A Note about p-adic Logarithms

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Let k be a perfect field of characteristic $p > 0$ and denote by $W(k)$ (resp. K_0) the ring of Witt vectors over k (resp. its field of fractions). In a joint paper with Valentino Cristante (cf. [PD]), we introduced a ring, denoted by $\mathbf{Biv}\,R$ (cf. § 1), in order to compute periods for differentials of the second kind on an abelian variety with good reduction, defined over K_0. One of the main features of $\mathbf{Biv}\,R$ is that it allows one to define a "good" logarithm, which, in particular, permits a realization of the Tate Module of the multiplicative group over K_0 as a submodule of the additive group of $\mathbf{Biv}\,R$. In this paper we generalize the constructions given in [PD], replacing the base field K_0 with a more general local field K and we extend the logarithm map. This generalization appears to be more suitable for treating the case of abelian varieties having semi-stable reduction.

The rings defined here have deep relations with the rings introduced by Fontaine for the study of different types of Galois representations, so a detailed comparison is given.

Here is a list of the content of the paper. In § 1 we extend the definition of $\mathbf{Biv}\,R$ and the results of [PD] to a more general base field K and we introduce the ring $\mathbb{B}$, which appears to be necessary to deal with the extension of the logarithm. This extended map is the content of § 2 and we note here that the proof of Theorem 2.3 adapts an idea of Fontaine (cf. [F88]) to the present situation. Finally, in § 3 we compare our constructions with Fontaine's rings, extending some results contained in [PD]. In an Appendix we prove that the elements of $\mathbf{Biv}\,R$ invariants under the Frobenius map are exactly the elements of $\mathbb{Q}_p$.

1. Definitions of the basic rings

Let K be a field of characteristic 0, complete with respect to a discrete valuation v, and denote by $\mathcal{O}_K$ its ring of integers and by k its residue field, perfect, with characteristic $p > 0$. Denote by W the ring $W(k)$ of Witt vectors over k and by K_0 its field of fractions. Let $[K : K_0] = e$ and denote by $\bar{K}$ (resp. C) an algebraic closure of K (resp. its completion) and let $\mathcal{O}_{\bar{K}}$ (resp. $\mathcal{O}_C$) be the corresponding ring of integers. Finally, denote by G_K the Galois group $Gal(\bar{K}/K)$, and consider the extension of the valuation v of K to a valuation v_C on C, normalized by setting $v_C(p) = 1$.

We start with a ring of characteristic p. Denote by $R = R_{\mathcal{O}_C}$ the k–algebra

$$R = \varprojlim_{\mathbb{N}}(\mathcal{O}_C/p\mathcal{O}_C \xleftarrow{x^p} \mathcal{O}_C/p\mathcal{O}_C \xleftarrow{x^p} \dots). \tag{1.1}$$

One can represent an element x of R either as a sequence $x = (x_n)_{n\in\mathbb{N}}$ of elements $x_n \in \mathcal{O}_C/p\mathcal{O}_C$, or as a sequence $x = (\hat{x}_n)_{n\in\mathbb{Z}}$, where $\hat{x}_n \in \mathcal{O}_C$ is defined as follows: for $n \geq 0$, choose any element of $\tilde{x}_n \in \mathcal{O}_C$ which projects to x_n modulo $p\mathcal{O}_C$, and set $\hat{x}_n = \lim_{m\to\infty} \tilde{x}_{m+n}^{p^m}$; then, for $n < 0$, put $\hat{x}_n = \hat{x}_0^{p^{-n}}$. One defines a valuation on (the fraction field of) R, by setting $v_R(x) = v_C(\hat{x}_0) \in \mathbb{Q} \cup \{\infty\}$. The properties of R can be summarized in the following (cf. [F82] 2.1, or [PD] Proposition 5.6)

1.2 Proposition. *Notations as above. The ring R is a perfect k-algebra. Moreover R is a complete valuation ring w.r.t v_R, having the algebraic closure $\bar{k}$ of k as residue field. Finally, there is a natural action of G_K on R, obtained by applying the elements of G_K to the entries of the sequences $(\hat{x}_n)_{n\in\mathbb{Z}} \in R$.*

Starting with the perfect ring R, one can construct the ring $W(R)$ of Witt vectors over R and one can put a valuation w on its field of fractions by setting

$$w(\xi) = \min\left\{ p^{-i} v_R(x_i) + i \,:\, i \geq 0 \right\} \quad \text{for any } \xi = (x_0, x_1, \dots) \neq 0,\ \xi \in W(R) \tag{1.3}$$

(cf. [PD] Proposition 5.15 or the following *Proposition 1.5* for a slight generalization). We observe that $W(R)$ is a complete subring of the valuation ring and that there is a natural (continuous) action of G_K on $W(R)$, induced by the action on R.

One can represent a vector $\xi = (x_0, x_1, \dots) \in W(R)$ by the (convergent) series $\xi = \sum_{n\geq 0} p^n [x_n^{p^{-n}}]$, where, for any $x \in R$, its Teichmüller representative in $W(R)$ is denoted by $[x] = (x, 0, 0, \dots)$. In what follows, we extend both the representations of vectors to the elements of $W(R)[\frac{1}{p}]$, which will be represented either as a sequence $\eta = (y_j)_{j\in\mathbb{Z}}$ or as the corresponding (convergent) series $\eta = \sum_{n\in\mathbb{Z}} p^n [y_n^{p^{-n}}]$.

Now take the ring $\mathcal{O}_K$ and consider $W_K(R) := W(R) \otimes_W \mathcal{O}_K$. Denoting by π any uniformizing parameter of $\mathcal{O}_K$, one can represent (uniquely) any element ξ of $W_K(R)$ in the form $\xi = \sum_{n\geq 0} \pi^n [x_n]$, with $x_n \in R$. In particular, the above sum converges for the tensor product topology and the ring $W_K(R)$ is complete for that topology. There is a generalization of the usual Witt polynomials (the theory of "ramified Witt vectors"), which relates the above representation of a sum (resp. product) with the representations of the summands (resp. factors). We summarize these facts in the following Proposition and we refer to [FV], Ch. 1, § 7 or to [Haz] § 6, for a proof.

1.4 Proposition. *Notations as above. Let $\xi = \sum_{n\geq 0} \pi^n [x_n^{p^{-n}}]$ and $\eta = \sum_{n\geq 0} \pi^n [y_n^{p^{-n}}]$ be elements of $W_K(R)$ and let $\xi + \eta = \sum_{n\geq 0} \pi^n [z(+)_n^{p^{-n}}]$ and $\xi\eta = \sum_{n\geq 0} \pi^n [z(\times)_n^{p^{-n}}]$.*

Then, there exist polynomials $S_n(X_0, \dots, X_n; Y_0, \dots, Y_n)$ *and* $P_n(X_0, \dots, X_n; Y_0, \dots, Y_n)$ *with coefficients in* $\mathcal{O}_K$ *such that, for any* $n \geq 0$*, one has*

$$[z(+)_n] \equiv S_n([x_0], \dots, [x_n]; [y_0], \dots, [y_n]) \mod \pi W_K(R),$$
$$[z(\times)_n] \equiv P_n([x_0], \dots, [x_n]; [y_0], \dots, [y_n]) \mod \pi W_K(R).$$

If, for any $i \geq 0$*, one sets weight* p^i *to the variables* X_i *and* Y_i*, then* $S_n(X;Y)$ *is an isobaric polynomial with weight* p^n*; and* $P_n(X;Y)$ *is isobaric with weight* p^n *with respect to each group of variables.*

The above proposition allows us to extend to $W_K(R)$ some of the constructions made on the ring $W(R)$. As a first consequence, we extend the valuation w as follows.

