Geometric Aspects of the Complex Heisenberg Group

Maurizio Candilera and Valentino Cristante

Dipartimento di Matematica Pura e Applicata
Via Belzoni, 7 — 35131 PADOVA (Italy)
e-mail: candilera@pdmat1.unipd.it or cristanv@pdmat1.unipd.it

Introduction.

In his article *Groupe de Heisenberg et Réalité*, P. Deligne relates the Heat Equation with the irreducible representation of the real Heisenberg group $\mathbb{C}_1^* \times \mathbb{R}^2$. First he defines a holomorphic vector bundle over the projective line, $\mathcal{W}$, on which the complex Heisenberg group $\mathbb{C}^* \times \mathbb{C}^2$ acts. Then, by using $\mathcal{W}$, he proves that the space Γ of solutions of the Heat Equation on an open disc is naturally endowed with a pre-Hilbert space structure and that its completion, $\mathcal{H}_1$, is the irreducible representation of $\mathbb{C}_1^* \times \mathbb{R}^2$. Moreover he shows that the Metaplectic group acts on Γ, and that its action extends to $\mathcal{H}_1$, normalizing the action of $\mathbb{C}_1^* \times \mathbb{R}^2$.

The present paper is motivated by the desire to understand the Deligne's ideas and to see how they apply to the analytic theory of the abelian varieties. In order to reach this goal, most of the results of the original paper are extended to the case of many variables, and the place occupied by the Riemann theta function in this picture is brought out. A pleasant feature of this point of view is that all the representations of the real Heisenberg group, used to obtain the Riemann theta function as a matrix coefficient (cf. [T3]), arise naturally from the regular right representation of the complex Heisenberg group.

Even though not treated here, there is another point which should be remarked: the Lie algebra of the universal vectorial extension of a polarized abelian scheme over the integers of a p-adic field has a non-degenerate alternating form. Thus the method used here for the complex case seems to give good suggestions for an analytic theory in a p-adic context. We hope to return soon on these arguments.

The paper consists of four sections whose contents can be summarized as follows:

1. Starting from a complex vector space V, of dimension $2n$, endowed with a non degenerate alternating form $\langle -, - \rangle$, and taking the associated complex Heisenberg group H, we define a holomorphic vector bundle $\mathcal{W}$ over the manifold S of the lagrangian sub-spaces of V. The fibers of $\mathcal{W}$ are spaces of functions on H and H acts on these fibers by means of the regular right representation. We then twist $\mathcal{W}$ with

the line bundle $\mathcal{D}et^{-\frac{1}{2}}$ and define a connection on the twisted bundle. This connection commutes with the action of H and coincides locally with the Heat Equations.

2. We consider the family of real Heisenberg groups coming from the maximal real subspaces of V on which $\langle -,-\rangle$ assumes real values. By restricting the previous representations of H to the groups of this family, in view of the uniqueness of the irreducible representation of a real Heisenberg group, we can define a connection on the projective bundle $\mathbb{P}(\mathcal{W})$. This turns out to be the projectivization of the connection defined in section 1. In this context, the Riemann theta function appears as a horizontal section of $\mathcal{W}$.

3. We consider solutions of Heat Equations on (symplectic) open discs, showing how the radius of the disc can be estimated by the growth of the value of the solution in the center of the disc. This allows us to characterize the germs of the horizontal sections of $\mathcal{W}$, and to define a pre-Hilbert pairing on the space Γ of horizontal sections on a disc. The completion of this space, $\mathcal{H}_1$, is shown to be an irreducible (Fock) representation of a group of the family introduced in section 2.

4. We show how the action of the real symplectic group on a disc lifts to the action of its double cover, $Mp(2n,\mathbb{R})$, on Γ, inducing the classical action on the irreducible representation of the real Heisenberg group. In this context the Riemann theta function appears as a section invariant, up to a constant, under the action (of the double cover) of $\Gamma_{1,2}$.

1. A representation of the complex Heisenberg group.

We start with the following

1.1 Definition. Given a complex vector space V of dimension $2n$, endowed with a non-degenerate alternating bilinear form $\langle -,-\rangle$, the *complex Heisenberg Group* over V, $H = H_V$, is the group-extension of V by the multiplicative group $\mathbb{C}^*$, whose composition law is

$$(\lambda, v)(\lambda', v') = (\lambda\lambda' \exp(-\pi i\langle v, v'\rangle), v + v'),$$

for any $\lambda, \lambda' \in \mathbb{C}^*$, $v, v' \in V$.

1.2 Remarks. (a). Let V and $\langle -,-\rangle$ be as above, we recall that a *symplectic basis* of V is a basis for which the alternating form $\langle -,-\rangle$ has matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. If one chooses a symplectic basis, H_V is identified with the *standard* Heisenberg Group $H_{\mathbb{C}^{2n}}$. The group law of $H_{\mathbb{C}^{2n}}$ is

$$(\lambda, x_1, x_2)(\mu, y_1, y_2) = (\lambda\mu \exp[\pi i({}^t x_1 y_2 - {}^t x_2 y_1)],\ x_1 + y_1, x_2 + y_2),$$

for any $x_1, x_2, y_1, y_2 \in \mathbb{C}^n$ and $\lambda \in \mathbb{C}^*$. The standard Heisenberg group acts on the space $\mathcal{O}(\mathbb{C}^n)$ of holomorphic functions on $\mathbb{C}^n$ by the rule

$$(U_{(\lambda,x,y)}f)(z) = \lambda \exp \pi i(2\,{}^t yz + {}^t yx) f(z + x),$$

for any $(\lambda, x, y) \in H_{\mathbb{C}^{2n}}$ and $f \in \mathcal{O}(\mathbb{C}^n)$. This action is called the *Schrödinger action.*
(b). Given a symplectic basis of V, the real subspace L spanned by that basis is naturally endowed with a non-degenerate, real-valued alternating form, so that the subgroup $H_L = \mathbb{C}_1^*.L$ is a real Heisenberg group.
(c). In what follows we shall use the same symbol D to denote either a maximal isotropic subspace of V or the corresponding level subgroup $\{\,(1, d) \,|\, d \in D\,\}$ of H.

1.3 Notations. Let $G(n, V)$ be the Grassmannian variety of the n-dimensional subspaces of V, and denote by S the subvariety of $G(n, V)$ whose points are isotropic subspaces of V. As E varies in S, the open sets $U'_E = \{\,D \in G(n, V) \,|\, V = E \oplus D\,\}$ form a covering of $G(n, V)$. If $U \in U'_E$ is fixed, then for any other D in U'_E there exists a unique $\phi_D \in \mathrm{Hom}(U, E)$ such that $D = \{\,u + \phi_D(u) \,|\, u \in U\,\}$. The correspondence $D \mapsto \phi_D$ is a bijection, so that after a choice of bases for U and E, the entries of the matrix T_D of ϕ_D become coordinates on U'_E. In particular, if the subspaces E and U are both in S, once the basis $\{e_1, \ldots, e_n\}$ of E is fixed, we shall choose the dual basis $\{e_{n+1}, \ldots, e_{2n}\}$ of U, which is determined by the conditions $\langle e_{n+i}, e_j\rangle = \delta_{ij}$ for $i, j = 1, \ldots, n$. After this choice, we get a symplectic identification of V with $\mathbb{C}^{2n}$ and we shall write $E = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $D = \begin{pmatrix} -T_D \\ 1 \end{pmatrix}$. More generally, we associate to the matrix $\begin{pmatrix} T_1 \\ T_2 \end{pmatrix}$ the subspace of $\mathbb{C}^{2n}$ spanned by its columns. Finally we observe that $D = \begin{pmatrix} T \\ 1 \end{pmatrix}$ belongs to $U_E := U'_E \cap S$ if, and only if, T is a symmetric matrix (${}^tT = T$).

In what follows we'll attribute the affine coordinate T to the point $\begin{pmatrix} -T \\ 1 \end{pmatrix} \in U_E$.

1.4 Remark. The objects described above have a natural interpretation in a geometric context. Let us give a brief sketch. Let V and $\langle -, -\rangle$ be as above and consider the n-dimensional subspaces D of V for which:

(i) there exists a free abelian group Λ spanned by a symplectic basis of V, such that its image in the quotient V/D is a lattice Λ_D;

(ii) the extension by $\mathbb{R}$-linearity of the alternating form induced by $\langle -, -\rangle$ on Λ_D is the imaginary part of a positive definite hermitian form h on V/D.

Once a lattice Λ as above is fixed, it's well known that for any one of the subspaces D satisfying the above conditions relative to Λ, the quotient B_D of the vector space V/D by the lattice Λ_D is a principally polarized abelian variety with an isomorphism $\Lambda \cong \mathrm{H}_1(B_D, \mathbb{Z})$, which identifies the bilinear form $\langle -, -\rangle$ with the (Chern) class in

$\mathrm{H}^2(B_D,\mathbb{Z})$ of any line bundle on B_D associated to the hermitian form h. Finally, the quotient V/Λ is canonically isomorphic to the maximal non-trivial vectorial extension $\mathrm{H}_1(B_D,\mathbb{C})/\mathrm{H}_1(B_D,\mathbb{Z})$ of any torus B_D of the family determined by Λ (cf. [MM] Cor. 4.4.2).

Let $\{e_1,\dots,e_{2n}\}$ be a symplectic basis of Λ; then in the frame given by the subspaces E and U spanned by $\{e_1,\dots,e_n\}$ and $\{e_{n+1},\dots,e_{2n}\}$, respectively, the subspaces $D\in U_E$ satisfying (i) and (ii) relative to Λ, are exactly those with affine coordinates Ω, where $\Omega\in\mathfrak{S}_n$, the Siegel upper half-space of degree n (cf. [T1], Ch.II, Lemma 4.1). Finally, by observing that the symplectic group $Sp(V)$ acts transitively on S, we can conclude that any element of S produces a principally polarized abelian variety for a suitable lattice Λ in V.

Our goal now is to construct a vector bundle $\mathcal{W}$, over S, of representations of H. The fiber of $\mathcal{W}$ over $D\in S$, is the complex vector space W_D of the holomorphic functions $\phi\colon H\to\mathbb{C}$ satisfying the condition $\phi(\lambda dh)=\lambda\phi(h)$ for any $\lambda\in\mathbb{C}^*$, $d\in D$ and $h\in H$; and the action of H on W_D is given by the regular right representation. The structure of $\mathcal{W}$ is described in the following

1.5 Proposition. *Notations as above.*

(a) *Let $E\in S$. For any $D\in U_E$, there exist isomorphisms $\varphi_D\colon H\to H_{\mathbb{C}^{2n}}$ and $\rho_D\colon W_D\to\mathcal{O}(\mathbb{C}^n)$ which fit into the commutative diagram*

$$\begin{array}{ccc} H\times W_D & \xrightarrow{\ R\ } & W_D \\ \Big\downarrow{\scriptstyle \varphi_D\times\rho_D} & & \Big\downarrow{\scriptstyle \rho_D} \\ H_{\mathbb{C}^{2n}}\times\mathcal{O}(\mathbb{C}^n) & \xrightarrow{\ U\ } & \mathcal{O}(\mathbb{C}^n), \end{array}$$

where R denotes the regular right action of H on W_D and U denotes the Schrödinger action (cf. Remark 1.2.(a)).

