p(\Omega)$, $f_k \in L_p(\Omega), k \in \mathbb{N}$ and $f_k \to f$ in $L_p(\Omega)$ as $k \to \infty$. If* $\operatorname{meas}\Omega < \infty$, *then $f_k \to f$ in $L_q(\Omega)$ as $k \to \infty$. Otherwise, $f_k \to f$ in $L_q(\Omega \cap B_r)$ as $k \to \infty$ for all $r > 0$.*

So, for sets of finite measure the convergence in $L_p(\Omega)$ is 'stronger' than the convergence in $L_q(\Omega)$ with $q < p$.

Idea of the proof Apply inequality (2.1.36). □

Proof If $\operatorname{meas}\Omega < \infty$, then by (2.1.36)

$$\|f_k - f\|_{L_q(\Omega)} \leqslant (\operatorname{meas}\Omega)^{\frac{1}{q}-\frac{1}{p}} \|f_k - f\|_{L(\Omega)} . \tag{2.4.2}$$

Passing to the limit as $k \to \infty$ we obtain that $\lim_{k\to\infty} \|f_k - f\|_{L_q(\Omega)} = 0$, hence $f_k \to f$ in $L_q(\Omega)$ as $k \to \infty$. If $\operatorname{meas}\Omega = \infty$, in (2.4.2) one should replace Ω by $\Omega \cap B_r$ with an arbitrary $r > 0$. □

Example 2.4.1. Let $0 < p < q \leqslant \infty$ and $f_k(x) := k^{\frac{1}{q}} \ln k$ if $0 < x < \frac{1}{k}$, $f_k(x) := 0$ if $\frac{1}{k} \leqslant x < 1$. Then $f_k \to 0$ in $L_p((0,1))$ as $k \to \infty$ since $\|f_k\|_{L_p((0,1))} = k^{\frac{1}{q}-\frac{1}{p}} \ln k \to 0$, but f_k do not converge in $L_q((0,1))$ since $\|f_k\|_{L_p((0,1))} = \ln k \to \infty$. Note also that $\lim_{k\to\infty} f_k(x) = 0$ for all $x \in (0,1)$.

Exercise 2.4.1. Let $0 < p, q \leqslant \infty, p \neq q$. Consider the set $(0,\infty)$ of infinite measure and construct a sequence $\{f_k\}_{k\in\mathbb{N}}$ of functions $f_k \in L_p(\Omega)$ such that it converges in $L_p(\Omega)$ but does not converge in $L_q(\Omega)$.

Example 2.4.2. For $k \in \mathbb{N}$ and $l \in \{1, \ldots, 2^k\}$ consider the characteristic functions of the intervals $(\frac{l-1}{k}, \frac{l}{k}]$. Put $f_1 := \chi_{11}, f_2 := \chi_{12}, f_3 := \chi_{22}$ and so on. Hence $f_m = \chi_{kl}$, where $m = 2^{k-1}+l-1$. Given $m \in \mathbb{N}$, $k = k(m)$ and $l = l(m)$ are defined by this equation uniquely and $k(m) \to \infty$ as $m \to \infty$. The sequence $\{f_m\}_{m\in\mathbb{N}}$ converges to 0 for any $0 < p < \infty$ because $\|f_m\|_{L_p((0,1))} = \left\|\chi_{k(m),l(m)}\right\|_{L_p((0,1))} = 2^{-k(m)} \to 0$ as $m \to \infty$. On the other hand, $\lim_{m\to\infty} f_m(x)$ does not exist for any $x \in (0,1)$ because for any $x \in (0,1)$ and for any $s \in \mathbb{N}$ there exist $m_0 > s$ and $m_1 > s$ such that $f_{m_0}(x) = 0$ and $f_{m_1}(x) = 1$.

So, Examples 2.4.1 and 2.4.2 show that pointwise convergence on Ω does not imply convergence in $L_p(\Omega)$ for any $0 < p \leqslant \infty$ and that it may happen that a sequence of functions converges in $L_p(\Omega)$ for some $0 < p < \infty$, but does not converge at any point of Ω. In particular, convergence in $L_p(\Omega)$ where $0 < p < \infty$ does not imply convergence almost everywhere on Ω. However, the situation changes if one looks at subsequences as the following statement shows.

Theorem 2.4.1. *Let $\Omega \subset \mathbb{R}^n$ be a measurable set and $0 < p < \infty$. Moreover, let $f \in L_p(\Omega)$ and $f_k \in L_p(\Omega), k \in \mathbb{N}$. If the sequence $\{f_k\}_{k\in\mathbb{N}}$ converges to f in $L_p(\Omega)$, then there exists its subsequence $\{f_{k_s}\}_{s\in\mathbb{N}}$ which converges to f almost everywhere on Ω.*

Idea of the proof Choose k_s in such a way that

$$\|f_{k_s}\|_{L_p(\Omega)} < 2^{-\frac{s}{p}} \tag{2.4.3}$$

and apply Corollary 1.3.4. □

Proof Equality (2.4.1) implies that for all $s \in \mathbb{N}$ there exist k_s such that inequality (2.4.3) holds. Consequently $\sum_{s=1}^{\infty} \int_\Omega |f_{k_s} - f|^p \, dx < \infty$. Therefore by Corollary 1.3.4 $\sum_{s=1}^{\infty} |f_{k_s}(x) - f(x)|^p < \infty$ for almost all $x \in \Omega$. Hence $\lim_{s\to\infty} f_{k_s}(x) = f(x)$ for almost all $x \in \Omega$. □

Exercise 2.4.2. Let $\Omega \subset \mathbb{R}^n$ be a measurable set. Moreover, let $f \in L_\infty(\Omega)$ and $f_k \in L_\infty(\Omega), k \in \mathbb{N}$. If the sequence $\{f_k\}_{k\in\mathbb{N}}$ converges to f in $L_\infty(\Omega)$, then it converges to f almost everywhere on Ω.