1.5 Proposition. *For any* $\zeta = \sum_{n \geq 0} \pi^n [z_n] \in W_K(R)$*,* $\zeta \neq 0$*, set*

$$w_K(\zeta) = \min \{ v_R(z_i) + i/e \, : \, i \geq 0 \} .$$

In this way one defines a valuation on the fraction field of $W_K(R)$*. In particular, the embedding of* $W(R)$ *into* $W_K(R)$ *is isometric and the topology induced by the valuation* w_K *on* $W_K(R)$ *is equivalent to the tensor product topology.*

Proof We maintain the notations of the preceding Proposition. If ξ and η are the elements of $W_K(R)$ described in 1.4 and $w_K(\xi) = a \leq w_K(\eta) = b$; then, for each $n \geq 0$, one has $v_R(x_n) \geq p^n(a - n/e)$ and $v_R(y_n) \geq p^n(b - n/e) \geq p^n(a - n/e)$. From the congruence in *Proposition 1.4* and the isobaricity of polynomials $S_n(X;Y)$, one deduces that $v_R(z(+)_n) \geq p^n(a - n/e)$, for each n, and this means $w_K(\xi + \eta) \geq a$. If, moreover, $a < b$ and i_0 is the least index such that $v_R(x_{i_0}) = p^{i_0}(a - i_0/e)$; then

$$S_{i_0}([x_0], \dots, [x_{i_0}]; [y_0], \dots, [y_{i_0}]) = [x_{i_0} + y_{i_0}] + \left({\text{terms of value} \atop > p^{i_0}(a - i_0/e)} \right) .$$

This suffices to conclude that $w_K(\xi + \eta) = a$.

Now suppose $w_K(\xi) = a$ and $w_K(\eta) = b$ and let i_0 (resp. j_0) be the least index such that $v_R(x_{i_0}) = p^{i_0}(a - i_0/e)$ (resp. $v_R(y_{j_0}) = p^{j_0}(b - j_0/e)$). The hypothesis on the isobaricity of $P_n(X;Y)$ with respect to both variables implies that $w_K(\xi\eta) \geq a + b$, and the remark that

$$P_{i_0+j_0}([x_0], \dots, [x_{i_0+j_0}]; [y_0], \dots, [y_{i_0+j_0}]) = [x_{i_0}^{p^{j_0}} y_{j_0}^{p^{i_0}}] + \left({\text{terms of value} \atop > p^{i_0+j_0}(a+b-(i_0+j_0)/e)} \right)$$

allows us to conclude that $w_K(\xi\eta) = a + b$.

The isometricity of the embedding of $W(R)$ into $W_K(R)$ follows easily from the remark that $p = \pi^e a$ for some $a \in \mathcal{O}_K^\times$. Finally, denoting by W_r^0 the neighborhood of 0 in $W(R)$ made by the vectors having value greater than r, and by W_r^K the neighborhood of 0 in $W_K(R)$ made by the elements having value greater than r, one has

$$W^K_{n/e+r} \subseteq W^0_r \otimes \mathcal{O}_K + W(R) \otimes \pi^n \mathcal{O}_K \subseteq W^K_{r'},$$

where $r' = \min\{r, n/e\}$. **QED** ■

Denote by $\mathbf{Biv}\, R$ the closure of the ring $W(R)[\frac{1}{p}]$ in the completion of the fraction field $Fr\, W(R)$ with respect to the topology induced by the valuation w. There are continuous endomorphisms F (Frobenius) and V (Verschiebung) of $\mathbf{Biv}\, R$, obtained extending by continuity the maps of $W(R)[\frac{1}{p}]$, defined by

$$(1.6) \qquad F((y_j)_{j\in\mathbb{Z}}) = (y_j^p)_{j\in\mathbb{Z}} = \sum_{n\in\mathbb{Z}} p^n [y_n^{p^{-n+1}}] \qquad \text{and} \qquad V((y_j)_{j\in\mathbb{Z}}) = \sum_{n\in\mathbb{Z}} p^n [y_{n-1}^{p^{-n}}].$$

These maps satisfy the relations $FV = VF = p$. Then, denote by $\Theta : \mathbf{Biv}\, R \to C$ the continuous and surjective homomorphism, which extends by continuity the map $\Theta : W(R)[\frac{1}{p}] \to C$, defined by

$$(1.7) \qquad \Theta(\xi) = \sum_{n\in\mathbb{Z}} p^n \hat{x}_{n,n}, \qquad \text{for any } \xi = (x_j)_{j\in\mathbb{Z}} \in W(R)[\tfrac{1}{p}].$$

Observing that $\mathcal{O}_K$ is a free W-module of finite rank, one deduces that the ring $\mathbf{Biv}_K R := \mathbf{Biv}\, R \otimes_{K_0} K = \mathbf{Biv}\, R \otimes_W \mathcal{O}_K$ is the completion of $W_K(R)[\frac{1}{p}] = W(R)[\frac{1}{p}] \otimes_W \mathcal{O}_K$ with respect to the topology of the valuation. One can define a continuous and surjective homomorphism $\Theta_K : \mathbf{Biv}_K R \to C$, extending the above defined map Θ by K-linearity; in particular this map commutes with the actions of G_K.

We summarize some properties of the ring $\mathbf{Biv}_K R$ and the map Θ_K in the following

1.8 Theorem. *Notations as above.*

(a) *The ring* $\mathbf{Biv}_K R$ *is a complete topological* K*–algebra, endowed with a continuous action of the Galois group* G_K*.*

(b) *The map* Θ_K *is a continuous and surjective ring–homomorphism, equivariant with respect to the actions of the Galois group* G_K*.*

(c) *The ideal* $\ker\Theta_K \subset \mathbf{Biv}_K R$ *is principal, and an element* $\alpha = [a_0] + \pi[a_1] + \cdots \in \ker\Theta_K \cap W_K(R)$ *is a generator for* $\ker\Theta_K$ *if, and only if,* $v_R(a_0) = 1/e$ *and* $v_R(a_1) = 0$*.*

(d) $\bigcap_{n\geq 1} (\ker\Theta_K)^n = \{0\}$.

Proof (sketch). All these results have been proved for the ring $\mathbf{Biv}\, R$ (cf. [PD], Theorem 6.10), by means of the valuation and of the properties of Witt polynomials. One can easily adapt those arguments to the present situation. For instance, we sketch the proof of (c). By a direct calculation one can prove that an element α as above is a generator of the kernel of the restriction of Θ_K to $W_K(R)$ and that this restriction maps $W_K(R)$ onto $\mathcal{O}_C$ (cf. [PD] Proposition 5.8.(b) or [F82] Proposition 2.4). In particular, such element α is again a generator of the kernel of the restriction of Θ_K to $W_K(R)[\frac{1}{p}]$. From the surjectivity of the restriction of Θ_K to $W_K(R)$, it follows that, for any element $\eta \in \ker\Theta_K$, there exists a sequence $(\zeta_n)_{n\in\mathbb{N}}$, with $\zeta_n \in W_K(R)[\frac{1}{p}] \cap \ker\Theta_K$, which converges to η. This means that, for each n, one has $\zeta_n = \alpha\xi_n$, for some $\xi_n \in W_K(R)[\frac{1}{p}]$, and we observe that the sequence $(\xi_n)_{n\in\mathbb{N}}$ converges, because $w_K(\xi_n - \xi_{n+1}) = w_K(\zeta_n - \zeta_{n+1}) - 1/e$ for each n. **QED** ■

The following Lemma gives an explicit relation between $\ker\Theta_K$ and the kernel of the restriction of Θ_K to $\mathbf{Biv}\,R$.