(b) *The collection of spaces W_D has a structure of holomorphic vector bundle, $\mathcal{W}$, such that $\mathcal{W}|_{U_E}\cong U_E\times\mathcal{O}(\mathbb{C}^n)$ by means of the maps ρ_D, as E varies in S. $\mathcal{W}$ is a bundle of representations.*

Proof. (a). Denote the space of the holomorphic functions on E by $\mathcal{O}(E)$ and consider the isomorphism $r_E\colon W_D\to\mathcal{O}(E)$ induced by the restriction to E of the functions on H (cf. *Remark 1.2.(c)*). Its inverse map can be written explicitly as $(r_E^{-1}f)(\lambda,d+e)=(r_E^{-1}f)(\lambda\exp(\pi i\langle d,e\rangle)(1,d)(1,e))=\lambda\exp(\pi i\langle d,e\rangle)f(e)$, for $f\in\mathcal{O}(E)$.

Now fix a symplectic basis of V, $\{e_1,\dots,e_{2n}\}$, in such a way that $\{e_1,\dots,e_n\}$ spans E and denote the subspace generated by $\{e_{n+1},\dots,e_{2n}\}$ by U. Then for any $D\in U_E$, the corresponding map $\phi_D\in\mathrm{Hom}(U,E)$ produces the basis ($d_i=e_{n+i}+$

$\phi_D(e_{n+i}))_{i=1,\dots,n}$, dual to $\{e_1,\dots,e_n\}$. After these choices, the isomorphisms ρ_D and φ_D are defined by

$$\rho_D(\phi)(z)=\phi(1,\sum_{i=1}^n z_ie_i)\quad\text{and}\quad\varphi_D(\lambda,\sum_{i=1}^n x_ie_i+\sum_{i=1}^n y_id_i)=(\lambda,x,y).$$

Now, let $e'=\sum_{i=1}^n z_ie_i$, $e=\sum_{i=1}^n x_ie_i$, $d=\sum_{i=1}^n y_id_i$ and $h=(\lambda,e+d)$; then, recalling the definition of the Schrödinger action (cf. *Remark 1.2.(a)*), by a direct computation one has

$$\begin{aligned}\rho_D(R_h\phi)(z)=&R_h\phi(1,e')=\phi((1,e')(\lambda,e+d))=\\=&\phi(\lambda\exp[-\pi i(2\langle e',d\rangle+\langle e,d\rangle)](1,d)(1,e'+e))=\\=&\lambda\exp\pi i(2{}^tyz+{}^tyx)\,\phi(1,e'+e)=[U_{\varphi_D(h)}\rho_D(\phi)](z).\end{aligned}$$

(b) To prove this point we must show that, for any $E,E'\in S$, the map $D\mapsto\rho_D\circ{\rho'_D}^{-1}$ defined on $U_E\cap U_{E'}$ with values in $Aut(\mathcal{O}(E))$, is holomorphic (ρ'_D is obtained from the couple E',D in the same way as ρ_D was obtained from E,D).

In order to compute $\rho_D\circ{\rho'_D}^{-1}$ let's identify V with $\mathbb{C}^{2n}$ by means of $\{e_1,\dots,e_{2n}\}$. Let $E=\begin{pmatrix}1\\0\end{pmatrix}$, $U=\begin{pmatrix}0\\1\end{pmatrix}$, and assume $E'=\begin{pmatrix}E_1\\E_2\end{pmatrix}$; then if $D=\begin{pmatrix}-T\\1\end{pmatrix}$, its basis dual to the columns of E' is given by the columns of the matrix $\begin{pmatrix}-T{}^t(E_1+TE_2)^{-1}\\{}^t(E_1+TE_2)^{-1}\end{pmatrix}$ ($D\in U_{E'}$ implies that $E_1+TE_2\in GL(n,\mathbb{C})$). Now let $(1,E'z)\in E'\subset H$, then $E'z=\begin{pmatrix}E_1\\E_2\end{pmatrix}z=\begin{pmatrix}-T\\1\end{pmatrix}u+\begin{pmatrix}1\\0\end{pmatrix}v$, where $u=E_2z$ and $v=(E_1+TE_2)z$. From this decomposition it follows that

$$\rho'_D(\rho_D^{-1}(f))(z)=\rho_D^{-1}(f)(1,E'z)=\exp(\pi i{}^tz{}^tE_2(E_1+TE_2)z)\,f((E_1+TE_2)z),$$

and this shows that the relation between f and $\rho'_D(\rho_D^{-1}(f))$ is holomorphic with respect to T.

The same holds for the automorphism $\varphi'_D\circ\varphi_D^{-1}$ of $H_{\mathbb{C}^{2n}}$: in fact by direct calculation, one sees that it induces the identity on $\mathbb{C}^*$ and acts on $\mathbb{C}^{2n}$ by the symplectic matrix

$$\begin{pmatrix}(E_1+TE_2)^{-1}&0\\-{}^t(E_1+TE_2)E_2(E_1+TE_2)^{-1}&{}^t(E_1+TE_2)\end{pmatrix},$$

so that also the family $\varphi'_D\circ\varphi_D^{-1}$ is holomorphic as D varies in $U_E\cap U_{E'}$.

The last claim follows from the commutativity of the above diagram. **QED** ■

The structural isomorphisms defining $\mathcal{W}$ are linear substitutions in the variables followed by multiplication by a quadratic exponential. Thus, for $v \in W_D$ the condition

$$\exists A \text{ such that } |r_E(v)(z)| \leq O(\exp A|z|^2), \tag{1.6}$$

doesn't depend on the choice of E or on the choice of the coordinate z on E. So we can state the following

1.7 Corollary. *The subspaces W^0_D of W_D consisting of the functions satisfying the above condition* (1.6), *form a sub-vector bundle $\mathcal{W}^0$ of $\mathcal{W}$.*

Our next task is to prove that the vector bundle $\mathcal{W}$, after a suitable twist, carries a connection ∇, for which the group H acts horizontally. We denote by $\mathcal{D}et$ the restriction to S of the n^{th} exterior power of the tautological vector bundle on $G(n,V)$ and by $\mathcal{D}et^{\frac{1}{2}}$ one of its *square roots*, *i.e.* a line bundle L with a fixed isomorphism $L^{\otimes 2} \cong \mathcal{D}et$. The remaining part of this section is devoted to proving the following

1.8 Theorem. *Let $\mathcal{W}(-\frac{1}{2}) := \mathcal{W} \otimes \mathcal{D}et^{-\frac{1}{2}}$, then there exists a connection*

$$\nabla \colon \mathcal{W}(-\tfrac{1}{2}) \to \mathcal{W}(-\tfrac{1}{2}) \otimes \Omega^1_S$$

for which the group H acts horizontally.

Proof. We shall carry out the construction of ∇ in three steps:

- (i) first we fix E and for any $D \in U_E$ we identify the tangent space $T_D(S) = \mathrm{Hom}^{sym}_{\mathbb{C}}(D, V/D)$ with $\mathrm{Hom}^{sym}_{\mathbb{C}}(D,E)$; then to each $t \in \mathrm{Hom}^{sym}_{\mathbb{C}}(D,E)$ we associate the Kodaira deformation of W_D and the corresponding connection ∇^E on $\mathcal{W}|_{U_E}$;
- (ii) we explain how t acts on $\mathcal{D}et$ and $\mathcal{D}et^{-\frac{1}{2}}$ and so we define a connection ∇^E on $\mathcal{W}(-\frac{1}{2})|_{U_E}$;
- (iii) finally we show that, given E and E' in S, the connections ∇^E and $\nabla^{E'}$ coincide on $\mathcal{W}(-\frac{1}{2})|_{U_E \cap U_{E'}}$.

Let's start with the first step. Given E and D as in (i), we choose a basis $\{e_1, \ldots, e_n\}$ of E and denote by $\{d_1, \ldots, d_n\}$ the dual basis of D; then we identify $t \in T_D S$ with the element $t_E \in \mathrm{Hom}(D,E)$, defined by

$$t_E(d_j) = \sum_{i=1}^n \langle d_i, t(d_j) \rangle e_i \qquad \text{for } j = 1, \ldots, n.$$

The symmetry condition implies that t_E coincides with its adjoint, so that the matrix $T = (T_{ij})$ defined by $T_{ij} = \langle d_i, t(d_j) \rangle$ is symmetric. Recalling that the tangent space to H at the identity is $\mathbb{C} \times V$, we shall denote by δ_v the right-invariant vector field on H determined by the tangent vector $(0, v)$; then

$$\sigma = \frac{1}{4\pi i} \sum_{1 \leq i,j \leq n} T_{ij} \delta_{e_i} \delta_{e_j} \tag{1.9}$$

is a symmetric operator on W_D. Observing that $[\delta_u, \delta_v] = 2\pi i \langle u, v\rangle \delta_{(1,0)}$, by a direct computation one checks that $[-\sigma, \delta_{d_h}] = \delta_{t_E(d_h)}$, for $h = 1, \ldots, n$, and then, by linearity, one gets

$$[-\sigma, \delta_d] = \delta_{t_E(d)}, \quad \text{for any } d \in D. \tag{1.10}$$

Now let $\phi \in W_D$; assuming $\epsilon^2 = 0$, we prove that $\phi - \epsilon(\sigma\phi) \in W_{D+\epsilon t}$. This, in view of the definition of W_D, amounts to verifying that

$$\delta_{(1,0)}(\phi - \epsilon(\sigma\phi)) = (\phi - \epsilon(\sigma\phi)) \quad \text{and} \quad \delta_{d+\epsilon t_E(d)}(\phi - \epsilon(\sigma\phi)) = 0, \text{ for all } d \in D.$$

The first check is immediate because $[\delta_{(1,0)}, \delta_v] = 0$, for any $v \in V$; the second one is a consequence of (1.10); in fact,

$$\delta_{d+\epsilon t_E(d)}(\phi - \epsilon(\sigma\phi)) = \epsilon\delta_{t_E(d)}(\phi) - \epsilon[-\sigma, \delta_d]\phi = 0.$$

The connection ∇^E on $\mathcal{W}|_{U_E}$ is defined by

$$(\nabla^E_t(\mu))(D) = (\partial_t\mu)(D) + \sigma(\mu(D)),$$

for every section μ and every $D \in U_E$.