Corollary 2.4.1. *Let $\Omega \subset \mathbb{R}^n$ be a measurable set and $0 < p, q \leqslant \infty$. Moreover, let $f, g \in L_p(\Omega)$ and $f_k \in L_p(\Omega) \cap L_q(\Omega), k \in \mathbb{N}$. If $f_k \to f$ in $L_p(\Omega)$ and $f_k \to f$ in $L_p(\Omega)$, then $f \sim g$ on Ω.*

Idea of the proof Apply the theorem twice. □

Proof If $p = q$ this is Lemma 2.4.1. Assume that $p \neq q$. By the theorem there exist a subset Ω_1 and a subsequence $\{f_{k_s}\}_{s\in\mathbb{N}}$ such that meas $(\Omega \setminus \Omega_1) = 0$ and $\lim_{s\to\infty} f_{k_s}(x) = f(x)$ for all $x \in \Omega_1$. Since $f_{k_s} \to g$ in $L_q(\Omega_1)$ as $s \to \infty$, by the theorem there exist a subset Ω_2 and a subsequence $\{f_{k_{s_r}}\}_{r\in\mathbb{N}}$ such that meas $(\Omega_1 \setminus \Omega_2) = 0$ and $\lim_{r\to\infty} f_{k_{s_r}}(x) = g(x)$ for all $x \in \Omega_2$. Hence $f(x) = g(x)$ for all $x \in \Omega_2$. Since meas$(\Omega \setminus \Omega_2) =$ meas $(\Omega \setminus \Omega_1) +$ meas $(\Omega_1 \setminus \Omega_2) = 0$, it follows that $f \sim g$ on Ω. □

Next we pass to proving the main result of the section, the completeness of the spaces $L_p(\Omega)$. We start with proving the completeness of the spaces l_p.

Theorem 2.4.2. *Let $0 < p \leqslant \infty$ and let $a_k := \{a_{kl}\}_{l\in\mathbb{N}} \in l_p, k \in \mathbb{N}$. If $\{a_k\}_{k\in\mathbb{N}}$ is a Cauchy sequence in l_p, i.e.*

$$\lim_{k,m\to\infty} \|a_k - a_m\|_{l_p} = 0\,, \tag{2.4.4}$$

then there exist $a \in l_p$ such that $a_k \to a$ in l_p.

Idea of the proof Deduce from (2.4.4) that for all $l \in \mathbb{N}$ the sequences $\{a_{kl}\}_{k\in\mathbb{N}}$ are Cauchy sequences of complex numbers and apply the completeness property of complex numbers. □

Proof Since $|a_{kl} - a_{ml}| \leqslant \|a_k - a_m\|_{l_p}$ for all $l \in \mathbb{N}$ equality (2.4.4) implies that $\lim_{k,m\to\infty} |a_{kl} - a_{ml}| = 0$. Hence all $\{a_{kl}\}_{k\in\mathbb{N}}$ are Cauchy sequences of complex numbers. By the General Principle of Convergence there exist $b_l \in \mathbb{C}$ such that $a_{kl} \to b_l$ as $k \to \infty$. Set $a := \{b_l\}_{l\in\mathbb{N}}$.

Equality (2.4.4) means that for all $\varepsilon > 0$ there exist $N \in \mathbb{N}$ such that for all $k, m > N$

$$\|a_k - a_m\|_{l_p} < \varepsilon\,. \tag{2.4.5}$$

We want to pass to the limit here as $m \to \infty$. Since the sum consists of an infinite number of summands, first we note that for all $s \in \mathbb{N}$

$$\Big(\sum_{l=1}^{s} |a_{kl} - a_{ml}|^p\Big)^{\frac{1}{p}} \leqslant \|a_k - a_m\|_{l_p} < \varepsilon\,. \tag{2.4.6}$$

This sum being finite, by passing to the limit as $m \to \infty$, we get that

$$\Big(\sum_{l=1}^{s} |a_{kl} - b_l|^p\Big)^{\frac{1}{p}} \leqslant \varepsilon\,. \tag{2.4.7}$$

Finally, passing to the limit as $s \to \infty$, we have $\|a_k - a\|_{l_p} \leqslant \varepsilon$ for all $k > N$. Hence $a_k \to a$ in l_p as $k \to \infty$. □

Theorem 2.4.3. (The Riesz—Fisher theorem on the completeness of the spaces $L_p(\Omega)$) *Let $\Omega \subset \mathbb{R}^n$ be a measurable set, $0 < p \leqslant \infty$ and $f_k \in L_p(\Omega), k \in \mathbb{N}$. If $\{f_k\}_{k\in\mathbb{N}}$ is a Cauchy sequence in $L_p(\Omega)$, i.e.*

$$\lim_{k,m\to\infty} \|f_k - f_m\|_{L_p(\Omega)} = 0\,, \tag{2.4.8}$$

then there exist $f \in L_p(\Omega)$ such that $f_k \to f$ in $L_p(\Omega)$.

Preliminary discussion Compared with the previous proof, the general plan of the proof of this theorem is similar. However, there are important distinctions. First of all one cannot prove, excluding the case $p = \infty$, that there exists a finite limit $\lim\limits_{k\to\infty} f_k(x)$ for almost all $x \in \Omega$. (See Example 2.4.2.) However, as in the case of Theorem 2.4.1, it is possible to prove that this holds for a subsequence $\{f_{k_s}\}_{s\in\mathbb{N}}$. Equality (2.4.8) means that for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $k, m \geqslant N$

$$\|f_k - f_m\|_{L_p(\Omega)} < \varepsilon\,. \tag{2.4.9}$$

One can take here $m = k_s$ and pass to the limit as $s \to \infty$. However, the argument used in the proof of Theorem 2.4.2 cannot be applied and one should use instead one of the Fatou theorem.