1.9 Lemma. *There exists a generator ζ of $\ker\Theta = \ker\Theta_K \cap W(R)$ which does not belongs to $(\ker\Theta_K)^2$.*

Proof Let π be a uniformizing parameter of $\mathcal{O}_K$, and let π' be an element of R such that $\hat{\pi}'_0 = -\pi$. Denote by $f(X)$ the minimal polynomial for $-\pi$ over K_0. It is well known that f is an Eisenstein polynomial of degree $e = [K : K_0]$. This implies that $f([\pi'])$ belongs to $\ker\Theta_K \cap W(R)$ and that

$$\zeta = f([\pi']) \equiv [\pi'^e] + p[a] \mod p^2 W(R), \quad \text{with } v_R(a) = 0.$$

Then ζ generates $\ker\Theta$ (cf. *Theorem 1.8.(c)* for $K = K_0$). Moreover, in $\mathcal{O}_K[X]$ one has the decomposition $f(X) = (X + \pi)g(X)$ with $g(-\pi) \neq 0$, hence in $W_K(R)$ we can write $\zeta = ([\pi'] + \pi)g([\pi'])$, where $[\pi'] + \pi$ is a generator of $\ker\Theta_K$ (cf. *Theorem 1.8.(c)*) and $\Theta_K(g([\pi'])) = g(-\pi) \neq 0$. This concludes the proof. **QED** ■

Now we introduce a sub–$W(R)$–module $biv\,R$ of $\mathbf{Biv}\,R$, defined as

$$biv\,R = \left\{ \sum_{n\in\mathbb{Z}} p^n[x_n] : \liminf_{n\to+\infty} p^{-n} v_R(x_{-n}) > 0 \right\}. \tag{1.10}$$

Following Barsotti's [MA], the elements of $biv\,R$ are called *bivectors*. If $\xi = \sum_{n\in\mathbb{Z}} p^n[x_n]$ is a bivector, then the series defining ξ converges in $\mathbf{Biv}\,R$ and in this way one defines an immersion of $biv\,R$ in $\mathbf{Biv}\,R$. In particular $W(R)[\frac{1}{p}] \subset biv\,R$ and we can extend both the representations introduced for the elements of $W(R)[\frac{1}{p}]$ to the bivectors, so that we can write $\xi = \sum_{n\in\mathbb{Z}} p^n[x_n] = (x'_j)_{j\in\mathbb{Z}}$ to denote a bivector, where $x'_j = x_j^{p^j}$ for any $j \in \mathbb{Z}$. Moreover $biv\,R$ is closed for the endomorphisms F and V defined in (1.6).

We introduce a logarithm map for the units of $W(R)$ by means of the usual logarithmic series and we quote the following result (cf. [PD] Theorem 7.1), which says, in particular, that the values of this map lie in the submodule $biv\,R$.

1.11 Theorem. *Let* $U_{W(R)} = \{\, \xi \in W(R) : w(\xi - 1) > 0 \,\}$. *The map* $\log\colon U_{W(R)} \to \mathbf{Biv}\, R$ *is defined by setting*

$$\log(\xi) = -\sum_{n\geq 1} \frac{(1-\xi)^n}{n}.$$

The above map is a continuous and injective group–homomorphism, with values in the submodule $biv\, R$. *In particular, it commutes with the actions of the Galois group* G_K *and there is a commutative diagram of* G_K*–equivariant homomorphisms*

$$\begin{array}{ccc} U_{W(R)} & \xrightarrow{\log} & biv\, R \\ \big\downarrow{\scriptstyle\Theta} & & \big\downarrow{\scriptstyle\Theta} \\ U_{\mathcal{O}_C} & \xrightarrow{\log} & C \end{array}$$

where $U_{\mathcal{O}_C} = \{\, x \in \mathcal{O}_C : v(x-1) > 0 \,\}$ *and the vertical arrows are the restrictions of the map* Θ *(cf.* (1.7)*).*

Of course, one can extend the definition of the above map to the multiplicative group $U_{W_K(R)} = \{\, \xi \in W_K(R) : w_K(\xi - 1) > 0 \,\}$ and we observe that its values are contained in $biv\, R + W_K(R)[\frac{1}{p}]$, as one can deduce easily by remarking that for any $\xi \in U_{W_K(R)}$ one has $\xi = [x] + \pi\zeta$ with $[x] \in U_{W(R)}$ and $\zeta \in W_K(R)$, and that, for $n > 0$, in $K[X, Y]$ the congruence

$$\frac{1}{n}(X + \pi Y)^n \equiv \frac{X^n}{n} \mod \pi^s \mathcal{O}_K[X, Y]$$

holds, where $s = s_e := \min\{\, p^r - re : r \geq 0 \,\}$. In what follows, we are mainly interested in the logarithm of the vectors $[u]$, with $u \in U_R = \{\, x \in R : v_R(x-1) > 0 \,\}$, so that the map of the above theorem suffices for our purposes.

We can prove some other results about the structure of the ring $\mathbf{Biv}_K R$. We begin by introducing an element of $\ker \Theta_K$. Let ε be an element of R such that $\hat{\varepsilon}_0 = 1 \neq \hat{\varepsilon}_1$ and consider $[\varepsilon] \in W(R)$. One can prove that $w(1 - [\varepsilon]) = \frac{p}{p-1}$ so that there exists $t = \log[\varepsilon] \in biv\, R \cap \ker \Theta_K$. In particular, the sub–$\mathbb{Z}_p$–module spanned by t is isomorphic (as $\mathbb{Z}_p[G_K]$–module) to $\mathbb{Z}_p(1)$, the Tate Module of the multiplicative group $\mathbb{G}_m$ (over K). This means that for any $g \in G_K$, one has $g(t) = \chi(g)t$, where $\chi : G_K \to \mathbb{Z}_p^\times$ is the cyclotomic character. As usual, we denote by $\mathbb{Z}_p(n)$ the $\mathbb{Z}_p[G_K]$–module obtained by raising $\mathbb{Z}_p(1)$ to the n^{th} tensor power.

1.12 Proposition. *Notations as above.*

(*a*) *The element* $t = \log[\varepsilon] \in \mathbf{Biv}\, R$ *belongs to* $\ker \Theta_K$ *and its image spans the* C*–vector space* $\ker \Theta / (\ker \Theta)^2 \cong \ker \Theta_K / (\ker \Theta_K)^2$.

(*b*) *For any integer $n \geq 0$ there is a homomorphism $\phi_n\colon (\ker\Theta_K)^n \to C$, which induces an isomorphism of G_K–modules $\phi_n\colon (\ker\Theta_K)^n/(\ker\Theta_K)^{n+1} \to C(n)$, where $C(n) = C \otimes \mathbb{Z}_p(n)$.*

(*c*) *For any integer $n \geq 0$, one has $B(n) = \{\, \xi \in \mathbf{Biv}_K R : g(\xi) = \chi(g)^n \xi \ \forall g \in G_K \,\} = t^n K$.*

Proof It is a consequence of *Lemma 1.9* that $\ker\Theta/(\ker\Theta)^2 \cong \ker\Theta_K/(\ker\Theta_K)^2$. By means of that lemma, one can reduce the proof to the case $K = K_0$ and for that case, we refer to [PD], Propositions 7.9, 7.11 and Corollary 7.10. **QED** ∎

The final part of this section is devoted to the description of a new ring. Denote by $\mathbb{B}_K$ the completion of $\mathbf{Biv}_K R$ with respect to the ideal $\ker\Theta_K$ i.e.

$$\mathbb{B}_K = \varprojlim_{n\in\mathbb{N}} \mathbf{Biv}_K R/(\ker\Theta_K)^n. \tag{1.13}$$

We endow $\mathbb{B}_K$ with the projective-limit-topology and consider $\mathbf{Biv}_K R$ as a subring of $\mathbb{B}_K$ (cf. *Theorem 1.8.(d)*). In particular, we denote by $\mathbb{B}$ the ring $\mathbb{B}_{K_0}$.