We now turn to (ii). The line bundle $\mathcal{D}et$ on S is the inverse image of $\mathcal{O}(1)$ under (the restriction to S of) the Plücker embedding $p: G(n, V) \to \mathbb{P}(\Lambda^n V)$. The $\mathcal{O}(U_E)$-module of its sections over U_E is generated by the section $D \mapsto (d_1 \wedge \cdots \wedge d_n)$; as a consequence, ∇^E is uniquely defined on the restriction of $\mathcal{D}et$ and $\mathcal{D}et^{-\frac{1}{2}}$ to U_E, by setting $\nabla^E_t(d_1 \wedge \cdots \wedge d_n) = \sum_{j=1}^n d_1 \wedge \cdots \wedge t_E(d_j) \wedge \cdots \wedge d_n$.

If $\mu \otimes s$ is a section of $\mathcal{W}(-\frac{1}{2})$ over U_E we put

$$\nabla^E_t(\mu \otimes s) = \nabla^E_t \mu \otimes s + \mu \otimes (\nabla^E_t s).$$

Finally we prove (iii). Let $E' \in U_D$ and $\{e'_1, \ldots, e'_n\}$ be its basis dual to $\{d_1, \ldots, d_n\}$; then there exists $\alpha \in \mathrm{Hom}(E, D)$ such that $e'_i = e_i + \alpha(e_i)$, $i = 1, \ldots, n$. Since E and E' are isotropic, as previously observed, the matrix $A = (a_{ij})$ of α, with respect to the basis $(e_i), (d_i)$, is symmetric. Now let's compute $\delta_{e'_i}\delta_{e'_j}$ as an operator on the space W_D:

$$\begin{aligned}\delta_{e'_i}\delta_{e'_j} =&(\delta_{e_i} + \sum_{h=1}^{n} a_{hi}\delta_{d_h})(\delta_{e_j} + \sum_{k=1}^{n} a_{kj}\delta_{d_k}) = \\ =&\delta_{e_i}\delta_{e_j} + \sum_h a_{hi}[\delta_{d_h}, \delta_{e_j}] = \\ =&\delta_{e_i}\delta_{e_j} + 2\pi i \sum_h a_{hi}\langle d_h, e_j\rangle = \delta_{e_i}\delta_{e_j} + 2\pi i a_{ji}.\end{aligned}$$

So, if σ' is built up from the $\delta_{e'_i}$'s in the same way as σ was built from the δ_{e_i}'s, by comparing the two operators on W_D, we get

$$\sigma' = \sigma + \tfrac{1}{2} \sum_{1\le i,j\le n} T_{ij}a_{ji} = \sigma + \tfrac{1}{2}\mathrm{tr}(t_E\alpha). \tag{1.11}$$

Now, since $t_{E'} = (1+\alpha)\circ t_E$, we get

$$\begin{aligned}\nabla_t^{E'}(d_1\wedge\cdots\wedge d_n) &= \nabla_t^{E}(d_1\wedge\cdots\wedge d_n) + \sum_{j=1}^{n} d_1\wedge\cdots\wedge\alpha(t_E(d_j))\wedge\cdots\wedge d_n = \\ &= \nabla_t^{E}(d_1\wedge\cdots\wedge d_n) + \mathrm{tr}(\alpha t_E)(d_1\wedge\cdots\wedge d_n),\end{aligned}$$

and more generally, if f is a section of $\mathcal{D}et$ over $U_E\cap U_{E'}$,

$$\begin{aligned}(\nabla_t^{E'} f^{-\frac{1}{2}}) &= -\tfrac{1}{2}f^{-\frac{3}{2}}(\nabla_t^{E'} f) = \\ &= (\nabla_t^{E} f^{-\frac{1}{2}}) - \tfrac{1}{2}\mathrm{tr}(\alpha t_E)f^{-\frac{1}{2}}.\end{aligned} \tag{1.12}$$

By applying (1.11) and (1.12) to a section $\mu\otimes s$ of $\mathcal{W}(-\frac{1}{2})$ over $U_E\cap U_{E'}$, we conclude that

$$\begin{aligned}\nabla_t^{E'}(\mu\otimes s) &= (\partial_t\mu + \sigma'\mu)\otimes s + \mu\otimes(\nabla_t^{E'} s)) = \\ &= (\partial_t\mu)\otimes s + (\sigma\mu)\otimes s + \mu\otimes(\nabla_t^{E} s)) = \\ &= \nabla_t^{E}(\mu\otimes s).\end{aligned}$$

Finally, the connection ∇ commutes with the action of the group H because the operators used in its construction are right-invariant. **QED** ■

1.13 Remarks. (a). After the identification of $\mathcal{W}(-\frac{1}{2})|_{U_E}$ with $\mathcal{O}(U_E\times\mathbb{C}^n)$, described in *Proposition 1.5 (b)*, the invariant vector fields δ_{e_i} become the partial derivatives ∂_{z_i} of $\mathcal{O}(\mathbb{C}^n)$, so that the sections horizontal with respect to ∇ are the solutions of the system of the heat equations $(\partial_{t_{ij}} - \frac{1}{4\pi i}\partial_{z_i}\partial_{z_j})g = 0$ for $1 \leq i \leq j \leq n$ (cf. *Notations 1.3* for the choice of affine coordinates on U_E).

(b). With the notations of *Remark 1.4*, let $\mathcal{A} = \left\{ B_D \,|\, D = \begin{pmatrix} -\Omega \\ 1 \end{pmatrix}, \Omega \in \mathfrak{S}_n \right\}$ be the family of principally polarized abelian varieties related to the lattice Λ spanned by $\{e_1, \ldots, e_{2n}\}$ and denote by $\vartheta = \vartheta(\Omega, z)$ its Riemann theta function. Then the deformation $\vartheta \mapsto \vartheta - \epsilon(\sigma\vartheta)$ induces on $\mathcal{A}$ the deformation given by t_E in the sense of Kodaira (cf. [KS], §14, (γ) and (δ)).

2. Building horizontal sections.

In this section we consider maximal real Heisenberg subgroup of H. Any such group G, defines an open subset $X \subset S$, with the property that for any $D \in X$, there exists an irreducible representation of G, $F_D \subset W_D$. Thus, as a consequence of the Stone-Von Neumann Theorem, $\mathcal{W}_{|X}$ carries a projective connection. Our goal is to explain the relation between this connection and the connection ∇ on $\mathcal{W}(-\frac{1}{2})$ defined in the previous section.

We start by fixing some notations regarding the real subgroups of H.

2.1 Definition. A *real structure* on the complex Heisenberg group H is a real subspace L of V spanned by a symplectic basis.

For later use let's recall some elementary and well known results about real structures.

2.2 Proposition. *Let L be a real structure on H. Then one has:*
(a) $\dim_{\mathbb{R}} L = 2n$ *and* $L \cap iL = \{0\}$*;*
(b) the restriction of $\langle -, - \rangle$ *to* L *is an alternating form with real values;*
(c) the subgroup $H_L := \mathbb{C}_1^*.L$ *is a real Heisenberg group;*
(d) the decomposition of real vector spaces $V = L \oplus iL$ *induces a complex conjugation* $\sigma_L \colon V \to V$ *defined by setting* $\sigma_L(l_1 + il_2) = l_1 - il_2$ *for any* $l_1, l_2 \in L$*.*

Now we define a subset of S canonically associated to a real structure L on H.

2.3 Definition. The *open disc* associated to the real structure L is the subset

$$X_L = \{\, D \in S \,|\, \tfrac{i}{2}\langle D, \sigma_L D\rangle < 0 \,\},$$

where $\frac{i}{2}\langle D, \sigma_L D\rangle < 0$ means that the pairing $\frac{i}{2}\langle d, \sigma_L d'\rangle$, for $d, d' \in D$, is a negative-definite hermitian form.

The real structures and the related open discs can be described by means of coordinates. In fact let's start with a fixed symplectic basis $\{e_1, \ldots, e_{2n}\}$ of V and denote by L the real structure spanned by that basis; then, if $Sp(V)$ denotes the group of symplectic automorphisms of V and $Sp(L)$ its subgroup which stabilizes L, the set of all the real structures on H can be parameterized by $Sp(V)/Sp(L)$. More explicitly, if V is identified with $\mathbb{C}^{2n}$ by means of $\{e_1, \ldots, e_{2n}\}$, then the real structure and its related disc corresponding to (the coset of) a matrix $\Gamma \in Sp(2n, \mathbb{C})$ are:

$$L_\Gamma = \left\{ \Gamma x \mid x \in \mathbb{R}^{2n} \right\} \subset \mathbb{C}^{2n} \quad \text{and} \quad X_{L_\Gamma} = \left\{ \Gamma \left(\begin{smallmatrix} -\Omega \\ 1 \end{smallmatrix} \right) \mid \Omega \in \mathfrak{S}_n \right\}.$$

Finally, the complex conjugation σ_{L_Γ} is given by $\sigma_{L_\Gamma}(\Gamma z) = \Gamma \bar{z}$, for any $z \in \mathbb{C}^{2n}$.

Now it's clear that the well-known result describing the action of $Sp(2n, \mathbb{R})$ on Siegel upper half-space can be rephrased as follows:

2.4 Proposition. *Let L be a real structure on H; then the group $Sp(L)$ acts transitively on the open disc X_L and the stabilizer of any point is isomorphic to the group of unitary transformations in $\mathbb{C}^n$.*

Here is a first result connecting $\mathcal{W}$ with the irreducible representation of a real subgroup of H.

2.5 Proposition. *Let L be a real structure on H and let $D \in S$ be such that $D \cap L = 0$. Given $E \in U_D$, denote by $\lambda : E \to D$ the isomorphism defined by $e + \lambda(e) \in L$, for any $e \in E$, and consider the following norm on $\mathcal{O}(E)$:*

$$\|f\|_E = \left(\int_E |f(e) \exp \pi i \langle \lambda(e), e \rangle|^2 \, dx \right)^{\frac{1}{2}}, \tag{2.6}$$

where dx denotes the Haar measure on E. Then:

(a) The action of the real group H_L on $\mathcal{O}(E)$ is unitary.

(b) Let $r_E \colon W_D \to \mathcal{O}(E)$ be the usual isomorphism of restriction, and define $\|\phi\|_D = \|r_E \phi\|_E$, for any $\phi \in W_D$; then $\|\phi\|_D$ is independent of the choice of E.

In particular, if $D \in X_L$, the subspace $F_D \subset W_D^0$ of elements with finite norm is a Hilbert space on which the action of H_L is irreducible.