Idea of the proof Choose k_s in such a way that

$$\big\|f_{k_{s+1}} - f_{k_s}\big\|_{L_p(\Omega)} < 2^{-s}\,. \tag{2.4.10}$$

By applying Jensen's and Hölder's inequalities prove that this implies that

$$\int\limits_\Omega \Big(\sum_{k=1}^{\infty} \big|f_{k_{s+1}} - f_{k_s}\big|\Big)^p \,dx < \infty\,, \quad 0 < p \leqslant 1\,, \tag{2.4.11}$$

and also, if $\operatorname{meas}\Omega < \infty$, that

$$\int\limits_\Omega \Big(\sum_{k=1}^{\infty} \big|f_{k_{s+1}} - f_{k_s}\big|\Big) \,dx < \infty\,, \quad 1 < p < \infty\,. \tag{2.4.12}$$

If $\operatorname{meas}\Omega = \infty$, here Ω should be replaced by $\Omega \cap B_r$ with an arbitrary $r > 0$. Deduce from (2.4.11) and (2.4.12) that for almost all $x \in \Omega$

$$\lim_{s\to\infty} f_{k_s}(x) =: f(x)\,. \tag{2.4.13}$$

Taking $m = k_s$ in inequality (2.4.9) and applying Theorem 1.3.9, prove that $f_k \to f$ in $L_p(\Omega)$. □

Proof *Step* 1. First, let $0 < p < \infty$. Condition (2.4.9) implies that for all $s \in \mathbb{N}$ there exist $k_s \in \mathbb{N}$ such that $k_1 < k_2 < \cdots$ and $\|f_k - f_m\|_{L_p(\Omega)} < 2^{-s}$ for all $k, m \geqslant k_s$. Taking here $k := k_{s+1}, m := k_s$ we obtain inequality (2.4.10).

Step 2. If $0 < p \leqslant 1$, then by Jensen's inequality (2.1.5) and Corollary 1.3.8

$$\int\limits_\Omega \Big(\sum_{k=1}^{\infty} \big|f_{k_{s+1}} - f_{k_s}\big|\Big)^p \,dx \leqslant \int\limits_\Omega \Big(\sum_{k=1}^{\infty} \big|f_{k_{s+1}} - f_{k_s}\big|^p\Big) \,dx$$

$$= \sum_{k=1}^{\infty} \int_{\Omega} |f_{k_{s+1}} - f_{k_s}|^p \, dx = \sum_{k=1}^{\infty} \|f_{k_{s+1}} - f_{k_s}\|^p_{L_p(\Omega)} < \sum_{k=1}^{\infty} 2^{-sp} < \infty . \quad (2.4.14)$$

If $1 < p \leqslant \infty$ and $\operatorname{meas}\Omega < \infty$, then by Corollary 1.3.8 and Hölder's inequality (2.2.8)

$$\int_{\Omega} \Big(\sum_{k=1}^{\infty} |f_{k_{s+1}} - f_{k_s}| \Big) \, dx = \sum_{k=1}^{\infty} \int_{\Omega} |f_{k_{s+1}} - f_{k_s}| \, dx$$

$$\leqslant (\operatorname{meas}\Omega)^{\frac{1}{p'}} \sum_{k=1}^{\infty} \|f_{k_{s+1}} - f_{k_s}\|_{L_p(\Omega)} < (\operatorname{meas}\Omega)^{\frac{1}{p'}} \sum_{k=1}^{\infty} 2^{-s} < \infty . \quad (2.4.15)$$

If $\operatorname{meas}\Omega = \infty$, this inequality holds if Ω is replaced by $\Omega \cap B_r$ with an arbitrary $r > 0$.

Step 3. Conditions (2.4.14) and (2.4.15) imply that the series

$$f_{k_1}(x) + \sum_{k=1}^{\infty} (f_{k_{s+1}}(x) - f_{k_s}(x))$$

converges for almost all $x \in \Omega$, which is equivalent to the existence of a finite limit (2.4.13). If $\operatorname{meas}\Omega = \infty$ this first follows for almost all $x \in \Omega \cap B_r$ and, since $r > 0$ is arbitrary, for almost all $x \in \Omega$. (See Exercise 1.5.8.)

Step 4. Let a function f be defined by (2.4.13) for those $x \in \Omega$ for which a finite limit in (2.4.13) exists, and in an arbitrary way for all other points $x \in \Omega$. Being an almost everywhere limit of a sequence of measurable functions, it is measurable on Ω. By (2.4.9) with any $k \geqslant N$ and $m = k_s$, where $s \geqslant N$ hence $k_s \geqslant N$ because $k_s \geqslant s$, we get $\|f_k - f_{k_s}\|_{L_p(\Omega)} < \varepsilon$. By applying the Fatou theorem, in particular inequality (1.3.17) we get

$$\|f_k - f\|_{L_p(\Omega)} = \Big\| \lim_{s \to \infty} (f_k - f_{k_s}) \Big\|_{L_p(\Omega)} \leqslant \sup_{s \in \mathbb{N}} \|f_k - f_{k_s}\|_{L_p(\Omega)} \leqslant \varepsilon .$$

This means that $f \in L_p(\Omega)$, because $f_k \in L_p(\Omega)$ and $f - f_k \in L_p(\Omega)$, and that $f_k \to f$ in $L_p(\Omega)$ as $k \to \infty$.

Step 5. Finally, let $p = \infty$. By Corollary 2.1.1 for all $k, m \in \mathbb{N}$

$$|f_k(x) - f_m(x)| \leqslant \|f_k - f_m\|_{L_\infty(\Omega)} \quad (2.4.16)$$

for almost all $x \in \Omega$, i.e. there exist sets $\omega_{km} \subset \Omega$ of zero measure such that (2.4.16) holds for all $x \in \Omega \setminus \omega_{km}$. Let $\omega := \bigcup_{k,m \in \mathbb{N}} \omega_{km}$. Since ω is a union contains of countable number of sets of zero measure, $\operatorname{meas}\omega = 0$. Moreover, for all $x \in \Omega \setminus \omega$ inequality (2.4.16) holds for all $k, m \in \mathbb{N}$. Hence condition (2.4.9) with $p = \infty$ implies that $\{f_k(x)\}_{k \in \mathbb{N}}$ are Cauchy sequences of complex numbers for all $x \in \Omega \setminus \omega$. Therefore, by the General Principle of Convergence, there exist finite limits $\lim_{k \to \infty} f_k(x) =: f(x)$ for all $x \in \Omega \setminus \omega$. For $x \in \omega$ we may define $f(x)$ in an arbitrary way.