Our first task is to give a description of the above ring and to show that its definition depends only on C and not on K.

1.14 Proposition. *The ring $\mathbb{B}_K$ is a discrete valuation ring, having C as its residue field and its maximal ideal is spanned by any element of $\ker\Theta_K \setminus (\ker\Theta_K)^2$. Moreover, there is a natural action of G_K on $\mathbb{B}_K$ and the inclusion $\mathbf{Biv} R \to \mathbf{Biv}_K R$ induces an isomorphism of rings $\mathbb{B} \to \mathbb{B}_K$.*

Proof The proof follows from the facts that θ_K is surjective, it commutes with the Galois actions and its kernel is a principal ideal (cf. *Theorem 1.8*). The last claim is a consequence of *Lemma 1.9* because the rings $\mathbb{B}$ and $\mathbb{B}_K$ have the same residue field and one can choose the image of the vector ζ of that lemma as uniformizing parameter for both. **QED** ∎

We remark here that, as a consequence of the above isomorphism, one deduces an embedding of the field K in $\mathbb{B}$, for any local field K having k as residue field.

We again denote by $\Theta : \mathbb{B} \to C$ the map induced by $\Theta : \mathbf{Biv} R \to C$ (cf. (1.7)), but, for $n > 0$, we will use the symbol Fil^n to denote the ideal $(\ker\Theta)^n$ of $\mathbb{B}$ and we set $Fil^0 := \mathbb{B}$. In the following Proposition, we summarize some properties of the ring $\mathbb{B}$ and of the map Θ, which follow from analogous properties of $\mathbf{Biv}_K R$ (cf. *Theorem 1.8* and *Proposition 1.12*).

1.15 Proposition. *Notations as above.*

(a) *The map $\Theta : \mathbb{B} \to C$ is a continuous, surjective ring–homomorphism, equivariant with respect to the actions of the Galois group G_K.*

(b) *For any integer $n \geq 0$ there is a homomorphism $\phi_n : Fil^n \to C$, which induces an isomorphism of G_K–modules*

$$\phi_n : \frac{Fil^n}{Fil^{n+1}} \to C(n),$$

where $C(n) = C \otimes \mathbb{Z}_p(n)$.

(c) *For any integer $n \geq 0$, one has*

$$B(n) = \{\, \xi \in \mathbb{B} : g(\xi) = \chi(g)^n \xi \ \forall g \in G_K \,\} = t^n K.$$

Proof (a) follows from *Theorem 1.8.(b)*. It follows from *Proposition 1.12* that, for any $n \geq 0$, the ideal $Fil^n \subset \mathbb{B}$ is spanned by the image of t^n, where $t = \log[\varepsilon]$ is the element of $\mathbf{Biv}\, R$ defined above. Then, the map ϕ_n is defined by setting $\phi_n(\xi) = \Theta(\zeta)$, for any $\xi = t^n\zeta \in Fil^n$. The proof of (c) is analogous to the proof of *Proposition 1.12.(c)*, and we now give a sketch of the proof for the convenience of the reader. Let's start with $B(0)$. If $g(\xi) = \xi$ for all $g \in G_K$, then $\Theta(\xi) \in C^{G_K} = K$. Hence, there exists $\xi' \in K \subset \mathbb{B}$ such that $\xi - \xi' = \eta \in Fil^1 \cap B(0)$. This implies $\phi_1(\eta) = \phi_1(g(\eta)) = \chi(g)g(\phi_1(\eta))$, and then $\phi_1(\eta) = 0$, because, for any $i > 0$, one has $C(i)^{G_K} = (0)$ (cf. Theorem 2 of [T]). This implies $\eta \in Fil^2$ and, by analogous arguments, one deduces that $\eta \in Fil^n$ for each n, which implies $\eta = 0$ (cf. *Theorem 1.8.(d)*) and then $B(0) = K$.

If $\xi \in B(n)$, one has $\phi_j(\xi) = 0$ for $j < n$ and $\xi \equiv t^n\xi' \mod Fil^{n+1}$, for some $\xi' \in K$. As before we can write $\xi - t^n\xi' = \eta_n \in Fil^{n+1} \cap B(n)$ where η_n satisfies to the condition $\chi(g)^n\phi_{n+1}(\eta_n) = \phi_{n+1}(g(\eta_n)) = \chi(g)^{n+1}g(\phi_{n+1}(\eta_n))$, for each $g \in G_K$, which implies $\phi_{n+1}(\eta_n) = 0$ and then $\eta_n \in Fil^{n+2}$. By the same argument used for η_1 we get $\eta_n \in Fil^k$ for each k and then $\eta_n = 0$, which gives $B(n) = t^nK$. **QED** ■

2. Logarithms

In this section we give an extension of the logarithm defined in *Theorem 1.11*. Let's recall some standard notations. If Γ is a commutative group, one defines

$$\begin{aligned} &\text{the } \textit{Tate Module} \text{ of } \Gamma \qquad T_p(\Gamma) := \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[\tfrac{1}{p}]/\mathbb{Z}, \Gamma), \\ &\text{the } \textit{Tate Space} \text{ of } \Gamma \qquad V_{(p)}(\Gamma) := \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[\tfrac{1}{p}], \Gamma). \end{aligned}$$

For any Γ, there is an exact sequence

$$0 \longrightarrow T_p(\Gamma) \longrightarrow V_{(p)}(\Gamma) \xrightarrow{\ \pi\ } \Gamma$$

and, in particular, if Γ is a p–divisible group, the map π is surjective.

By applying the above definitions to the multiplicative groups $\mathcal{O}_C^\times$ and $C^\times$, one gets

$$V_{(p)}(\mathcal{O}_C^\times) = R^\times, \qquad V_{(p)}(C^\times) = (Fr\,R)^\times, \qquad T_p(\mathcal{O}_C^\times) = T_p(C^\times) \cong \varepsilon^{\mathbb{Z}_p} = \mathbb{Z}_p(1),$$

where the element $\varepsilon \in R$ is defined in the previous section. The relations among these groups can be summarized in the following diagram with exact rows and columns (cf. [F88])

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathbb{Z}_p(1) \arrow[r] \arrow[d, equal] & R^\times \arrow[r] \arrow[d] & \mathcal{O}_C^\times \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathbb{Z}_p(1) \arrow[r] & (Fr\,R)^\times \arrow[r] \arrow[d, "v_R"] & C^\times \arrow[r] \arrow[d, "v_C"] & 0 \\
 & & \mathbb{Q} \arrow[r, equal] \arrow[d] & \mathbb{Q} \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd} \quad . \tag{2.1}$$

Now we prove the following result about the extension of the logarithm.

2.2 Theorem. *Let* $\log : C^\times \to C$ *be a determination of the usual* p*-adic logarithm and let* $x \mapsto \log[x]$ *be the injective map* $U_R \to biv\,R$ *defined by means of Theorem 1.11. Then, there is a unique homomorphism* $\log : V_{(p)}(C^\times) \to \mathbb{B}$*, which extends the above map and which fits into the following commutative diagram of* G_K*–equivariant maps*

$$\begin{tikzcd}
V_{(p)}(C^\times) \arrow[r, "\log"] \arrow[d, "\pi"] & \mathbb{B} \arrow[d, "\Theta"] \\
C^\times \arrow[r, "\log"] & C
\end{tikzcd} \quad .$$

Proof As a first approach, suppose that x belongs to $V_{(p)}(\mathcal{O}_C^\times) = R^\times$. We decompose x as the product $x = x_0 u$, where $x_0 \in \bar{k} \subset R$ and $u \in U_R$ and we set $\log(x) := \log[u]$. In this way, we define a homomorphism which commutes with Galois actions and has the desired relations with the usual p-adic logarithm of $\mathcal{O}_C^\times$.