Proof. (a). Denote by U the action of H on $\mathcal{O}(E)$ which renders r_E an isomorphism of representations. Given $f = r_E \phi \in \mathcal{O}(E)$ and $(\tau, l) \in H_L$, we must prove that

$$\|U_{(\tau,l)} f\|_E^2 = \int_E |U_{(\tau,l)} f(e) \exp \pi i \langle \lambda(e), e \rangle|^2 \, dx = \|f\|_E^2.$$

Let $l = \lambda(e') + e'$ be the decomposition of $l \in L$, relative to the sum $D \oplus E$, then one has

$$\begin{aligned} U_{(\tau,l)}f(e) =&[r_E^{-1}f]((1,e)(\tau,\lambda(e')+e')) = \\ =&[r_E^{-1}f](\tau \exp \pi i\langle\lambda(e'), e'+2e\rangle(1,\lambda(e'))(1,e+e')) = \\ =&\tau \exp \pi i\langle\lambda(e'), e'+2e\rangle\, f(e+e'). \end{aligned}$$

Now by observing that $\tau \in \mathbb{C}_1^*$ and $\langle\lambda(e'), e\rangle - \langle\lambda(e), e'\rangle = \langle\lambda(e') + e', \lambda(e) + e\rangle \in \mathbb{R}$, the previous integral may be rewritten as follows:

$$\|U_{(\tau,\lambda(e')+e')}f\|_E^2 = \int_E |f(e+e')\exp \pi i\langle\lambda(e+e'), e+e'\rangle|^2\, dx,$$

and so it coincides with $\|f\|_E^2$ since the measure is invariant by translations.
(b). Let $f = r_E\phi$, in order to check that $\|r_E\phi\|_E$ doesn't depend on E, it suffices to observe that

$$f(e)\exp \pi i\langle\lambda(e), e\rangle = \phi(1, \lambda(e) + e),$$

and actually this shows that $\|r_E\phi\|_E$ is just the L^2-norm of the restriction of ϕ to L.

To prove the last claim we make use of coordinates. Let's fix a symplectic basis of L so that $L = \mathbb{R}^{2n}$, $D = \begin{pmatrix} -\Omega \\ 1 \end{pmatrix} \in X_L$ and $E = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$; then $\lambda(x_1+\Omega x_2, 0) = (-\Omega x_2, x_2)$, for any $(x_1, x_2) \in L$, so that

$$\|f\|^2 = \int_{\mathbb{R}^{2n}} |f(\underline{x})|^2 \exp \pi i\,{}^t x_2(\Omega - \overline{\Omega})x_2\, dx_1\, dx_2, \tag{2.7}$$

where $\underline{x} = x_1 + \Omega x_2$. This expression shows that the subspace of elements of $\mathcal{O}(E)$ of finite norm, endowed with the action U of H_L, is the *Fock Representation* of the real group H_L, which is well-known to be irreducible (cf. [T3], §.2). Finally the convergence of the last integral implies that f satisfies the condition (1.6) defining the subspace W_D^0. **QED** ■

2.8 Remark. For any $D \in S$ denote by W_D^- the vector space of holomorphic functions ϕ on H, such that $\phi(\lambda dh) = \lambda^{-1}\phi(h)$, for any $\lambda \in \mathbb{C}^*$, $d \in D$ and $h \in H$. Assume that H acts on W_D^- by means of the regular right representation. Then, using the arguments of §.1, we can see that $\mathcal{W}^- := (W_D^-)_{D\in S}$ has a structure of holomorphic vector bundle of representations over S, and that $\mathcal{W}(-\frac{1}{2})^- := \mathcal{W}^- \otimes \mathcal{D}et^{-\frac{1}{2}}$ carries a connection commuting with the action of H.

The conjugation σ_L on V gives an antilinear isomorphism between $\mathcal{W}(-\frac{1}{2})^-$ and $\mathcal{W}(-\frac{1}{2})$ defined by $\sigma_L\phi(h) = \overline{\phi(\sigma_L h)}$, where $\sigma_L h$ denotes the image of h by the automorphism of H defined by $\sigma_L(\lambda, v) = (\overline{\lambda^{-1}}, \sigma_L v)$. Observing that $R_g(\sigma_L\phi) = \sigma_L(R_{\sigma_L g}\phi)$, we conclude that $\phi \mapsto \sigma_L\phi$ is an isomorphism of representations.

Each W_D^-, like W_D, possess a Fock norm (cf. *Proposition 2.5*) obtained by taking the L^2-norm of the functions restricted to L, and its subspace of elements with finite norm, F_D^-, is an irreducible representation of H_L. By restricting σ_L to F_D we get a unitary isomorphism $\sigma_L: F_D \to F_{\sigma_L D}^-$, thus the Hilbert pairing on F_D gives a non degenerate bilinear paring $(-,-): F_D \times F_{\sigma_L D}^- \to \mathbb{C}$.

By *Proposition 2.5*, for any $D \in X_L$ there exists the irreducible representation F_D of H_L. Since all the irreducible representations of H_L are unitarily equivalent (Theorem of Stone-Von Neumann), the restriction of the projective bundle $\mathbb{P}(\mathcal{W})$ to X_L carries a connection. More explicitly: if H_L is identified with $H_{\mathbb{R}^{2n}}$ by means of a symplectic basis of L, and $L^2(\mathbb{R}^n)$ is endowed with the Schrödinger action of $H_{\mathbb{R}^{2n}}$, for any fixed $D \in X_L$, there exists an isomorphism of representations, $\psi_D: L^2(\mathbb{R}^n) \to F_D$, unique up to a constant in $\mathbb{C}_1^*$. The last part of this section is devoted to showing how the projective connection induced by the maps ψ_D is related to the connection ∇ defined in the previous section.

In view of the irreducibility of $L^2(\mathbb{R}^n)$, ψ_D is completely determined by its value on a non-zero element; so it is sufficient to consider its restriction to the (dense) subspace $L^2(\mathbb{R}^n)^{hol}$ of holomorphic vectors. By holomorphic vector we mean an element $f \in L^2(\mathbb{R}^n)$ such that $h \mapsto U_h f$ extends to a holomorphic map from H to $L^2(\mathbb{R}^n)$ (the definition and the basic properties of holomorphic maps in infinite dimensional vector spaces can be found in [H], Ch. III).

Let $\epsilon: W_D \to \mathbb{C}$ be the map defined by $\epsilon(\phi) = \phi(1_H)$; then, by observing that the previous identification $H_L \cong H_{\mathbb{R}^{2n}}$ extends to $H \cong H_{\mathbb{C}^{2n}}$, we have

$$[\psi_D(f)](h) = [R_h\psi_D(f)](1_H) = [\psi_D(U_h f)](1_H) = (\epsilon \circ \psi_D)(U_h f), \tag{2.9}$$

for any $h \in H$, $f \in L^2(E_{\mathbb{R}})^{hol}$. This shows that ψ_D is determined by the linear form $\omega_D = \epsilon \circ \psi_D$. From (2.9) it follows that $\omega_D(U_h f) = \omega_D(U_{dh} f)$, for any $f \in L^2(\mathbb{R}^n)^{hol}$ and $d \in D$; so ω_D is a left D-invariant form.

In order to compute ω_D, let's describe the identification $H_L \cong H_{\mathbb{R}^{2n}}$ more precisely. Let $\{e_1, \dots, e_{2n}\}$ be a symplectic basis of L, denote by $E_{\mathbb{R}}$ the real subspace spanned by $\{e_1, \dots, e_n\}$, and consider the the corresponding identifications $V \cong \mathbb{C}^{2n}$, $E_{\mathbb{R}} \cong \mathbb{R}^n$ Then $X_L = \left\{ \begin{pmatrix} -\Omega \\ 1 \end{pmatrix} \mid \Omega \in \mathfrak{S}_n \right\}$; moreover, if $E = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $U = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, it's easy to check that $L^2(E_{\mathbb{R}})^{hol} \subset \mathcal{O}(E)$, where the action of H on $\mathcal{O}(E)$ is induced by the regular right representation on W_U by means of $r_E : W_U \to \mathcal{O}(E)$.

Now we can prove the following result (cf. [T3] Thm. 2.2):

2.10 Proposition. *Notations as above. Let $D = \begin{pmatrix} -\Omega \\ 1 \end{pmatrix} \in X_L$, and ω_D be a left D-invariant form on $L^2(E_{\mathbb{R}})^{hol}$. Then there exists an element $e_\Omega \in L^2(E_{\mathbb{R}})^{hol}$, invariant* *with respect to the level subgroup lifting* $D' = \binom{\Omega}{1}$, *such that*

$$\omega_D(g) = \int_{E_{\mathbb{R}}} g(x) e_\Omega(x)\, dx,$$

for any $g \in L^2(E_{\mathbb{R}})^{hol}$.

Proof. By general facts about Fock representations (e.g. cf. [I], Ch.I, §.8), it follows that the restriction to F_D of the linear form ϵ is induced by a function; thus the same must happen for ω_D. Since ω_D is left D-invariant, our task amounts to finding the left D'-invariant functions on $L^2(E_{\mathbb{R}})^{\mathrm{hol}}$. Let $f \in L^2(E_{\mathbb{R}})^{\mathrm{hol}} \subset \mathcal{O}(E)$, $\phi = r_E^{-1} f \in W_U$, and denote by ρ_d the left invariant vector field on H determined by $d \in D'$. Then f is left D'-invariant if, and only if, $\rho_d(\phi) = 0$, for each $d \in D'$. Recalling the relations among right- and left-invariant vector fields, one sees that

$$\rho_d(\phi)(1, v) = \delta_d \phi(1, v) - 2\pi i \langle v, d\rangle \phi(1, v) = 0.$$

By using the coordinate z on E given by $\{e_1, \ldots, e_n\}$, the last relation may be rewritten as follows:

$$\sum_{i=1}^{n} t_{ij} \partial_{z_i} f(z) + 2\pi i z_j f(z) = 0,$$

for $j = 1, \ldots, n$ and $(t_{ij})_{1 \le i,j \le n} = \Omega$. So by a direct computation we get $f(z) = c \exp -\pi i\, {}^t z \Omega^{-1} z$, where c is a constant. **QED** ∎

2.11 Remarks. (a) From now on we shall denote $\omega_D(g)$ by (g, e_Ω). Let's observe that for any $g \in L^2(E_{\mathbb{R}})$ and $h = (\lambda, e + u) \in H_L$, where $\lambda \in \mathbb{C}^*$, $u \in U$ and $e \in E$, the following identity holds

$$\begin{aligned} (U_h g, e_\Omega) &= \int_{E_{\mathbb{R}}} \lambda \exp \pi i \langle u, 2x + e\rangle g(e + x) e_\Omega(x)\, dx = \\ &= \int_{E_{\mathbb{R}}} \lambda \exp \pi i \langle u, 2y - e\rangle g(y) e_\Omega(y - e)\, dy = (g, U_{h'} e_\Omega), \end{aligned} \tag{2.12}$$

where $h' = (\lambda, -e + u) \in H$; the map $h \mapsto h'$ is an anti-isomorphism of H.