Inequalities (2.4.5) with $p = \infty$ and (2.4.16) imply that $|f_k(x) - f_m(x)| < \varepsilon$ for all $x \in \Omega \setminus \omega$ and for all $k, m \geqslant N$. Passing to the limit as $m \to \infty$ we get that $|f_k(x) - f(x)| \leqslant \varepsilon$ for almost all $x \in \Omega$. So, by Definition 2.1.4 $\|f_k - f\|_{L_\infty(\Omega)} \leqslant \varepsilon$ for all $k \geqslant N$, which means that $f \in L_\infty(\Omega)$ and $f_k \to f$ in $L_\infty(\Omega)$ as $k \to \infty$. □

Let functions f and $f_k, k \in \mathbb{N}$, be measurable on a measurable set $\Omega \subset \mathbb{R}^n$. It is said that the sequence $\{f_k\}_{k\in\mathbb{N}}$ converges to f *in measure* on Ω ?? if for all $\sigma > 0$

$$\lim_{k\to\infty} \operatorname{meas}\{x \in \Omega\colon\ |f_k(x) - f(x)| \geqslant \sigma\} = 0\,.$$

Example 2.4.3. The sequence $\{f_l\}_{l\in\mathbb{N}}$ of Example 2.4.2 converges to 0 in measure on $(0,1)$ because $\operatorname{meas}\{x \in (0,1)\colon\ |f_l(x)| \geqslant \sigma\} \leqslant 2^{-k(l)}$ for all $\sigma > 0$.

Exercise 2.4.3. By estimating the integral $\int\limits_\Omega |f_k - f|^p\, dx$ below via the integral of the same integrand over the set $\{x \in \Omega\colon\ |f_k(x) - f(x)| \geqslant \sigma\}$, prove that if for some $0 < p < \infty$ a sequence $\{f_k\}_{k\in\mathbb{N}}$ of functions $f_k \in L_p(\Omega)$ converges in $L_p(\Omega)$ to a function f measurable on Ω, then it also converges to f in measure on Ω. Construct an example showing that the converse is not true.

2.5 Classification of the spaces $L_p(\Omega)$

In this section we summarize the basic information about L_p-spaces from the point of view of the classification of various spaces in the functional analysis, briefly describes in Section 1.1.

We start with the spaces of sequences l_p. If $1 \leqslant p \leqslant \infty$, then l_p are linear (vector) spaces and $\|a\|_{l_p}$ are norms. Moreover, they are Banach spaces. Indeed, properties 1 – 3 of a normed linear space are clearly satisfied. Property 4, the triangle inequality, is proved in Corollary 2.3.3, and the completeness in Theorem 2.4.2.

If $0 < p < 1$, then properties 1—3 of a normed linear space are satisfied, but the triangle inequality is not as proved in Section 2.3.2. It should be replaced by inequality (2.3.19) in which the factor $2^{\frac{1}{p}-1} > 1$ is the smallest possible. As for Theorem 2.4.2, it is valid for $0 < p < 1$ as well. Hence in this case l_p are complete quasi-normed spaces.

Next, let $\Omega \subset \mathbb{R}^n$ be a measurable set and let $\operatorname{meas}\Omega > 0$. If $1 \leqslant p \leqslant \infty$, then by Lemma 2.1.13 the spaces $L_p(\Omega)$ are linear spaces, and for the quantity $\|f\|_{L_p(\Omega)}$ properties 1, 3 and 4 are satisfied. Minkowski's inequality (2.3.2) is the triangle inequality for these spaces. Moreover, by the Riesz-Fisher theorem, they are complete spaces. This important property holds because the Lebesgue integration is considered, and this is the main reason why the Lebesgue integral is used in the theory of L_p-spaces.

However, property 2 is not satisfied, since the condition $\|f\|_{L_p(\Omega)} = 0$ does not imply that $f = 0$ on Ω, i.e. $f(x) = 0$ for all $x \in \Omega$. By Lemma 2.1.10 this holds if and only if $f(x) = 0$ for almost all $x \in \Omega$. In other words, in the notation of Section 1.1, the set $\tilde{\theta}$ is the the set of all functions f equivalent to 0 on Ω. So, $L_p(\Omega)$ are semi-Banach spaces and $\|f\|_{L_p(\Omega)}$ semi-norms respectively.

One of course should note that the distinction from being Banach spaces is 'very tiny', since functions equivalent on Ω from point of view of applications in the majority of cases may be treated as equal because in applications of L_p-spaces normally various integrals are considered and, by the properties of the Lebesgue integration, integrals involving equivalent functions have the same value.

In many cases, together with the spaces $L_p(\Omega)$, the factor-spaces $\widetilde{L}_p(\Omega) = L_p(\Omega)/\tilde{\theta}$ are considered, consisting of the disjoint classes $\tilde{f}$ of all functions which are equivalent to each other on Ω: $f_1, f_2 \in \tilde{f} \Longleftrightarrow f_1 - f_2 \in \tilde{\theta} \Longleftrightarrow f_1 \sim f_2$ on Ω. For them, by definition, $\|\tilde{f}\|_{\widetilde{L}_p(\Omega)} := \|f\|_{L_p(\Omega)}$, where f is any function in the class $\tilde{f}$. The spaces $\widetilde{L}_p(\Omega)$ are Banach spaces, and $\|\tilde{f}\|_{\widetilde{L}_p(\Omega)}$ norms respectively. (The class $\tilde{\theta}$ is a zero element in $\widetilde{L}_p(\Omega)$.)

If $0 < p < 1$, then the triangle inequality for the spaces $L_p(\Omega)$ does not hold and should be replaced by inequality (2.3.15) in which the factor $2^{\frac{1}{p}-1} > 1$ is the smallest possible. In this case $L_p(\Omega)$ is a semiquasi-normed space and $\widetilde{L}_p(\Omega)$ is a quasi-normed space.

If $1 \leqslant p \leqslant \infty$, then the spaces $L_p(\Omega)$ are complete semi-metric linear spaces if the semi-distance of $f \in L_p(\Omega)$ to $g \in L_p(\Omega)$ is defined by

$$d(f,g) := \|f - g\|_{L_p(\Omega)} \,.$$

(This is a standard way of defining the semi-distance for any semi-normed space.) Respectively, the spaces $\widetilde{L}_p(\Omega)$ are complete metric linear spaces.