Now fix an element $\pi \in R$ such that $\hat{\pi}_0 = p$. Then, for any $x \in (Fr\,R)^\times = V_{(p)}(C^\times)$, there exist –and are uniquely determined– two integers r and s, and an element $y \in R^\times$, such that $s > 0$, $(r,s) = 1$ and $\pi^r y = x^s$ (in particular one has $v_R(x) = \frac{r}{s}$). This implies that, for any extension of the above logarithm, we must set

$$\log(x) := \frac{r}{s}\log(\pi) + \frac{1}{s}\log[y].$$

We have reduced our problem to defining a value for $\log(\pi)$, such that it satisfies the compatibility conditions stated above. To satisfy the compatibility condition with the p–adic logarithm, we set $\log(\pi) = \log\frac{[\pi]}{p} + \log p$, where the latter summand is the determination of p-adic logarithm, which we can suppose in $K \subset \mathbb{B}$ (cf. *Proposition 1.15*) because the p-adic logarithm commutes with Galois actions. It remains to observe that $1 - \frac{[\pi]}{p}$ belongs to $\ker\Theta$, which implies that the usual logarithmic series converges in $\mathbb{B}$ to an element of Fil^1; hence, we set

$$\log\frac{[\pi]}{p} := -\sum_{n\geq 1}\frac{(1-[\pi]/p)^n}{n}$$

and this concludes our proof. **QED** ∎

We remark here that the series defining $u = \log\frac{[\pi]}{p}$ does not converge in $\mathbf{Biv}_K R$, because $w_K(1-[\pi]/p) = 0$. Moreover, we may prove a stronger result about the relations between the element u and the ring $\mathbf{Biv}_K R$.

2.3 Theorem. *Notations as above. The element* $u = -\sum_{n\geq 1}\frac{(1-[\pi]/p)^n}{n}$ *of* $\mathbb{B}$ *is not algebraic over* $Fr(\mathbf{Biv}_K R)$.

Because any ring $\mathbf{Biv}_K R \cong \mathbf{Biv}\,R \otimes_W \mathcal{O}_K$ is a finite module over $\mathbf{Biv}\,R$, it suffices to prove the claim for the ring $\mathbf{Biv}\,R$. The proof will follow from two Lemmas. We first fix here some notations. Let $\pi \in R$, with $\hat{\pi}_0 = p$ and denote by $\alpha = p - [\pi]$ this generator of the ideal $I := \ker\Theta_{|W(R)} = Fil^1 \cap W(R)$; then, set $\beta := \alpha/p \in W(R)[\frac{1}{p}] \cap Fil^1$. Moreover, we set $I^n := Fil^n \cap W(R)$ for any $n \geq 0$, so that $I = I^1$. We start with the following observation, which gives another description of the image of $\mathbf{Biv}\,R$ in $\mathbb{B}$.

2.4 Lemma. *Notations as above. Denote by S the subring of $\mathbb{B}$ consisting of the series* $\displaystyle\sum_{n\geq 0} a_n\beta^n$, *with $a_n \in W(R)$ for each $n \geq 0$. Then* $\mathbf{Biv}\,R \subseteq S[\frac{1}{p}]$.

Proof By definition, one can write any element of $\mathbf{Biv}\,R$ in the form $\sum_{n\geq 0} b_n$, with $b_n \in W(R)[\frac{1}{p}]$ for each $n \geq 0$, and the above sum converges in the topology induced by the valuation w. We observe that, for any positive integer n, one has

$$W_n^0 := \{\, x \in W(R) : w(x) \geq n \,\} = \sum_{j=0}^{n} p^j I^{n-j}$$

(cf. [PD], Proposition 5.10.(b)) and from this decomposition one deduces

$$W_k := \left\{ x \in W(R)[\tfrac{1}{p}] : w(x) \geq k \right\} = \sum_{j \geq 0} p^{-j} W^0_{k+j} = \sum_{j \geq -k} p^{-j} I^{k+j}.$$

Given the above sum $\sum_{n\geq 0} b_n$, for any positive integer k, there exists n_k such that $b_n \in W_k$ for each $n \geq n_k$ and then, from the above decomposition, one gets

$$b_n = \sum_{j \geq -k} p^{-j} a_{n,j+k} \alpha^{k+j},$$

for $n \geq n_k$, where $a_{n,j+k} \in W(R)$ and there is only a finite number of summands different from 0. By means of β one can write $b_n = \sum_{h\geq 0} a'_{n,h}\beta^h$, with $a'_{n,h} \in p^k W(R)$ for each $n \geq n_k$. This implies that, for any fixed value of h, the series $\sum_{n\geq 0} a'_{n,h}$ converges to an element $a_h \in W(R)[\frac{1}{p}]$ (with respect to the p–adic topology) and then $\sum_{n\geq 0} b_n = \sum_{h\geq 0} a_h \beta^h$ which belongs to $S[\frac{1}{p}]$. **QED** ■

2.5 Lemma. *Let $u = \log \frac{\{\pi\}}{p}$ in $\mathbb{B}$; then $u \notin Fr(\mathbf{Biv}\, R)$.*

Proof Let $\alpha = p - [\pi]$ and $\beta = \alpha/p$ as above; then $u = -\sum_{n\geq 1} \frac{\beta^n}{n}$. From *Lemma 2.4*, it follows that it suffices to prove that $u \notin Fr\, S$. Now consider the ideals (cf. § 1)

$$Fil^j S := S \cap Fil^j = \beta^j S \qquad \text{for } j \geq 1;$$

and denote by $\phi_j : Fil^j S \to \mathcal{O}_C$ the map which sends $a\beta^j \in Fil^j S$ to $\Theta(a)$. We observe that it suffices to prove that, if $0 \neq a \in S$, then $au \notin S$; and observing that $\bigcap_{n\geq 0} p^n S = (0)$, we can write $a = p^r x$ with $r \geq 0$ and $x \in S \setminus pS$.

If $b \in W(R)$ and $\Theta(b) \in p\mathcal{O}_C$, then $b \in pW(R) + I \subseteq pS$ (cf. *Theorem 1.8.(c)*). This implies that, given $x = \sum_{n\geq 0} b_n \beta^n \in S \setminus pS$, there exists an integer $i \geq 0$ such that

$$\Theta(b_h) \in p\mathcal{O}_C \text{ for } h < i \text{ and } \Theta(b_i) \notin p\mathcal{O}_C.$$

Fix an integer j such that $j > r$ and $p^j > i$. From the assumption $p^r xu \in S$, we obtain a contradiction. In fact, $p^r xu \in S$ implies that

$$-p^{j-1}xu = x \left(\sum_{m\geq 1} p^{j-1} \frac{\beta^m}{m} \right) \in S;$$

and then one deduces $x\beta^{p^j}/p \in S + Fil^{2p^j}$. Hence, by the hypothesis on i and j, one gets $b_i\beta^{p^j+i}/p \in S + Fil^{i+p^j+1}$; and then

$$\phi_{i+p^j}(b_i\frac{\beta^{p^j+i}}{p}) \in \phi_{i+p^j}(Fil^{i+p^j}S) = \mathcal{O}_C.$$

But by a direct calculation one has

$$\phi_{i+p^j}(b_i\frac{\beta^{p^j+i}}{p}) = \frac{\Theta(b_i)}{p} \notin \mathcal{O}_C$$

which gives the desired contradiction. **QED** ∎

Now we can conclude this section with

Proof of Theorem 2.3. In view of the two previous lemmas, it suffices to prove that, if u is algebraic over $Fr(\mathbf{Biv}\,R)$, then $u \in Fr(\mathbf{Biv}\,R)$.

