Convex Combinations

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Abstract

This module is appropriate for low level linear algebra students. It can be used after linear combinations have been introduced. The goal is to introduce various kinds of combinations of points/vectors specifically affine, integer and convex combinations. Students develop the ability to understand new algebraic definitions and use them to describe geometric objects.

This module is written as a set of exercises. Each instructor can choose to present some of the material, use some as in class work or give as homework as appropriate.

Introduction

In class we have looked at linear combinations of points, and learned that the linear combination of two different points gives a line. Here we learn about other kinds of combinations. In our examples we will look at combinations of the form $\{(x, y)|(x, y) = \lambda_1(0, 1)+\lambda_2(1, 0)\}$ for various restrictions on the coefficients λ_i.

Affine Combinations

What happens if we require the coefficients to sum to 1? These are called *affine* combinations.

(i) Describe the set of all affine combinations of the points $(0, 1)$ and $(1, 0)$. In other words find $\{(x, y)|(x, y) = \lambda(0, 1) + (1 - \lambda)(1, 0)\}$.

(ii) Find another two points that determine the same set as in (i), or determine that this is impossible.

(iii) Describe the set of all affine combinations of the points $(0, 1)$ and $(1, 1)$.

(iv) Describe the set of all affine combinations of the points $(0, 1)$, $(1, 0)$, and $(1, 1)$. In other words find $\{(x, y)|(x, y) = \lambda_1(0, 1) + \lambda_2(1, 0) + (1 - \lambda_1 - \lambda_2)(1, 1)\}$.

(v) Speculate on the shape of the set of affine combinations of two random points in R^2. What about the set of affine combinations of 3 random points?

Convex Combinations

What happens if we require the coefficients to sum to 1 and that they are non-negative? These are called *convex* combinations.

(i) Describe the set of all convex combinations of the points $(0,1)$ and $(1,0)$. In other words find $\{(x,y)|(x,y)=\lambda(0,1)+(1-\lambda)(1,0), \text{ where } 0 \leq \lambda \leq 1\}$.

(ii) Find another two points that determine the same set as in (i) or determine that this is impossible.

(iii) Describe the set of all convex combinations of the points $(0,1)$ and $(1,1)$.

(iv) Describe the set of all convex combinations of the points $(0,1)$, $(1,0)$, and $(1,1)$. In other words find $\{(x,y)|(x,y)=\lambda_1(0,1)+\lambda_2(1,0)+\lambda_3(1,1) \text{ where } 0 \leq \lambda_i \leq 1 \text{ and } \lambda_1+\lambda_2+\lambda_3 = 1\}$.

(v) Speculate on the shape of the set of convex combinations of two random points in R^2. What about the set of convex combinations of 3 random points?

Integer Combinations

What happens if we take "integer" combinations?

(i) Describe the set of all integer combinations of the points $(0,1)$ and $(1,0)$. In other words find $\{(x,y)|(x,y)=\lambda_1(0,1)+\lambda_2(1,0) \text{where } \lambda_1, \lambda_2 \text{ are integers }\}$.

(ii) Find another two points that determine the same set as in (i) or determine that this is impossible.

(iii) Describe the set of all integer combinations of the points $(0,1)$ and $(1,1)$.

(iv) Describe the set of all integer combinations of the points $(0,1)$, $(1,0)$, and $(1,1)$.

(v) Speculate on the shape of the set of integer combinations of two random points in R^2. What about the set of integer combinations of 3 random points?

Combinations and solutions to sets of linear equations

Consider the set of equations $Ax=b$, for matrix A, and column vectors x, b. What happens if we take combinations of two vectors x and y which both satisfy this equation?

(i) If $Ax=b$ and $Ay=b$ and $z=\lambda_1 x+\lambda_2 y$. For what values of λ_1, λ_2 will it be true that $Az=b$. Specifically, decide whether this is true for all affine combinations, all integer combinations and all convex combinations.

(ii) Suppose we consider the special case where $b=0$. Does this change your answer to part (i)?

Follow up suggestions

Discuss the solutions and generalizations of affine, convex and integer combinations of n points in R^d as appropriate.

Convex combinations define polyhedra. A polyhedron is a feasible region for a set of linear inequalities and equalities. We can define a function we wish to maximize or minimize over this region. The problem of optimizing a linear function over a polyhedron is called *Linear Programming.*