Exercise 2.5.1. Prove that for $0 < p < 1$, the spaces $L_p(\Omega)$ are also complete semi-metric linear spaces with the semi-distance of $f \in L_p(\Omega)$ to $g \in L_p(\Omega)$ defined by

$$d(f,g) := \|f - g\|^p_{L_p(\Omega)} = \int\limits_\Omega |f - g|^p \, dx \,.$$

The case $p = 2$ is a very special case, because the expression

$$(f,g) := \int\limits_\Omega f\bar{g} \, dx \tag{2.5.1}$$

is a semi-inner product on $L_2(\Omega)$ satisfying $\sqrt{(f,f)} = \|f\|_{L_2(\Omega)}$. So, $L_2(\Omega)$ is a semi-Hilbert space and $\widetilde{L}_2(\Omega)$ is a Hilbert space.

If $p \neq 2$, then it it not possible to define a semi-inner product on $L_p(\Omega)$ satisfying $\sqrt{(f,f)} = \|f\|_{L_p(\Omega)}$ as the following example shows.

Example 2.5.1. Let $0 < p \leqslant \infty$, $p \neq 2$. For $\Omega = (-1,1)$ consider $f := \chi_{(-1,0]}$ and $g := \chi_{(0,1)}$. Then

$$\|f+g\|^2_{L_p(\Omega)} + \|f-g\|^2_{L_p(\Omega)} = 2^{\frac{2}{p}+1}, \quad 2\big(\|f\|^2_{L_p(\Omega)} + \|g\|^2_{L_p(\Omega)}\big) = 4 \,.$$

Therefore, the parallelogram identity is not satisfied and hence the statement follows. (See Section 1.1.)

Moreover, the following stronger assertion holds.

Lemma 2.5.1. *Let $\Omega \subset \mathbb{R}^n$ be a measurable set,* meas $\Omega > 0$, $0 < p \leqslant \infty$ *and* $p \neq 2$. *Then there does not exist a semi-inner product on $L_p(\Omega)$ such that $\sqrt{(f,f)}$ is equivalent to $\|f\|_{L_p(\Omega)}$, i.e. for some $c_1, c_2 > 0$ the inequality*

$$c_1 \|f\|_{L_p(\Omega)} \leqslant \sqrt{(f,f)} \leqslant c_2 \|f\|_{L_p(\Omega)}$$

for all $f \in L_p(\Omega)$.

Idea of the proof Assuming the converse deduce that there exist $c_3, c_4 > 0$ such that

$$c_3\left(\|f\|^2_{L_p(\Omega)}+\|g\|^2_{L_p(\Omega)}\right) \leqslant \|f+g\|^2_{L_p(\Omega)}+\|f-g\|^2_{L_p(\Omega)} \leqslant c_4\left(\|f\|^2_{L_p(\Omega)}+\|g\|^2_{L_p(\Omega)}\right)$$

for all $f, g \in L_p(\Omega)$. □

Moreover, $L_0(\Omega)$, being a linear space, is not a semiquasi-normed space. Indeed, if $\Omega = \Omega_1 \cup \Omega_2$, where Ω_1 and Ω_2 are disjoint measurable subsets of Ω of positive measures, $f = \chi(\Omega_1)$ and $g = \chi(\Omega_2)$, then $\|f\|^*_{L_0(\Omega)} = \|g\|^*_{L_0(\Omega)} = 0$ whilst $\|f+g\|^*_{L_0(\Omega)} = 1$. Hence property 4′ in Section 1.1 does not hold for any $c \geqslant 1$.

Remark 2.5.1. It should be noted that in many books $L_p(\Omega)$ is understood as $\widetilde{L}_p(\Omega)$ in our book, or the same notation $L_p(\Omega)$ is used both for the spaces $L_p(\Omega)$ and for the spaces $\widetilde{L}_p(\Omega)$, sometimes without stating this explicitly. One can often come across the statement 'For $1 \leqslant p \leqslant \infty$ the spaces $L_p(\Omega)$ are Banach spaces', the actual meaning of which being 'For $1 \leqslant p \leqslant \infty$ the spaces $\widetilde{L}_p(\Omega)$ are Banach spaces'. As explained above the distinction of $\widetilde{L}_p(\Omega)$ and $L_p(\Omega)$ is, in fact, not essential, nevertheless the formulations of some theorems depend on this distinction, and one should be certain of what is meant by $L_p(\Omega)$, the spaces $L_p(\Omega)$ or the spaces $\widetilde{L}_p(\Omega)$ of their equivalence classes.

Chapter 3

Solutions and hints to exercises

In this chapter we give detailed solutions to exercises included into the main text. As for other exercises we mostly give hints.

3.1 Exercises in Chapter 1

3.1.1 Exercises in Section 1.1

Exercise 3.1.1. If $x \in \tilde{\theta}$, i.e. $x \in X$ and $\|x\| = 0$, then $\|ax\| = |a| \cdot \|x\| = 0$ for all $a \in \mathbb{C}$, hence $ax \in \tilde{\theta}$. If $x_1, x_2 \in \tilde{\theta}$, then $0 \leqslant \|x_1 + x_2\| \leqslant \|x_1\| + \|x_2\| = 0$, hence $\|x_1 + x_2\| = 0$ and $x_1 + x_2 \in \tilde{\theta}$.

Exercise 3.1.2. Assume that $x_1, x_2 \in \tilde{x}$, then, by the triangle inequality,

$$\|x_2\| = \|x_2 - x_1 + x_1\| \leqslant \|x_2 - x_1\| + \|x_1\| = \|x_1\| \ .$$

Similarly, $\|x_1\| \leqslant \|x_2\|$, hence $\|x_1\| = \|x_2\|$.

Property 1 is clear. Properties 3 and 4 are proved similarly. We prove, for example, property 3. Let $x \in \tilde{x}$, then

$$\|a\tilde{x}\|_{\widetilde{X}} = \|\widetilde{ax}\|_{\widetilde{X}} = \|ax\|_X = |a| \cdot \|x\|_X = |a| \cdot \|\tilde{x}\|_{\widetilde{X}} \ .$$

Finally, to prove property 2 we first note that $\tilde{\theta}$ is the null element of $\widetilde{X}$, since for all $\tilde{x} \in \widetilde{X} \quad \tilde{x} + \tilde{\theta} = \widetilde{x + \theta} = \tilde{x}$, and $\|\tilde{\theta}\|_{\widetilde{X}} = \|\theta\|_X = 0$. Furthermore, if $\|\tilde{x}\|_{\widetilde{X}} = 0$, then for all $x \in \tilde{x} \quad \|x\|_X = \|\tilde{x}\|_X = \|\tilde{x}\|_{\widetilde{X}} = 0$, hence $x \in \tilde{\theta}$. So $\tilde{x} \subset \tilde{\theta}$. Since, by Exercise 1, $\tilde{\theta}$ is a linear space, it follows that for all $x \in \tilde{x} \quad -x \in \tilde{\theta}$. By the definition of $\tilde{x}$, $\theta = x + (-x) \in \tilde{x}$ and for all $y \in \tilde{\theta} \quad y = \theta + y \in \tilde{x}$. Hence $\tilde{\theta} \subset \tilde{x}$ and $\tilde{x} = \tilde{\theta}$.