Let $c_d + c_{d-1}X + \cdots + c_1X^{d-1} + X^d$ be the minimal polynomial for u over $Fr(\mathbf{Biv}\,R)$. Given $g \in G_{K_0}$, one has $g(\frac{[\pi]}{p}) = \frac{[\pi]}{p}[\varepsilon]^{\eta(g)}$, where $\eta : G_{K_0} \to \mathbb{Z}_p$ is a continuous map and $\varepsilon \in R$ is defined in § 1. If $t = \log[\epsilon] \in \mathbf{Biv}\,R$, from the above relation, one obtains $g(u) = u + \eta(g)t$ because the logarithm map commutes with the actions of G_{K_0}.

By observing that $\mathbf{Biv}\,R$ and its fraction field are closed with respect to the action of G_{K_0}, from the relation

$$g(c_d) + g(c_{d-1})(u + \eta(g)\tau) + \cdots + g(c_1)(u + \eta(g)\tau)^{d-1} + (u + \eta(g)\tau)^d = 0$$

and the uniqueness of the minimal polynomial, one finds

$$g(c_1) + d\eta(g)\tau = c_1 \qquad \text{for all } g \in G_{K_0}.$$

Then the element $c = c_1 + du$, satisfies $g(c) = c$ for all $g \in G_{K_0}$; and so $c \in \mathbb{B}^{G_{K_0}} = K_0 \subset \mathbf{Biv}\,R$ (cf. *Proposition 1.15.(c)*). This implies that $u = \frac{c-c_1}{d} \in Fr(\mathbf{Biv}\,R)$ and concludes the proof. **QED** ∎

3. Comparison with Fontaine's rings

This section is devoted to explaining the relations among the ring $\mathbf{Biv}\,R$, the module $biv\,R$ and the rings B^+, B^+_{cris}, B^+_{st}, B^+_{dR}, B^+_{HT} described by Fontaine in [F88]. Some results of this section are contained also in Remark 7.12 of [PD], but we reproduce them here for the sake of completeness. Throughout this section, we suppose that the base field is K_0; this makes it easier to read all definitions and notations. Of course, by means of the results of § 1, one can easily work over a more general base field K.

We start with the definition of the rings B^+_{cris} and B^+. We denote by $\alpha \in W(R)$ a generator of the ideal $I = \ker\Theta \cap W(R)$ and let A be the subring of $W(R)[\frac{1}{p}]$ generated over $W(R)$ by the elements $\frac{\alpha^n}{n!}$ for $n \geq 0$. The ring A does not depend on the choice of α and we denote by J the kernel of the map Θ restricted to A. In particular, there is a natural structure of divided powers on the ideal J of A, defined by setting $\gamma_n(x) = \frac{x^n}{n!}$ for each x in J. We denote by A_{cris} the completion of A with respect to the p-adic topology. Finally one sets (cf. [F88] 2.3)

$$B^+_{cris} := A_{cris}[\tfrac{1}{p}]. \tag{3.1}$$

The ring B^+_{cris} is endowed with a Frobenius endomorphism F, obtained on extending by continuity the analogous map of $W(R)[\frac{1}{p}]$ (cf. (1.6)).

Now we recall the definition of the ring B^+ (cf. [F82] § 4 and [F88] 4.1.4). For any rational number $c \geq 1$, denote by J_c the principal ideal $J_c = \{\, x \in R : v_R(x) \geq c \,\}$ and denote by y_c one of its generators. Let $S_{J_c} := W(R)[\frac{[y_c]}{p}] \subseteq W(R)[\frac{1}{p}]$ and denote by $\hat{S}_{J_c}$ the completion of S_{J_c} with respect to the p–adic topology and set $B^+_{J_c} := \hat{S}_{J_c}[\frac{1}{p}]$. It's easy to prove that for $1 \leq c \leq c'$, one has $B^+_{J_{c'}} \subseteq B^+_{J_c}$. Finally, we define

$$B^+ = \bigcap_{c \geq 1} B^+_{J_c}. \tag{3.2}$$

We intend to prove the following

3.3 Proposition. *Notations as above. Then*

(*a*) $B^+_{cris} \subset \mathbf{Biv}\,R$.

(*b*) $B^+ = \bigcap_{n \geq 1} F^n(B^+_{cris})$.

(*c*) $biv\,R \subseteq B^+$.

The above inclusions commute with Frobenius maps and with the actions of the Galois group G_{K_0}.

Proof (*a*).Denoting by α a generator of the ideal I of $W(R)$, any element of A_{cris} can be written in the form

$$\sum_{n \geq 0} a_n \frac{\alpha^n}{n!},$$

where the sequence of the coefficients $a_n \in W(R)$ tends p–adically to zero. The above sum converges in $\mathbf{Biv}\,R$ because for any generator α of I one has $w(\alpha) = 1$ (cf. *Theorem 1.8.(c)*). This implies

$$w(a_n \frac{\alpha^n}{n!}) \geq w(a_n) + n - v_p(n!) \geq w(a_n) + n\tfrac{p-2}{p-1};$$

which suffices to conclude, because the coefficients $a_n \in W(R)$ tend p–adically to zero and so they tend to zero in the topology of the valuation w.
(b). We start by proving that $\hat{S}_{J_p} \subset A_{cris} \subset \hat{S}_{J_1}$. In fact, taking the generator $\alpha = [\pi]-p$ of I, with $\pi \in R$ such that $\hat{\pi}_0 = p$, we observe that such $\pi \in R$ generates J_1 and then π^p generates J_p. This implies that S_{J_p} is generated over $W(R)$ by the powers of

$$\frac{[\pi^p]}{p} = \frac{(\alpha+p)^p}{p} = \frac{\alpha^p}{p} + p\lambda,$$

for some $\lambda \in W(R)$. This implies that

$$\left(\frac{\alpha^p}{p} + p\lambda\right)^n = \sum_{j=0}^{n} \binom{n}{j} p^{n-2j} \lambda^{n-j} \alpha^{pj} = \sum_{j=0}^{n} \binom{n}{j} p^{n-2j} (pj)! \lambda^{n-j} \frac{\alpha^{pj}}{(pj)!};$$

and, for $j = 0, \dots, n$, the coefficient $\binom{n}{j}(pj)!p^{n-2j}\lambda^{n-j}$ is in $W(R)$ because $n - 2j + v_p((pj)!) = n - j + v_p(j!) \geq 0$. This means that all the powers of $\frac{[\pi^p]}{p}$ are in A and then $\hat{S}_{J_p} \subset A_{cris}$.

In order to prove that $A_{cris} \subset \hat{S}_{J_1}$, it suffices to show that $\frac{\alpha^n}{n!} \in S_{J_1}$ for $n \geq 0$. In fact, one has

$$\frac{\alpha^n}{n!} = \frac{([\pi]-p)^n}{n!} = \sum_{j=0}^{n} (-1)^{n-j} \frac{p^{n-j}}{(n-j)!} \frac{[\pi]^j}{j!} = \sum_{j=0}^{n} (-1)^{n-j} \frac{p^n}{j!\,(n-j)!} \frac{[\pi]^j}{p^j}$$

and $\frac{p^n}{j!(n-j)!} = \frac{p^n}{n!}\binom{n}{j} \in W(R)$.

Then, from the inclusions $\hat{S}_{J_p} \subset A_{cris} \subset \hat{S}_{J_1}$, one deduces $B^+_{J_p} \subset B^+_{cris} \subset B^+_{J_1}$.