(b) Since the restriction of $U_h e_\Omega$ to $\mathbb{R}^n = E_{\mathbb{R}}$, for any $h \in H$, is a Schwartz-function (e.g. cf. [I], Ch.I, §.9), the linear forms $(-, U_h e_\Omega)$ extend to the space of tempered distributions on $E_{\mathbb{R}}$.

The following theorem explains how to choose e_Ω to get horizontal sections of $\mathcal{W}(-\frac{1}{2})$.

2.13 Theorem. *Notations as above. Let* $(\det\Omega)^{1/2}$ *be the function on* $\mathfrak{S}_n$ *defined by*

$$(-i)^{n/2}(\det\Omega)^{1/2} = (\det -i\Omega)^{1/2} = \int_{\mathbb{R}^n} \exp(-\pi i\, {}^t x\Omega^{-1}x)dx,$$

and let

$$e_\Omega(z) = (\det\Omega)^{-\frac{1}{2}} \exp(-\pi i\, {}^t z\Omega^{-1}z). \tag{2.14}$$

Then, if $f \in L^2(E_{\mathbb{R}})$ *and* g_Ω *denotes the function on* H *defined by* $g_\Omega(h) = (f, U_{h'}e_\Omega)$, *the map* $D = \binom{-\Omega}{1} \mapsto g_\Omega$ *is a horizontal section of* $\mathcal{W}(-\frac{1}{2})$ *over* X_L. *The above map* $L^2(E_{\mathbb{R}}) \to \Gamma(X_L, \mathcal{W}(-\frac{1}{2})^{\nabla=0})$ *extends naturally to the dual space of* $L^2(E_{\mathbb{R}})^{\mathrm{hol}}$.

Proof. The function g_Ω is D-invariant; thus by *Remark 1.13.(a)*, it suffices to prove that $g_\Omega(1,z,0)$ lies in the kernel of the operators $\partial_{t_{ij}} - \frac{1}{4\pi i}\partial_{z_i}\partial_{z_j}$, for $1 \le i \le j \le n$, and this can be done directly. **QED** ■

Now we shall describe the link between the above construction of horizontal section of $\mathcal{W}(-\frac{1}{2})_{|X_L}$ and the geometry of the family of principally polarized abelian varieties associated to X_L (cf. *Remark 1.4*). Maintaining the notation given above, denote by Λ the free abelian group spanned by the symplectic basis of L used in the previous identification, and for any $\lambda \in \Lambda$ write $\lambda = \lambda_E + \lambda_U$, where $\lambda_E \in E = \binom{1}{0}$, $\lambda_U \in U = \binom{0}{1}$. Let $\Lambda_E = \{\,\lambda_E \mid \lambda \in \Lambda\,\}$, and denote by ω_Λ the tempered distribution on $E_{\mathbb{R}}$ defined by $\omega_\Lambda = \sum_{\lambda\in\Lambda_E} \Delta_\lambda$, where Δ_λ denotes the Dirac measure centered at the point λ. Then we can prove the following

2.15 Proposition. *Let* Θ *be the section of* $\mathcal{W}(-\frac{1}{2})_{|X_L}$, *horizontal with respect to* ∇, *defined by setting*

$$\Theta(D,h) = (\omega_\Lambda, U_{h'}e_\Omega),$$

for each $h \in H$ *and* $D \in X_L$ *(cf. Remark 2.11). Then* $r_E(\Theta(h,D)) = c\vartheta(z,\Omega)$, *where* c *is a suitable root of* 1 *and* $\vartheta(z,\Omega)$ *is the Riemann theta function.*

Proof. From the above definitions it follows that

$$\begin{aligned} r_E(\Theta(D,h)) &= (\omega_\Lambda, U_{(1,-z,0)}e_\Omega) = \sum_{\nu\in\mathbb{Z}^n} e_\Omega(\nu - z) = \\ &= (\det\Omega)^{-\frac{1}{2}} \sum_{\nu\in\mathbb{Z}^n} \exp \pi i[-{}^t(\nu - z)\Omega^{-1}(\nu - z)]. \end{aligned} \tag{2.16}$$

Now, since

$$\vartheta(\Omega^{-1}z, -\Omega^{-1}) = \sum_{\nu\in\mathbb{Z}^n} \exp \pi i[2{}^t\nu\Omega^{-1}z - {}^t\nu\Omega^{-1}\nu],$$

the conclusion follows from (2.16) on recalling the classical identity

$$\vartheta(\Omega^{-1}z, -\Omega^{-1}) = (\det -i\Omega)^{\frac{1}{2}} \exp \pi i[{}^tz\Omega^{-1}z]\vartheta(z,\Omega),$$

where the branch of the square root has positive value when Ω is purely imaginary (cf. [T1] Ch.II, p. 195). **QED** ■

2.17 Remark. Notations as in *Proposition 2.15*. If one considers the restriction of the function $h \mapsto (\omega_\Lambda, U_{h'}e_\Omega)$ to L, one gets a slight variation of the Riemann theta function (cf. [T3], §.2, Corollary 2.4). In fact we have the identity

$$(\omega_\Lambda, U_{(1,-Ex_1+Ux_2)}e_\Omega) = c \exp \pi i({}^tx_2(x_1+\Omega x_2))\,\vartheta(x_1+\Omega x_2,\Omega),$$

where c is a root of 1.

3. Local characterization of horizontal sections.

Here we shall characterize the germs of horizontal sections of $\mathcal{W}(-\frac{1}{2})$, and show how the growth of their values determines the extent to which they can be continued. To achieve our task we need some notations of symplectic geometry (cf. [S]). Let $M(n,\mathbb{C})$ be the space of the $n\times n$ complex matrices, then $A>0$ means that A is a positive-definite hermitian matrix. Let $r>0$ be a real number and $\Omega_0 = {}^t\Omega_0 \in M(n,\mathbb{C})$, then the *open disc* $D(\Omega_0, r)$, with center Ω_0 and radius r, is the subset of $M(n,\mathbb{C})$ defined as follows:

$$D(\Omega_0,r) = \left\{ \Omega = {}^t\Omega \,|\, r^2\mathbf{1} - (\Omega-\Omega_0)\overline{(\Omega-\Omega_0)} > 0 \right\}, \tag{3.1}$$

where $\mathbf{1}$ denotes the identity matrix.

Since there exists a "linear fractional transformation" which sends the Siegel upper half plane $\mathfrak{S}_n$ onto the open unit disc $D(0,1)$, given a real structure L, there exists a symplectic frame on V such that $X_L = D(\Omega_0, r)$ for any Ω_0, r as above (cf. the proof of *Corollary 3.5*).

Let $\Omega = {}^t\Omega$, and let Δ denote the diagonal real matrix having as entries the non-negative square roots of the characteristic values of $\Omega\bar\Omega$. Then there exists a unitary matrix V such that $({}^tV\Delta\bar V)^2 = \Omega\bar\Omega$; the matrix ${}^tV\Delta\bar V$ will be denoted by $|\Omega|$ and called the *modulus* of Ω.

Let's observe that $\Omega\in D(0,r)$ if, and only if, $\Delta\in D(0,r)$, *i.e.* all the characteristic values of $|\Omega|$ are less than r. For later use we recall the following elementary result (cf. [S], II §.7, Lemma 1).

3.2 Lemma. *Let Ω and Δ be as above; then there exists a unitary matrix U such that $\Omega = {}^tU\Delta U$.*

With the notations of the above Lemma, the matrix $X = {}^tU\Delta^{1/2}$ is called *a square root* of Ω.

Now we are ready to state the main result.

3.3 Theorem. *Let T be a positive real number, $f \in \mathcal{O}(\mathbb{C}^n)$, and consider the following conditions:*

$(i)_T$ *There exists a holomorphic function $g(\Omega, z)$, where $z \in \mathbb{C}^n$ and $\Omega \in D(0,T)$, which satisfies the Heat Equations $(\partial_{t_{ij}} - \partial_{z_i}\partial_{z_j})g = 0$, for $1 \le i \le j \le n$, with initial condition $g(0,z) = f(z)$;*

$(ii)_T$ $|f(z)| \le O(\exp(\frac{1}{4T}|z|^2))$;

$(iii)_T$ $\int_{\mathbb{C}^n} f(z)\overline{f(z)}\exp(-\frac{1}{2T}|z|^2)|dz \wedge d\bar{z}| < \infty$.

Then, if $0 < T < T_1$, any one of $(i)_{T_1}$, $(ii)_{T_1}$, $(iii)_{T_1}$ implies all of $(i)_T$, $(ii)_T$, $(iii)_T$.

Proof. $(i)_{T_1} \Rightarrow (ii)_T$ Let μ, α, $\beta \in \mathbb{N}^n$, such that $\mu = 2\alpha + \beta$ with $0 \le \beta_i \le 1$ for $i = 1, \dots, n$, and denote by $\alpha' = (\alpha'_{ij})$ the element of of $\mathbb{N}^{n^2}$ defined by $\alpha'_{ij} = \alpha_i\delta_{ij}$, where δ_{ij} is the Kronecker symbol.