Exercise 3.1.3. By applying the reverse triangle inequality we have

$$|\|x_k\| - \|x\|| \leqslant \|x_k - x\| \ ,$$

and the statement follows by passing to the limit in this inequality.

Exercise 3.1.4. Since $x - \frac{(x,y)}{\|y\|^2} y \perp y$, by the Pythagoras theorem

$$\|x\|^2 = \|x - \frac{(x,y)}{\|y\|^2} y\|^2 + \|\frac{(x,y)}{\|y\|^2} y\|^2 \Longleftrightarrow \|x - \frac{(x,y)}{\|y\|^2} y\|^2 = \|x\|^2 - \frac{|(x,y)|^2}{\|y\|^2} .$$

Since the left-hand side is non-negative, it follows that

$$|(x,y)| \leqslant \|x\| \cdot \|y\|$$

for $y \neq \theta$. (If $y = \theta$, then the inequality is trivial.) Also

$$|(x,y)| = \|x\| \cdot \|y\| \Longleftrightarrow \|x - \frac{(x,y)}{\|y\|^2} y\| = 0 \Longleftrightarrow x = \frac{(x,y)}{\|y\|^2} y ,$$

which, for $y \neq \theta$, is equivalent to the proportionality of x and y. (Again the case $y = \theta$ is trivial: the equality holds for all $x \in X$ and all $x \in X$ are proportional to θ.)

Exercise 3.1.5. By Exercise 1.1.3 there exists a finite limit $\lim\limits_{k\to\infty} \|y_k\|$ (equal to $\|x\|$), hence the sequence $\{\|y_k\|\}_{k\in\mathbb{N}}$ is bounded, i. e. for some $M > 0$ for all $k \in \mathbb{N}$ $\|y_k\| \leqslant M$. By applying the Cauchy—Bunyakovskiĭ inequality (1.1.4) we get

$$\begin{aligned} |(x_k, y_k) - (x,y)| &= |(x_k - x, y_k) - (x, y_k - y)| = |(x_k - x, y_k)| + |(x, y_k - y)| \\ &\leqslant \|x_k - x\| \cdot \|y_k\| + \|x\| \cdot \|y_k - y\| \leqslant M \cdot \|y_k\| + \|x\| \cdot \|y_k - y\| . \end{aligned}$$

The statement follows by passing to the limit as $k \to \infty$.

3.1.2 Exercises in Section 1.3

Exercise 3.1.1. The proof follows from the equality

$$\sum_{k=1}^{s} \int_{\Omega_k} f \,\mathrm{d}x = \sum_{k=1}^{s} \int_{\bigcup\limits_{k=1}^{s} \Omega_k} \chi_k f \,\mathrm{d}x = \int_{\bigcup\limits_{k=1}^{s} \Omega_k} \Big(\sum_{k=1}^{s} \chi_k\Big) f \,\mathrm{d}x = \int_{\bigcup\limits_{k=1}^{s} \Omega_k} N f \,\mathrm{d}x ,$$

because $1 \leqslant N(x) \leqslant \varkappa$ for all $x \in \bigcup\limits_{m=1}^{s} \Omega_m$.

Exercise 3.1.2. Definitions 1.3.16, 1.3.17 and parts 2 and 3 of Definition 1.3.18 of the Lebesgue integral, together with Definition 1.3.12 imply that it suffices to prove the theorem for the case of bounded Ω and non-negative bounded functions f, because the difference of measurable functions is measurable and existing almost everywhere on G limit of a sequence of functions measurable on G is a measurable function.

So, let $\Omega \subset \mathbb{R}^n$ and $G \subset \mathbb{R}^m$ be measurable sets, Ω be bounded, and let a function f be measurable on $\Omega \times G$ and for some $M \geqslant 0$ the inequality $0 \leqslant f(x,y) \leqslant M$ be satisfied for all $x \in \Omega, y \in G$. By Definition 1.3.12 there exist step-functions f_k defined on $\mathbb{R}^{m+n}$ such that $f(x,y) = \lim\limits_{k\to\infty} f_k(x,y)$ for almost all $(x,y) \in \Omega \times G$. Here

$$f_k := \sum_{l=1}^{m_k} a_{kl} \chi(\Delta_{kl}^{(1)} \times \Delta_{kl}^{(2)}) ,$$

where $m_k \in \mathbb{N}, a_{kl} > 0$, and $\chi(\Delta_{kl}^{(1)} \times \Delta_{kl}^{(2)})$ is the characteristic function of the direct product of cuboids $\Delta_{kl}^{(1)} \subset \mathbb{R}^n$ and $\Delta_{kl}^{(2)} \subset \mathbb{R}^m$, whose faces are parallel to the coordinate planes. Moreover, without loss of generality, one may assume that $0 \leqslant f_k(x, y) \leqslant M$ for all $k \in \mathbb{N}$ and for all $x \in \Omega, y \in G$. Otherwise, one may replace the functions f_k by the functions $\tilde{f}_k$ defined for all $x \in \Omega, y \in G$ by $\tilde{f}_k(x, y) := \max\{\min\{f_k(x, y), M\}, 0\}$. Therefore by the Dominated Convergence Theorem (see Theorem 1.3.11) for almost all $y \in G$

$$\int_\Omega f(x, y)\,\mathrm{d}x = \lim_{k\to\infty} \int_\Omega f_k(x, y)\,\mathrm{d}x = \lim_{k\to\infty} \sum_{l=1}^{m_k} a_{kl}\,\mathrm{meas}\,(\Delta_{kl}^{(1)})\chi(\Delta_{kl}^{(2)})(y)\,.$$

Hence, by Definition 1.3.12, the function $\int_\Omega f(x, \cdot)\,\mathrm{d}x$, being an almost everywhere limit of step-functions, is measurable on G.