Denote by F the Frobenius map on $W(R)[\frac{1}{p}]$ and remark that, for any $c \geq 1$, one has $FS_{J_c} = S_{J_{pc}}$, and then $F\hat{S}_{J_c} = \hat{S}_{J_{pc}}$ and $FB^+_{J_c} = B^+_{J_{pc}}$. The above inclusions and (3.2) allow us to conclude.
(c). Fix $c \geq 1$. We will prove that for any $\zeta = \sum_{n\in\mathbb{Z}} p^n [z_n^{p^{-n}}] \in biv\,R$, there exists an integer n_0 such that $p^{n_0}\zeta \in \hat{S}_{J_c}$. We observe that $\zeta = a + \sum_{n>0} \frac{[z_{-n}^{p^n}]}{p^n}$ for some $a \in W(R)$,

so it suffices to prove the claim for $\zeta - a$. By (1.10) there exists a positive real number b such that $v_R(z_{-m}) \geq b$ for each $m \gg 0$. Then there exists a positive integer n_0 such that $\frac{[z_{-n}^{p^n}]}{p^{n-n_0}} \in S_{J_c}$ for all $n > 0$. By noting that $v_R(z_{-n}^{p^n}) - (n-n_0)c \geq p^n b - (n-n_0)c$ diverges with n, one deduces that the sum $p^{n_0} \sum_{n>0} \frac{[z_{-n}^{p^n}]}{p^n}$ converges for the p–adic topology of $\hat{S}_{J_c}$. This concludes the proof of the inclusion of $biv\,R$ in $B^+_{J_c}$ for any $c \geq 1$, and thus the inclusion of $biv\,R$ in B^+. **QED** ∎

We recall that the rings B^+ e B^+_{cris} are endowed with decreasing filtrations

$$Fil^n B^+ := (\ker\Theta)^n \cap B^+ \qquad \text{and} \qquad Fil^n B^+_{cris} := (\ker\Theta)^n \cap B^+_{cris},$$

with $n \geq 0$ and $\ker\Theta \subset \mathbf{Biv}\,R$. We recall the definition of the rings B and B_{cris}. Let ε be the element of R described in § 1 and let $t = \log[\varepsilon] \in biv\,R \subseteq B^+ \subseteq B^+_{cris}$. By definition, one sets

$$B = B^+[\tfrac{1}{t}] \qquad \text{and} \qquad B_{cris} = B^+_{cris}[\tfrac{1}{t}].$$

Now we recall the definitions of the rings B^+_{dR} and B^+_{HT} (cf. [F82], § 2.8 and [F88], § 1.5). One denotes by Θ_0 the restriction of Θ to $W(R)$ and one sets

$$B^+_{dR} := \varprojlim_{n\geq 1} \left(\frac{W(R)}{(\ker\Theta_0)^n} \left[\frac{1}{p}\right] \right), \tag{3.4}$$

endowed with the projective-limit-topology of the p–adic topologies on the quotients (Fontaine's canonical topology).

There is a surjective homomorphism $\Theta : B^+_{dR} \to C$ which can be obtained as limit of the surjective homomorphisms $\frac{W(R)}{(\ker\Theta_0)^n}[\frac{1}{p}] \to C$ (cf. *Theorem 1.8*). One defines a decreasing filtration on B^+_{dR} by setting

$$Fil^n B^+_{dR} := (\ker\Theta)^n \quad \text{for } n \geq 1 \qquad \text{and} \qquad Fil^0 B^+_{dR} := B^+_{dR}.$$

In particular $Fil^n B^+_{dR}/Fil^{n+1} B^+_{dR} \cong C$, for each n.

The ring B^+_{HT} is the graded ring associated to the above filtration of B^+_{dR}, that is

$$B^+_{HT} = \bigoplus_{n\geq 0} Fil^n B^+_{dR}/Fil^{n+1} B^+_{dR}. \tag{3.5}$$

In particular, B^+_{HT} is isomorphic (in a non-canonical way) to the polinomial ring $C[s]$, where one can choose as indeterminate s the image of any element of $Fil^1 \setminus Fil^2$. Finally, we recall that B_{dR} is the field of fractions of B^+_{dR} and $B_{HT} = B^+_{HT}[\frac{1}{s}]$.

3.6 Proposition. *Notations as above.*

(*a*) *There is a bicontinuous isomorphism between* $\mathbb{B}$ *(cf.* (2.6)*) and* B^+_{dR}*.*

(*b*) *The graded ring associated to the filtration*

$$\cdots \subseteq (\ker\Theta)^{n+1} \subseteq (\ker\Theta)^n \subseteq \cdots \subseteq \ker\Theta \subseteq \mathbf{Biv}\, R$$

is isomorphic to B^+_{HT}*.*

Proof (*a*). From the above definition it follows that B^+_{dR} is the completion of $W(R)[\frac{1}{p}]$ with respect to the kernel of the restriction of Θ to $W(R)[\frac{1}{p}]$. In particular, the inclusion of $W(R)[\frac{1}{p}]$ into $\mathbf{Biv}\, R$ induces a continuous injection of B^+_{dR} into $\mathbb{B}$. This injection is continuous because both the p–adic topology and the w–topology on $W(R)$ induce the same topology on the quotients $W(R)/(\ker\Theta_0)^n[\frac{1}{p}]$. In order to prove that this map is a bicontinuous isomorphism, it suffices to prove that for any $\xi \in \mathbf{Biv}\, R$ and any integer $n \geq 0$, there exists $\zeta \in W(R)[\frac{1}{p}]$ such that $\xi - \zeta \in (\ker\Theta)^n$. First of all we observe that there exists $\eta \in W(R)[\frac{1}{p}]$ such that $w(\xi - \eta) \geq n$ in $\mathbf{Biv}\, R$ and then $v_C(\Theta(\xi - \eta)) \geq w(\xi - \eta) \geq n$. Then there exists $\lambda_0 \in W(R)$ such that $\Theta(\xi - \eta) = p^n\Theta(\lambda_0) = \Theta(p^n\lambda_0)$ and this implies $\xi - \eta - p^n\lambda_0 \in \ker\Theta$. This gives $\xi - \eta - p^n\lambda_0 = \alpha\xi_1$, with $w(\xi_1) \geq n-1$ and then by applying the above arguments one determines $\lambda_1 \in W(R)$ such that $\Theta(\xi_1) = p^{n-1}\Theta(\lambda_1) = \Theta(p^{n-1}\lambda_1)$. This shows $\xi - \eta - p^n\lambda_0 - p^{n-1}\alpha\lambda_1 \in (\ker\Theta)^2$. Following the same path one gets

$$\xi - (\eta + p^n\lambda_0 + p^{n-1}\alpha\lambda_1 + \cdots + \alpha^n\lambda_n) \in (\ker\Theta)^n,$$

and $\zeta = \eta + p^n\lambda_0 + p^{n-1}\alpha\lambda_1 + \cdots + \alpha^n\lambda_n$ belongs to $W(R)[\frac{1}{p}]$. This concludes the proof of the first claim and we observe that (*b*) follows easily from (*a*). **QED** ■

The above results can be summarized by the following chain of inclusions

$$biv\, R \subset B^+ \subset B^+_{cris} \subset \mathbf{Biv}\, R \subset \mathbb{B} \cong B^+_{dR}. \tag{3.7}$$

There remains to describe the relations with the ring B^+_{st}, and we start by recalling the definition of B^+_{st} (cf. [F88]).