Since the derivatives ∂_{z_i} and $\partial_{t_{ij}}$ commute, $\partial_z^\beta g$ is again a solution of the Heat Equations, so $\partial_z^\mu g = \partial_t^{\alpha'}(\partial_z^\beta g)$. Now, since $(\partial_z^\beta g)_{z=0}$ is holomorphic on $D(0,T_1)$, by taking its Taylor expansion at $\Omega = 0$, we have

$$|(\partial_z^\mu g(0,z))_{z=0}| = |[\partial_t^{\alpha'}(\partial_z^\beta g)_{z=0}]_{\Omega=0}| \le O(\alpha'!T_2^{-|\alpha'|}),$$

for any $0 < T_2 < T_1$, where $\alpha'! = \alpha! = \alpha_1! \cdots \alpha_n!$ and $|\alpha'| = |\alpha| = \alpha_1 + \cdots + \alpha_n$. As a consequence, the Taylor expansion of f at $z = 0$ shows the following asymptotic behavior:

$$|f(z)| \le O\left(\sum_{\alpha \in \mathbb{N}^n} \frac{\alpha!}{(2\alpha)!} T_2^{-|\alpha|}|z_1|^{2\alpha_1} \cdots |z_n|^{2\alpha_n}\right).$$

Now let's recall that $\frac{(m!)^2}{(2m)!} \le \frac{2m}{2^{2m}}$, for any positive integer $m \in \mathbb{N}$; so, if $|\alpha| = k$, one has $\frac{(\alpha!)^2}{(2\alpha)!} \le \frac{(2k)^n}{2^{2k}}$. This, for $0 < T < T_2$, implies that

$$\sum_{|\alpha|=k} \frac{\alpha!}{(2\alpha)!} T_2^{-|\alpha|}|z_1|^{2\alpha_1} \cdots |z_n|^{2\alpha_n} \le \frac{(2k)^n}{2^{2k}} \frac{1}{k!} \sum_{|\alpha|=k} \frac{k!}{\alpha!} T_2^{-|\alpha|}|z_1|^{2\alpha_1} \cdots |z_n|^{2\alpha_n} =$$

$$= (2k)^n \frac{1}{k!}(\tfrac{1}{4T_2}{}^tz\bar{z})^k \le O\left(\frac{1}{k!}(\tfrac{1}{4T}{}^tz\bar{z})^k\right),$$

so that $|f(z)| \leq O(\exp(\frac{1}{4T}|z|^2))$.
$(ii)_{T_1} \Rightarrow (i)_T$ Given $\Omega \in D(0,T)$, let X be one of its square roots (in the sense of *Lemma 3.2*) and consider the following integral

$$g(\Omega, z) = \int_{X\mathbb{R}^n} f(z+x) \frac{1}{\det(2\sqrt{\pi}X)} \exp(-\tfrac{1}{4}{}^t x \Omega^{-1} x) dx.$$

By a substitution of variables, recalling that $(\det X)^2 = \det \Omega$, one checks that $g(\Omega, z)$ is independent of the choice of X; thus, if we prove that the integral converges, we get a function. By hypothesis $|f(z+Xy)| \leq O(\exp(\frac{1}{4T_1}{}^t y {}^t X \overline{Xy}))$, for any $y \in \mathbb{R}^n$; so the conclusion follows by observing that ${}^t X \overline{X}$ has the same characteristic values as $|\Omega|$ and therefore lies in $D(0,T)$. In fact we have

$$\tfrac{1}{4T_1}{}^t y {}^t X \overline{Xy} < \tfrac{T}{4T_1}{}^t y y < \tfrac{1}{4}{}^t y y = \tfrac{1}{4}{}^t y {}^t X \Omega^{-1} X y \,,$$

which implies the convergence of the above integral for $\Omega \in D(0,T)$.
$(ii)_{T_1} \Rightarrow (iii)_T$ is clear. Now we prove that $(iii)_{T_1} \Rightarrow (ii)_T$. Denote by B_z the ball in $\mathbb{C}^n$ with center in z and radius $\frac{1}{|z|}$, and by V_z its volume. Since f is a holomorphic function, $f(z) = \frac{1}{V_z}\int_{B_z} f(u)|du \wedge d\bar{u}|$; then

$$|f(z)| \leq \frac{1}{V_z}\int_{B_z} |f(u)||du \wedge d\bar{u}| = \frac{1}{V_z}\int_{B_z} |f(u)| \exp(-\tfrac{1}{4T_1}{}^t u\bar{u}) \exp(\tfrac{1}{4T_1}{}^t u\bar{u})|du \wedge d\bar{u}|.$$

Now, by applying the Cauchy-Schwarz inequality for the L^2-norm of the functions on B_z, we get

$$\begin{aligned}
&\frac{1}{V_z}\int_{B_z} |f(u)| \exp(-\tfrac{1}{4T_1}{}^t u\bar{u}) \exp(\tfrac{1}{4T_1}{}^t u\bar{u})|du \wedge d\bar{u}| \leq \\
&\leq \left(\frac{1}{V_z}\int_{B_z} |f(u)|^2 \exp(-\tfrac{1}{2T_1}{}^t u\bar{u})|du \wedge d\bar{u}|\right)^{\frac{1}{2}} \left(\frac{1}{V_z}\int_{B_z} \exp(\tfrac{1}{2T_1}{}^t u\bar{u})|du \wedge d\bar{u}|\right)^{\frac{1}{2}} \leq \\
&\leq V_z^{-\frac{1}{2}} \left(\int_{\mathbb{C}^n} |f(u)|^2 \exp(-\tfrac{1}{2T_1}{}^t u\bar{u})|du \wedge d\bar{u}|\right)^{\frac{1}{2}} (\max_{u \in B_z} \exp(\tfrac{1}{2T_1}{}^t u\bar{u}))^{\frac{1}{2}}.
\end{aligned}$$

Finally, since $T < T_1$ and $(\max_{u\in B_z} \exp(\frac{1}{2T_1}{}^t u\bar{u}))^{\frac{1}{2}} = \exp(\frac{1}{4T_1}(1+\frac{1}{|z|})^2|z|^2)$, we conclude that $|f(z)| \leq O(\exp(\frac{1}{4T}|z|^2))$. **QED** ■

Now, we collect a couple of consequences of *Theorem 3.3*; the first one concerns the sub-vector bundle $\mathcal{W}^0$ described in *Corollary 1.6*.

3.4 Corollary. *Let $D \in S$. For any element $\phi \in W_D^0$ there exists a horizontal section s of $\mathcal{W}(-\frac{1}{2})$ such that $s(D) = \phi$.*

Proof. The condition (1.6) says that an element $\phi \in W_D$ belongs to the subspace W_D^0 if, and only if, there exists a constant such that $|r_E(\phi)(z)| \leq O(\exp A|z|^2)$ for some, and hence for any, complement E of D. By point (ii) of the previous theorem, this condition implies that ϕ is the value in D of a horizontal section s of $\mathcal{W}(-\frac{1}{2})$ and the constant A gives an estimate of the radius of the disc centered in D on which s is defined. **QED** ∎

The second fact concerns the spaces of horizontal sections of $\mathcal{W}(-\frac{1}{2})$ on the open discs X_L and $\sigma_L X_L$ associated to a real structure L on H, so let's observe first that $\sigma_L X_L = X_{iL}$.

3.5 Corollary. *Let X_L be the open disc, H_L the real group, and σ_L the conjugation associated to a real structure L on H; then the following chain of inclusions*

$$\Gamma(X_{iL}, \mathcal{W}(-\tfrac{1}{2})^{\nabla=0}) \subseteq \mathcal{H}_1 \subseteq \Gamma(X_L, \mathcal{W}(-\tfrac{1}{2})^{\nabla=0})$$

holds, where $\mathcal{H}_1$, is an irreducible unitary representation of H_L.

The space $\Gamma(X_{iL}, \mathcal{W}(-\frac{1}{2})^{\nabla=0})$ is dense in $\mathcal{H}_1$, so that it has structure of pre-Hilbert space.

Proof. Let's start by explaining how $\mathcal{H}_1$ arises. Let $\{e_1, \dots, e_{2n}\}$ be a symplectic basis of L, and set $e'_i = \frac{1}{\sqrt{2}}(e_i - ie_{n+i})$, $e'_{n+i} = \frac{1}{\sqrt{2}}(e_{n+i} - ie_i)$, for $i = 1, \dots, n$; then $\{e'_1, \dots, e'_{2n}\}$ is the symplectic basis we will use to identify V with $\mathbb{C}^n$. After this choice, let $E = \binom{1}{0}$ and $U = \binom{0}{1}$, then

$$H_L = \{\, (\lambda, z, i\bar{z}) \mid \lambda \in \mathbb{C}_1^*,\ z \in \mathbb{C}^n \,\}, \text{ and } X_L = \left\{ \begin{pmatrix} (\Omega-i)(\Omega+i)^{-1} \\ 1 \end{pmatrix} \mid \Omega \in \mathfrak{S}_n \right\} \subset U_E.$$

Since $\{\, (\Omega - i)(\Omega + i)^{-1} \mid \Omega \in \mathfrak{S}_n \,\} = D(0,1)$ (cf. (3.1) and [S], II §.4), we shall write $X_L = D(0,1)$.

Now consider the fibre W_U with the regular right action of H_L; then, the action on $\mathcal{O}(E)$ induced by $r_E \colon W_U \to \mathcal{O}(E)$ is as follows:

$$(U_{(\lambda,z,i\bar{z})} f)(y) = \lambda \exp[-\pi(2{}^t y\bar{z} + {}^t z\bar{z})]\, f(y+z).$$

As one can directly check, the action U is unitary with respect to the (Fock) norm

$$\|f\|_{(1)}^2 = \int_{\mathbb{C}^n} |f(y)|^2 \exp(-2\pi {}^t y\bar{y})\, |dy \wedge d\bar{y}|. \tag{3.6}$$

Then, (cf. *Proposition 2.5*) the space of the elements of $\mathcal{O}(E)$ having finite norm is an irreducible unitary representation of H_L: this is $\mathcal{H}_1$.

Now we describe the inclusion of $\mathcal{H}_1$ into $\Gamma(X_L, \mathcal{W}(-\frac{1}{2})^{\nabla=0})$. Let $f(y) \in \mathcal{H}_1$, then the function $\tilde{f}(z) := f(\frac{z}{2\sqrt{\pi i}})$ satisfies the condition $(iii)_T$ of *Theorem 3.3* with $T = 1$, so there exists a function $\tilde{g}(\Omega, z) \in \mathcal{O}(D(0,1) \times E)$ such that $\tilde{f}(z) = \tilde{g}(0,z)$, and $(\partial_{t_{ij}} - \partial_{z_i}\partial_{z_j})\tilde{g} = 0$, for $1 \le i \le j \le n$. As a consequence, $g(\Omega, y) := \tilde{g}(\Omega, 2\sqrt{\pi i}\, y)$ satisfies the equations $(\partial_{t_{ij}} - \frac{1}{4\pi i}\partial_{y_i}\partial_{y_j})g = 0$, for $1 \le i \le j \le n$, and then g corresponds to a section of $\mathcal{W}(-\frac{1}{2})$ on X_L, horizontal with respect to ∇ (cf. *Remark 1.13.(a)*).

Now there remains only to prove that $\Gamma(X_{iL}, \mathcal{W}(-\frac{1}{2})^{\nabla=0}) \subseteq \mathcal{H}_1$. By a direct computation one checks that $X_{iL} \cap U_E = \left\{ \begin{pmatrix} T \\ 1 \end{pmatrix} \mid |T| - \mathbf{1} > 0 \right\}$. As a consequence, if $r > 0$ is a real number and $D_0 = \begin{pmatrix} S \\ 1 \end{pmatrix}$, with $|S| - (r+2)\mathbf{1} > 0$, it can be shown that $D(D_0, r) \subseteq X_{iL} \cap U_E$(*). Let $s \in \Gamma(X_{iL}, \mathcal{W}(-\frac{1}{2})^{\nabla=0})$ and denote by $g(\Omega, z)$ the function it induces on $\mathcal{O}(D(D_0,r) \times \mathbb{C}^n)$ (cf. *Proposition 1.5 (b)*). Since $(\partial_{t_{ij}} - \frac{1}{4\pi i}\partial_{z_i}\partial_{z_j})g = 0$, for $1 \le i \le j \le n$, the function $\tilde{g}(\Omega, z)$ defined on $\mathcal{O}(D(0,r) \times \mathbb{C}^n)$ by $\tilde{g}(\Omega, z) = g(\Omega + S, \frac{z}{2\sqrt{\pi i}})$, satisfies the Heat Equations $(\partial_{t_{ij}} - \partial_{z_i}\partial_{z_j})\tilde{g} = 0$. Now assume $r > 1$, then *Theorem 3.3* tells us that

$$\int_{\mathbb{C}^n} |\tilde{g}(0,z)|^2 \exp(-\tfrac{{}^t z \bar{z}}{2})\, dz \wedge d\bar{z} < \infty,$$

i.e. $\|\tilde{g}(0,z)\|_{(1)} < \infty$; so $s \mapsto \tilde{g}(0,z)$ is the required inclusion.