3.1.3 Exercises in Section 1.5

Exercise 3.1.3. Let $m(r) := \mathrm{meas}\,(\Omega \cap B_r)$. Since $\bigcap_{r>0} (\Omega \cap B_r) = \emptyset$ and $\bigcup_{r>0} (\Omega \cap B_r) = \Omega$, by the properties of measurable sets (see Section 1.3.1) $\lim_{r\to 0+} m(r) = 0$ and $\lim_{r\to\infty} m(r) = \mathrm{meas}\,\Omega$. The function m is continuous on $(0, \infty)$. Indeed, if say $\Delta r > 0$, then $0 \leqslant m(r + \Delta r) - m(r) = \mathrm{meas}\,(\Omega \cap B_{r+\Delta r} \setminus \Omega \cap B_r) \leqslant \mathrm{meas}\,(B_{r+\Delta r} \setminus B_r) = v_n\big((r + \Delta r))^n - r^n\big) \to 0$ as $\Delta r \to 0+$.

1. If $\mathrm{meas}\,\Omega = \infty$, we choose $\xi_k > 0$ such that $\mathrm{meas}\,(\Omega \cap B_{\xi_k}) = k$, and obtain a required sequence of subsets by setting $\Omega_1 := \Omega \cap B_{\xi_1}$ and $\Omega_k = \Omega \cap (B_{\xi_2} \setminus B_{\xi_1})$ for $k \geqslant 2$.

2. Let $0 < \mathrm{meas}\,\Omega < \infty$. a) We choose $\eta > 0$ such that $m(\eta) = \frac{1}{2}\,\mathrm{meas}\,\Omega$. Let $\Omega_1 := \Omega \cap Q_\eta$ and $\Omega_2 = \Omega \setminus \Omega_1$. Then $\Omega = \Omega_1 \cup \Omega_2$, $\Omega_1 \cap \Omega_2 = \emptyset$ and $\mathrm{meas}\,\Omega_1 = \mathrm{meas}\,\Omega_2 = \frac{1}{2}\,\mathrm{meas}\,\Omega$.

b) We choose $\eta_k > 0$ such that $\mathrm{meas}\,(\Omega \cap B_{\eta_k}) = 2^{-k}$, and obtain a required sequence of subsets by setting $\Omega_1 := \Omega \cap B_{\eta_1}$ and $\Omega_k = \Omega \cap (B_{\eta_2} \setminus B_{\eta_1})$ for $k \geqslant 2$.

Exercise 3.1.4. The desired set D_a is constructed similarly to Cantor's set D (corresponding to the case $a = 0$). The only distinction is that, in the first step, from the closed interval $[0, 1]$ an open interval centered at $\frac{1}{2}$ of length $\alpha := \frac{1-a}{3-2a}$ is cut out. In the second step from each of the two remaining closed intervals, open intervals centered at their midpoints of length α^2 are cut out, and so on. Hence $\mathrm{meas}\,D_a = 1 - \sum_{k=1}^{\infty} 2^{k-1}\alpha^k = a$.

Exercise 3.1.5. Assume that such set D_1 exists. Then the open set $[0, 1] \setminus D_1 = (0, 1) \setminus D_1$ is of zero measure, hence empty. Consequently $D_1 = [0, 1]$, which contradicts the assumption that D_1 is nowhere dense in $[0, 1]$.

3.2 Exercises in Chapter 2

3.2.1 Exercises in Section 2.1

Exercises in Section 2.1.2

Exercise 3.2.1. Given $q < p < \infty$, define $a_k = k^{-\frac{1}{q}}$ for $k \in \mathbb{N}$. Then

$$\|a\|_{l_p} = \Big(\sum_{k=1}^{\infty} k^{-\frac{p}{q}}\Big)^{\frac{1}{p}} < \infty \qquad \|a\|_{l_p} = \Big(\sum_{k=1}^{\infty} k^{-1}\Big)^{\frac{1}{p}} = \infty\,.$$

Hence $l_p \not\subset l_q$.

Exercises in Section 2.1.5

Exercise 3.2.2. The statement follows since by taking the spherical coordinates (see formula (1.3.29))

$$\frac{1}{v_n r^n}\int\limits_{B_r} \ln|x|^\gamma \,\mathrm{d}x = \frac{\gamma\sigma_n}{v_n r^n}\int\limits_0^r \varrho^{n-1}\ln\varrho\,\mathrm{d}\varrho = \frac{\gamma n}{r^n}\Big(\varrho^n\ln\varrho\Big|_0^r - \int\limits_0^r \varrho^{n-1}\,\mathrm{d}\varrho\Big) = \gamma\ln r - \frac{\gamma}{n}\,.$$

Exercise 3.2.3. By Example 2.1.1 $\||x|^\gamma\|^*_{L_p(B_r)} = \big(\frac{n}{n+\gamma p}\big)^{\frac{1}{p}} r^\gamma$ for $0 < p < \frac{n}{|\gamma|}$. Since $\big(1+\frac{\gamma p}{n}\big)^{\frac{1}{p}} \to \mathrm{e}^{-\frac{\gamma}{n}}$ Theorem 2.1.4 implies the equality $\||x|^\gamma\|^*_{L_0(B_r)} = \mathrm{e}^{-\frac{\gamma}{n}} r^\gamma$.

3.2.2 Exercises in Section 2.2

Exercises in Section 2.2.5

Exercise 3.2.4. First let $0 < \operatorname{meas}\Omega \leqslant \infty$ and $q < p$. Consider a measurable subset G of positive finite measure, and let the subsets $G_k, k \in \mathbb{N}$, be constructed as in the part 2 b) of Exercise 1.5.5. Assume that $f(x) = 2^{\frac{k}{p}}$ on G_k and $f(x) = 0$ on $\Omega \setminus G$. Then

$$\|f\|_{L_q(\Omega)} = (\operatorname{meas} G)^{\frac{1}{q}}\Big(\sum_{k=1}^{\infty} 2^{\left(\frac{q}{p}-1\right)k}\Big)^{\frac{1}{q}} < \infty\,, \quad \|f\|_{L_p(\Omega)} = \infty\,.$$

Hence $L_q(\Omega) \not\subset L_p(\Omega)$.