Finally, $\Gamma(X_{iL}, \mathcal{W}(-\frac{1}{2})^{\nabla=0})$ is dense in $\mathcal{H}_1$ because it is a non-zero sub-representation of an irreducible representation. **QED** ■

(*) In fact, from *Lemma 3.2* it follows that $|S| > s \iff |Sz| > s|z| \forall z \in \mathbb{C}^n$. Then, if $|S| > r+2$ and $|T| < 1$, one has $|(S+T)z| \ge ||Sz| - |Tz|| > (r+1)|z|$ for all $z \in \mathbb{C}^n$.

4. Actions of the Metaplectic Group.

Let L be a real structure on H, X_L and H_L the corresponding disc and Heisenberg group. In this section we shall lift the action of $Sp(L)$ on X_L to an action of its double cover $Mp(L)$ on $\Gamma(X_L, \mathcal{W}(-\frac{1}{2})^{\nabla=0})$, and then we shall show how this induces the classical action of $Mp(L)$ on the irreducible representation of H_L.

First of all let's explain how the group $Sp(V)$ of symplectic automorphisms of $(V, \langle -, - \rangle)$, acts on the sections of the bundle $\mathcal{W}$. Let $\gamma \in Sp(V)$ and denote by the same name both the automorphism $(\lambda, v) \mapsto (\lambda, \gamma(v))$ of H, and the bijection $D \mapsto \gamma(D)$ of S. Then, for any open set $U \subseteq S$ and any $s \in \Gamma(U, \mathcal{W})$, we get a section $s_\gamma \in \Gamma(\gamma^{-1}(U), \mathcal{W})$, by setting

$$[s_\gamma(D)](h) = [s(\gamma(D))](\gamma(h)), \tag{4.1}$$

for any $D \in \gamma^{-1}(U)$ and $h \in H$.

Now fix a real structure L on H and consider the group $Sp(L)$ acting on the corresponding open disc X_L (cf. *Definition 2.3*). By means of the above definition one gets an action of $Sp(L)$ on $\Gamma(X_L, \mathcal{W})$; in fact it is immediate to check that $(s_\gamma)_\tau = s_{\gamma\tau}$ for any $\gamma, \tau \in Sp(L)$ and $s \in \Gamma(X_L, \mathcal{W})$.

If V is identified with $\mathbb{C}^{2n}$ by means of a symplectic basis of L, so that $X_L = \left\{ \left(\begin{smallmatrix} -\Omega \\ 1 \end{smallmatrix} \right) \mid \Omega \in \mathfrak{S}_n \right\} \subset U_E$, where $E = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix} \right)$, $\Gamma(X_L, \mathcal{W}) = \mathcal{O}(\mathfrak{S}_n \times \mathbb{C}^n)$ and $Sp(L) = Sp(2n, \mathbb{R})$, we can write the action of $Sp(L)$ on $\Gamma(X_L, \mathcal{W})$ explicitly. In fact let $\gamma = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix} \right) \in Sp(2n, \mathbb{R})$, $D = \left(\begin{smallmatrix} -\Omega \\ 1 \end{smallmatrix} \right) \in X_L$, and $g(\Omega, z) \in \mathcal{O}(\mathfrak{S}_n \times \mathbb{C}^n)$; then $\gamma(D) = \left(\begin{smallmatrix} -\gamma(\Omega) \\ 1 \end{smallmatrix} \right)$, where $\gamma(\Omega) = (A\Omega - B)(D - C\Omega)^{-1}$; thus using the decomposition

$$(1, \gamma(z, 0)) = \exp[-\pi i {}^t z (D - C\Omega)^{-1} C z] \, (1, -\gamma(\Omega) C z, C z) \, (1, {}^t(D - C\Omega)^{-1} z, 0),$$

from (4.1) we get the formula

$$g_\gamma(\Omega, z) = \exp[-\pi i {}^t z (D - C\Omega)^{-1} C z] \, g(\gamma(\Omega), {}^t(D - C\Omega)^{-1} z). \tag{4.2}$$

The group $Sp(L)$ also acts on the line bundle $\mathcal{D}et$, but its action doesn't extend to $\mathcal{D}et^{-\frac{1}{2}}$. In order to have an action on $\mathcal{W}(-\frac{1}{2})$ we must use $Mp(L)$. Now let's describe $Mp(L)$: let $s \in \Gamma(X_L, \mathcal{D}et)$ be a section such that $s(D) \neq 0$ for each $D \in X_L$; denote by f_γ the element of $\mathcal{O}^*(X_L)$ defined by $s_\gamma(D) = f_\gamma(D) s(\gamma(D))$, for any $\gamma \in Sp(L)$, and by $f_\gamma^{\frac{1}{2}}$ one of its square roots. Then the *Metaplectic group* is defined as follows:

$$Mp(L) = \left\{ (\gamma, \pm f_\gamma^{\frac{1}{2}}) \mid \gamma \in Sp(L) \right\},$$

with the group law $(\tau, f_\tau^{\frac{1}{2}})(\gamma, f_\gamma^{\frac{1}{2}}) = (\tau\gamma, f_\tau^{\frac{1}{2}}(f_\gamma^{\frac{1}{2}})^\tau)$, where $(f_\gamma^{\frac{1}{2}})^\tau(D) = f_\gamma^{\frac{1}{2}}(\tau(D))$. Now, using the coordinates and the identification $\Gamma(X_L, \mathcal{D}et^{-\frac{1}{2}}) \cong \mathcal{O}(X_L)$ given by $s^{-\frac{1}{2}} \mapsto 1$, we have:

$$Mp(L) \cong Mp(2n, \mathbb{R}) = \left\{ (\begin{pmatrix} A & B \\ C & D \end{pmatrix}, \pm[\det(D - C\Omega)]^{\frac{1}{2}}) \mid \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(2n, \mathbb{R}) \right\},$$

and $f_{\tilde\gamma}(\Omega) = [\det(D - C\Omega)]^{-\frac{1}{2}} f(\gamma(\Omega))$, for $\tilde\gamma = (\begin{pmatrix} A & B \\ C & D \end{pmatrix}, [\det(D - C\Omega)]^{\frac{1}{2}})$. Finally, identifying $\Gamma(X_L, \mathcal{W}(-\frac{1}{2}))$ with $\mathcal{O}(\mathfrak{S} \times \mathbb{C}^n)$ by means of the trivializations of $\mathcal{W}_{|X_L}$ and $\mathcal{D}et_{|X_L}^{-\frac{1}{2}}$ defined above, and combining both the actions of $Mp(L)$, we get

$$g_{\tilde\gamma}(\Omega, z) = [\det(D - C\Omega)]^{-\frac{1}{2}} \exp -\pi i\, {}^t z (D - C\Omega)^{-1} C z\, g(\gamma(\Omega), {}^t(D - C\Omega)^{-1} z). \quad (4.3)$$

Now we shall explain how the action of $Mp(L)$ on $\Gamma(X_L, \mathcal{W}(-\frac{1}{2})^{\nabla=0})$ induces a unitary representation $A : Mp(L) \to \mathrm{Aut}\,(L^2(E_\mathbb{R}))$ which normalizes the action of H_L. Let $\tilde\gamma \in Mp(L)$ and $\gamma \in Sp(L)$ be its projection. Since we are asking that $A_{\tilde\gamma}(U_h f) = U_{\gamma(h)}(A_{\tilde\gamma} f)$, for any $h \in H_L$ and $f \in L^2(E_\mathbb{R})$, in view of the irreducibility of $L^2(E_\mathbb{R})$ it's sufficient to define $A_{\tilde\gamma} v$ for a vector $v \neq 0$ in $L^2(E_\mathbb{R})$: we shall use the holomorphic vector e_Ω given by the formula (2.14).

Recall that in *Theorem 2.13* we described how to build a horizontal section of $\mathcal{W}(-\frac{1}{2})$ starting with an element of $L^2(E_\mathbb{R})$: given $f \in L^2(E_\mathbb{R})$, the corresponding horizontal section s was defined by setting $s(D, h) = (f\,, U_{h'} e_\Omega)$, for any $D = \begin{pmatrix} -\Omega \\ 1 \end{pmatrix} \in X_L$, where the map $\rho : h \mapsto h'$ is the involution of H defined in *Remark 2.11.(a)*. Now we are able to define the action A:

let f, s be as above, and denote by γ' the element of $Sp(L)$ which gives the automorphism $\rho\gamma\rho$ of H; then $A_{\tilde\gamma} e_\Omega$ is defined by requiring that

$$s_{\tilde\gamma'}(D, 1_H) = (f, A_{\tilde\gamma} e_\Omega), \quad (4.4)$$

for any f.

Now we can prove the following

4.5 Proposition. *Notations as above. There exists a (unique) unitary representation $A : Mp(L) \to \mathrm{Aut}\, L^2(E_\mathbb{R})$ satisfying (4.4) and normalizing the action of H_L on $L^2(E_\mathbb{R})$.*

Let A^* be the action of $Mp(L)$ on the dual space of $L^2(E_\mathbb{R})^{\mathrm{hol}}$ defined by setting

$$(A^*_{\tilde\gamma}\omega, f) = (\omega, A_{\tilde\gamma'} f),$$

for any $f \in L^2(E_{\mathbb{R}})^{\mathrm{hol}}$ and ω in the dual space. Then, the map described in Theorem 2.13, which sends the dual space of $L^2(E_{\mathbb{R}})^{\mathrm{hol}}$ to $\Gamma(X_L, \mathcal{W}(-\frac{1}{2})^{\nabla=0})$, is $Mp(L)$-equivariant.

Proof. Let $I = \{\, U_h e_\Omega \mid h \in H_L \,\}$ and denote by G the set of generators of $Sp(L)$ given by the following matrices:

$$\begin{pmatrix} A & 0 \\ 0 & {}^t\!A^{-1} \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ C & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

where $A, C = {}^tC \in GL_n(\mathbb{R})$.