If $\operatorname{meas}\Omega = \infty$ and $q > p$, then we consider the subsets Ω_k constructed in part 1 of Exercise 1.5.5. Let $f(x) = a_k$ on $\Omega_k, k \in \mathbb{N}$. Then $\|f\|_{L_p(\Omega)} = \|a\|_{l_p}$, and the statement follows by Exercise 2.1.6.

Appendix

Axiom of choice and determination axiom

The existence of sets $\Omega \subset \mathbb{R}^n$ which are not Lebesgue measurable and of non-measurable functions $f\colon \Omega \to \mathbb{C}$ is based on the axiom of choice. We recall the formulation of the axiom and one of the methods of proving of the existence of non-measurable sets.

Definition 3.2.1. (Axiom of choice AC) *For each family $S \equiv \{S_\alpha\}_{\alpha \in A}$ of non-empty sets S_α of any kind, where A is a non-empty set of indices α, there exists a* function of choice $f : A \to S$, *i.e. a function satisfying $f(\alpha) \in S_\alpha$ for all $\alpha \in A$.*

In other words it is stated that there exists a rule f allowing to choose an element $f(\alpha)$ in each of the sets S_α, $\alpha \in A$. On one hand, the statement seems to be 'clear' enough to be accepted as an axiom. On the other hand, A is assumed to be an arbitrary set of indices, whilst our intuitive understanding of clearness is, in fact, not reliable for sets which are not countable.

The *countable axiom of choice AC_ω* is a weakened form of the axiom of choice stating the existence of a function of choice only for countable sets A. The axiom AC_ω is, in fact, used to prove that the union of a countable family of countable sets is countable, to prove the countable additivity of the Lebesgue measure (1.3.3) etc.

The axiom AC is used to prove the Zermelo theorem on possibility to partially order any set and the Zorn lemma. (In fact, these three statements are equivalent.) Moreover, it is used to prove the existence of non-measurable sets. This can be done, for example, in the following way.

Let $\Omega \subset \mathbb{R}^n$ be an arbitrary measurable set in $\mathbb{R}^n$ of positive measure. Choose $R > 0$ such that the subset $\Omega_1 := \Omega \cap B_R$ is also of positive measure. Consider disjoint subsets of Ω_1, such that two points x and y belong to one such subset if and only if all coordinates of the difference $x - y$ are rational. By the axiom AC one can choose one point in each of such sets. The subset $\Omega_2 \subset \Omega$, consisting of all such 'chosen' points is non-measurable. Indeed

$$\Omega_1 \subset \bigcup_{r \in \mathbb{Q}_r} (\Omega_2 + r) \subset B_{2R}\,,$$

where $\mathbb{Q}_R$ is the set of all points in the ball B_R with rational coordinates. If Ω_2 is measurable, then by the properties of measurable sets (see Section 1.3.1)

$$0 < \operatorname{meas}\Omega_1 \leqslant \operatorname{meas} \bigcup_{r \in \mathbb{Q}_r} (\Omega_2 + r) = \sum_{r \in \mathbb{Q}_R} \operatorname{meas}(\Omega_2 + r) = \sum_{r \in \mathbb{Q}_R} \operatorname{meas}\Omega_2 \leqslant \operatorname{meas} B_{2R},$$

which is impossible. If meas $\Omega_2 = 0$, then the left inequality does not hold. If meas $\Omega_2 > 0$, then the right one does not hold.

The axiom AC also allows proving the existence of even more 'paradox' sets. For this reason the axiom of choice was criticized as being non-constructive, because only the existence of a function of choice is stated, and there is no information how to construct it. On the other hand replacing it by axiom AC_ω appears to be too restrictive. Therefore the question arose of replacing the axiom AC by another axiom, which on one hand would imply the axiom AC_ω and on the other hand would lead to opposite statements in the cases in which the axiom AC implies undesirable consequences. In 1962 Mychelski and Steingaus suggested an axiom satisfying these requirements, which states a countable analogue of the law of excluded third ??

Definition 3.2.2. (The determination axiom AD**)** *For each set A of sequences $a \equiv \{a_k\}_{k\in\mathbb{N}}$ of natural numbers a_k*

$$\text{either}\quad \exists\, a_1 \,\forall\, a_2 \,\exists\, a_3 \cdots \text{ such that } a \in A \quad \text{or} \quad \forall\, a_1 \,\exists\, a_2 \,\forall\, a_3 \cdots \text{ such that } a \notin A\,.$$

The axiom AD implies the axiom AC_ω. Moreover, it implies that each set $\Omega \subset \mathbb{R}^n$ is Lebesgue measurable, hence each function $f\colon \Omega \to \mathbb{C}$ is measurable. This is the content of the Mychelski—Sverchkovski theorem.

It also implies the positive solution of the *continuum problem* (in formulation of Cantor): each uncountable set of real numbers has cardinality continuum. (This follows by the Davies theorem.) Further consequences of the axiom AD and other axiomatic approaches are still under investigation. Details and further results can be found in ??

The exposition in the present book is based on the traditional assumption that the Zermelo—Fraenkel system of axioms of the theory of sets is considered together with the axiom AC. It has been established by Hödel that this does not lead to a contradiction. (If the Zermelo—Fraenkel system of axioms of the theory of sets is considered together with the negation of the axiom AC, it also, as has been established by Cohen, does not lead to a contradiction. Thus, the axiom AC cannot be either proved or disproved.)

Finally, we note that if the axiom AD were considered instead the axiom AC, then the exposition of the theory of L_p-spaces would be considerably simplified without changing its actual contents. Sections 1.3.1 and 1.3.2 could be completely omitted, and also Theorem 1.3.13. Moreover, it would be possible to integrate the inequality $f \leqslant g$ for non-negative functions f and g defined on any set $\Omega \subset \mathbb{R}^n$ without reserve, which in its turn would simplify a number formulations of theorems and proofs.

The same refers to real analysis in general and some other mathematical disciplines such as, say ordinary and partial differential equations, where L_p-spaces and other function spaces are widely used. On the other hand in some other mathematical disciplines, say general topology, this would lead to essential changes in the content. The problems related to the final choice in favour of the axiom of choice, the determination axiom or some other axiom are currently still open.

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