Since the subspace spanned by I is dense $L^2(E_{\mathbb{R}})$, there exists a unique A such that

$$A_{\tilde\gamma}(U_h e_\Omega) = U_{\gamma(h)}(A_{\tilde\gamma} e_\Omega), \tag{4.6}$$

for any $\gamma \in G$ and $h \in H$, where $A_{\tilde\gamma} e_\Omega$ is the element defined by (4.4). By construction this A normalizes the action of H_L; thus all that remains to check is the unitarity of the restriction to I of the maps $A_{\tilde\gamma}$, as γ moves in G.

First, let's compute an explicit expression for $A_{\tilde\gamma} e_\Omega$. From (4.1) and (4.3) it follows that $s_{\tilde\gamma'}(\Omega, 1_H) = [\det(D + C\Omega)]^{-\frac{1}{2}}(f, e_{\gamma'(\Omega)})$, thus using (2.14) we get

$$A_{\tilde\gamma} e_\Omega(y) = [\det(A\Omega + B)]^{-\frac{1}{2}} \exp(-\pi i {}^t y (D + C\Omega)(A\Omega + B)^{-1} y), \tag{4.7}$$

for any $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(2n, \mathbb{R})$. In particular:

(i) $A_{\tilde\gamma} e_\Omega(y) = (\det A)^{-\frac{1}{2}} e_\Omega(A^{-1}y)$, for any $\gamma = \begin{pmatrix} A & 0 \\ 0 & {}^t\!A^{-1} \end{pmatrix} \in G$;

(ii) $A_{\tilde\gamma} e_\Omega(y) = e_\Omega(y) \exp(-\pi i {}^t y C y)$, for any $\gamma = \begin{pmatrix} 1 & 0 \\ C & 1 \end{pmatrix} \in G$;

(iii) $A_{\tilde\gamma} e_\Omega(y) = \exp(\pi i {}^t y \Omega y) = (i)^{\frac{n}{2}} \hat e_\Omega(y)$, for $\gamma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, where $\hat e_\Omega(y)$ denotes the Fourier transform.

Now it's straightforward to check that for any $f \in I$ and $\gamma \in G$ the element $A_{\tilde\gamma} f(y)$ defined by (4.6) is related to $f(y)$ in the same way as $A_{\tilde\gamma} e_\Omega(y)$ was to e_Ω. Since all the transformations (i), (ii) and (iii) are unitary, the proof of the first claim is complete.

Given an element ω of the dual space, consider the section $s \in \Gamma(X_L, \mathcal{W}(-\frac{1}{2})^{\nabla=0})$ defined by $s(D, h) = (\omega\,, U_{h'} e_\Omega)$, where $D = \begin{pmatrix} -\Omega \\ 1 \end{pmatrix} \in X_L$. Then, for any $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(2n, \mathbb{R})$, one has

$$\begin{aligned} (A^*_{\tilde\gamma}\omega\,, U_{h'} e_\Omega) &= (\omega\,, A_{\tilde\gamma'} U_{h'} e_\Omega) = (\omega\,, U_{\gamma'(h')} A_{\tilde\gamma'} e_\Omega) = \\ &= [\det(D - C\Omega)]^{-\frac{1}{2}} (\omega\,, U_{\gamma(h)'} e_{\gamma(\Omega)}) = s_{\tilde\gamma}(D, h)\,. \end{aligned}$$

This concludes the proof. **QED** ■

4.8 Remark. Let's maintain the above notations. Recall that in *Proposition 2.15* we obtained the Riemann theta function (up to multiplication by a root of 1) as a horizontal section of $\mathcal{W}(-\frac{1}{2})$ on the open disc X_L. Precisely, the horizontal section Θ was the image of the (tempered) distribution ω_Λ and ϑ was obtained by restriction: $\vartheta(\Omega, z) = c(\Theta(D,(1,z,0))$, where c is a suitable root of 1.

Now observe that the distribution ω_Λ is characterized (up to a constant) by its invariance under the action of the subgroup $\tilde{\Lambda} = \{\, (\exp \pi i \langle \lambda_E, \lambda_U\rangle \,,\, \lambda) \mid \lambda \in \Lambda \,\}$ of H, where Λ is the free abelian group generated by a (fixed) symplectic basis of the real structure L and λ_E and λ_U are the projections of $\lambda \in \Lambda$ on the subspaces $E = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $U = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ (cf. [T3], Theorem 2.3).

By means of *Proposition 4.5*, one can compute (up to a constant) the classical functional equations for the Riemann theta function. In fact, denote by $\Gamma_{1,2}$ the subgroup of $Sp(L)$ consisting of the elements γ such that $\gamma(\tilde{\Lambda}) \subseteq \tilde{\Lambda}$, and let $\tilde{\gamma}$ be an element of $Mp(L)$ which projects onto $\gamma \in \Gamma_{1,2}$; then the distribution $A^*_{\tilde{\gamma}}\omega_\Lambda$ is again invariant under the action of the subgroup $\tilde{\Lambda}$, therefore there exists a constant $a_{\tilde{\gamma}}$ such that $A^*_{\tilde{\gamma}}\omega_\Lambda = a_{\tilde{\gamma}}\omega_\Lambda$, so that the horizontal sections Θ and $\Theta_{\tilde{\gamma}}$ are related by

$$\Theta_{\tilde{\gamma}}(D,h) = (A^*_{\tilde{\gamma}}\omega_\Lambda \,,\, U_{h'}e_\Omega) = a_{\tilde{\gamma}}(\omega_\Lambda \,,\, U_{h'}e_\Omega) = a_{\tilde{\gamma}}\Theta(D,h),$$

for any $D \in X_L$ and $h \in H$. Now, by means of (4.3), one can write these identities in terms of the Riemann theta function; in particular, if $\gamma \in \Gamma_{1,2}$ is represented by the matrix $\begin{pmatrix} A & -B \\ -C & D \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, the above relation assumes the classical form

$$\begin{aligned} &\vartheta((A\Omega + B)(C\Omega + D)^{-1}, {}^t(C\Omega + D)^{-1}z) = \\ &\qquad = a_{\tilde{\gamma}}[\det(C\Omega + D)]^{\frac{1}{2}} \, \exp[\pi i {}^t z (C\Omega + D)^{-1}Cz]\, \vartheta(\Omega, z). \end{aligned}$$

One can obtain the last equations directly using the quasi-periodicity of the Riemann theta function. In fact, by its construction, the section Θ satisfies the identities

$$\Theta(D, \lambda d h \xi) = \lambda(\omega_\Lambda \,,\, U_{(dh\xi)'}e_\Omega) = \lambda(\omega_\Lambda \,,\, U_{\xi'}U_{h'}U_{d'}e_\Omega) = \lambda\Theta(D,h), \tag{4.9}$$

for any $D \in X_L$, $\lambda \in \mathbb{C}$, $d \in D$, $h \in H$ and $\xi \in \tilde{\Lambda}$. These identities imply the quasi-periodicity of ϑ; in fact, for any $n_1, n_2 \in \mathbb{Z}^n$ we have the decomposition

$$\begin{aligned} &(1, z + n_1 + \Omega n_2, 0) = \\ &\quad = \exp[-\pi i({}^t n_2 \Omega n_2 + 2\,{}^t n_2 z)]\,(1, \Omega n_2, -n_2)\,(1, z, 0)\,(\exp \pi i {}^t n_2 n_1 \,,\, n_1, n_2). \end{aligned} \tag{4.10}$$

Thus, by observing that $d = (1, \Omega n_2, -n_2) \in D$ and $\xi = (\exp \pi i {}^t n_2 n_1, n_1, n_2) \in \tilde{\Lambda}$, in view of (4.9) one gets

$$\begin{aligned} \vartheta(\Omega, z + n_1 + \Omega n_2) &= c\Theta(D, \exp[-\pi i({}^t n_2 \Omega n_2 + 2 {}^t n_2 z)]\, d(1, z, 0)\xi) = \\ &= \exp[-\pi i({}^t n_2 \Omega n_2 + 2 {}^t n_2 z)]\, \vartheta(\Omega, z). \end{aligned} \tag{4.11}$$

Given $\tilde{\gamma}$, which projects on $\gamma \in \Gamma_{1,2}$, let $\vartheta_{\tilde{\gamma}} = r_E \Theta_{\tilde{\gamma}}$; then $\vartheta_{\tilde{\gamma}}$ and ϑ have the same quasi-periodicity. In fact from (4.9) and (4.10) one deduces

$$\begin{aligned} \vartheta_{\tilde{\gamma}}(\Omega, z + n_1 + \Omega n_2) &= f_\gamma(\Omega)^{-\frac{1}{2}} \Theta(\gamma(D), \gamma(1, z + n_1 + \Omega n_2, 0)) \\ &= f_\gamma(\Omega)^{-\frac{1}{2}} \Theta(\gamma(D), \exp[-\pi i({}^t n_2 \Omega n_2 + 2 {}^t n_2 z)]\, \gamma(d)\gamma(1, z, 0)\gamma(\xi)) \\ &= \exp[-\pi i({}^t n_2 \Omega n_2 + 2 {}^t n_2 z)]\, \vartheta_{\tilde{\gamma}}(\Omega, z), \end{aligned}$$

because $\gamma(d) = \gamma(1, \Omega n_2, -n_2) \in \gamma(D)$ and, after the hypothesis on γ, $\gamma(\xi) = \gamma(\exp \pi i {}^t n_2 n_1, n_1, n_2) \in \tilde{\Lambda}$. As a consequence there exists a function k on X_L, such that $\vartheta_{\tilde{\gamma}}(\Omega, z) = k(\Omega)\vartheta(\Omega, z)$. Finally, since ϑ comes from a horizontal section, the same holds for $\vartheta_{\tilde{\gamma}}$, therefore k is a constant.

References

[D] P. Deligne, *Groupe de Heisenberg et Réalité*, Journal of AMS 4 (1991) p.197–206

[H] M. Hervé, *Analytic and Plurisubharmonic Functions in Finite and Infinite Dimensional Spaces*, LNM 198, Springer Verlag 1971

[I] J-I. Igusa, *Theta Functions*, Grundlehren der Math. Wiss. Band 194, Springer Verlag 1972

[KS] K. Kodaira, D.C. Spencer, *On Deformations of complex analytic Structures, II*, Annals of Math. 67 (1958) p.403–466

[MM] B. Mazur, W. Messing, *Universal Extensions and One Dimensional Crystalline Cohomology*, LNM 370, Springer Verlag 1974

[S] C.L. Siegel, *Symplectic Geometry*, Academic Press 1964

[T1] D. Mumford, *Tata Lectures on Theta I*, Progr. in Math. 28, Birkhäuser 1983

[T3] D. Mumford, *Tata Lectures on Theta III*, Progr. in Math. 97, Birkhäuser 1991