By means of the map defined in *Theorem 1.11* we introduced a homomorphism $\log : V_{(p)}(\mathcal{O}_C^\times) \to biv\, R$ (cf. *Theorem 2.2*) and, by composing this map with the inclusions $biv\, R \subset B^+ \subset B^+_{cris}$ (cf. *Proposition 3.3*), we get a homomorphism

$$\lambda : V_{(p)}(\mathcal{O}_C^\times) \to B^+_{cris}.$$

Then one can consider the category of couples (S, λ_S), where S is a B^+_{cris}–algebra and $\lambda_S : V_{(p)}(C^\times) \to S$ is a homomorphism which extends the λ defined above. In this category the morphisms are B^+_{cris}–algebra-homomorphisms satisfying a natural compatibility. The ring B^+_{st} (endowed with a map $\lambda' : V_{(p)}(C^\times) \to B^+_{st}$) is by definition the initial object of the above category.

We have observed in the previous section that the quotient $V_{(p)}(C^\times)/V_{(p)}(\mathcal{O}_C^\times)$ is a 1-dimensional $\mathbb{Q}$–vector space (cf.diagram (2.1)), so a homomorphism $\lambda_S : V_{(p)}(C^\times) \to S$, extending $\lambda : V_{(p)}(\mathcal{O}_C^\times) \to B^+_{cris}$, is completely determined by its value on any element $b \in V_{(p)}(C^\times)$, which spans the quotient $V_{(p)}(C^\times)/V_{(p)}(\mathcal{O}_C^\times)$. This implies that the above category has an initial object, isomorphic to the couple (B^+_{st}, λ'), where

$$B^+_{st} \cong B^+_{cris}[X], \qquad \text{and} \qquad \lambda'(b) = X, \tag{3.8}$$

and X is an indeterminate over B^+_{cris}.

In particular, the algebra B^+_{st} is endowed with a natural extension of the Frobenius map of B^+_{cris}, given by setting $F(X) := pX$. Moreover, there is a natural extension to B^+_{st} of the action of G_{K_0} on B^+_{cris}, which makes λ' an equivariant map. Finally, there is a unique B^+_{cris}–linear derivation

$$N : B^+_{st} \to B^+_{st}, \text{ such that } N(\lambda'(b)) = v_R(b) \quad \text{for each } b \in V_{(p)}(C^\times). \tag{3.9}$$

The above map N is called the *monodromy operator* associated to v_R. The fundamental relation between this derivation and the Frobenius map is $NF = pFN$.

Finally, one sets $B_{st} := B_{cris} \otimes_{B^+_{cris}} B^+_{st}$.

In the previous section, we proved that the homomorphism $\lambda : V_{(p)}(\mathcal{O}_C^\times) \to B^+_{cris}$, has an extension $\log : V_{(p)}(C^\times) \to \mathbb{B}$, by setting $\log[\pi] = u + \log p$, where $\pi \in R$ and $\hat{\pi}_0 = p$ and $u = -\sum_{n\geq 1} \frac{(1-[\pi]/p)^n}{n} \in \mathbb{B}$ and $\log p$ denotes a determination of p–adic logarithm (cf. *Theorem 2.2*).

In particular, we proved in *Theorem 2.3* that the above $u \in \mathbb{B}$ is not algebraic over $Fr(\mathbf{Biv}\, R)$. Thus we have

3.10 Proposition. *Notations as above. The embedding of B^+_{cris} in $\mathbf{Biv}\, R$ extends to an embedding of B^+_{st} in $\mathbb{B}$, whose image is contained in the polynomial ring $\mathbf{Biv}\, R[u]$.*

This embedding commutes with the actions of G_{K_0} and with the Frobenius maps. Finally, the $\mathbf{Biv}\, R$–linear derivation $N : \mathbf{Biv}\, R[u] \to \mathbf{Biv}\, R[u]$, defined by setting $N(u) = 1$, extends the monodromy operator of B^+_{st}.

As a final remark we illustrate in the diagram below the relations between the couple (B^+_{st}, λ') and the logarithm of bivectors defined in section 2, where the horizontal maps are all inclusions

$$\begin{array}{ccccccccc} biv\,R & \longrightarrow & B^+_{cris} & \longrightarrow & B^+_{st} & \longrightarrow & \mathbf{Biv}\,R[u] & \longrightarrow & B^+_{dR} \\ \big\uparrow{\scriptstyle \log} & & \big\uparrow{\scriptstyle \lambda} & & \big\uparrow{\scriptstyle \lambda'} & & \big\uparrow{\scriptstyle \log} & & \\ V_{(p)}(\mathcal{O}_C^\times) & = & V_{(p)}(\mathcal{O}_C^\times) & \longrightarrow & V_{(p)}(C^\times) & = & V_{(p)}(C^\times) & & \end{array}$$

4. Appendix.

In this section we'll prove that the elements of $\mathbf{Biv}\,R$ which are stable with respect to the Frobenius map are the elements of $\mathbb{Q}_p \cong biv\mathbb{F}_p \subset biv\,R \subset \mathbf{Biv}\,R$. This improves the result of Proposition 6.16.(b) of [PD], in which an analogous statement was proved for the elements of $biv\,R$.

We start by proving two Lemmas.

4.1 Lemma. *Let $(\xi_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $W(R)[\frac{1}{p}]$ and write $\xi_n = (x_{n,j})_{j\in\mathbb{Z}}$ for each n. Then, there exists an integer j_0 such that $v_R(x_{n,j}) > 0$ for each $j < j_0$ and for each $n \in \mathbb{N}$.*

Proof If we suppose the claim false, then for any $j_0 \in \mathbb{Z}$, there exists $n \in \mathbb{N}$ and $j < j_0$ such that $v_R(x_{n,j}) = 0$. In particular, this implies that $w(\xi_n) \le j < j_0$, and this is a contradiction, because the values of the terms of a Cauchy sequence are bounded below. **QED** ■

4.2 Lemma. *Hypotheses as above and suppose that $\lim_{n\to\infty} \xi_n = \lim_{n\to\infty} F(\xi_n)$. Then there exists an integer j_0 with the following property: for any positive number k, there exists $n_k \in \mathbb{N}$ such that*

$$n \ge n_k,\ j < j_0 \ \Rightarrow\ v_R(x_{n,j}) > p^j(k-j).$$

Proof Fix j_0 as in the previous Lemma. For any $k > 0$, there exists a positive integer n_k such that $w(F(\xi_n) - \xi_n) > k$ for $n > n_k$. Now fix $n > n_k$ and $i < j_0$ and write $F(\xi_n) - \xi_n = (z_{n,j})_{j\in\mathbb{Z}}$. Since $\xi_n \in W(R)[\frac{1}{p}]$, we may make the inductive hypothesis that $v_R x_{n,j}) > p^j(k-j)$ for $j < i$ and we must prove that this implies $v_R(x_{n,i}) > p^i(k-i)$.

From the condition $w(F(\xi_n) - \xi_n) > k$, one deduces $v_R(z_{n,i}) > p^i(k-i)$. By the usual conditions on polynomials defining the operations on Witt vectors, $z_{n,i}$ is a sum of monomials in the components $x_{n,j}$ with $j \le i$ of the kind $a x_{n,i-m}^{p\lambda_0+\mu_0} \cdots x_{n,i}^{p\lambda_m+\mu_m}$, for some $m \gg 0$, with $\sum_{j=0}^m p^j(\lambda_j + \mu_j) = p^m$ and $a \in \mathbb{F}_p$. Any monomial in which $x_{n,i}$ does not appear has value strictly greater than $p^i(k-i)$. The above conditions imply either $v_R(x_{n,i}) > p^i(k-i)$ or $v_R(x_{n,i}) = v_R(x_{n,i}^p) = 0$, but the latter cannot happen because $j < j_0$. **QED** ■

Finally we are able to prove the main result of this section

4.3 Theorem. *Let $\xi \in \mathbf{Biv}\,R$ such that $F(\xi) = \xi$. Then $\xi \in \mathbb{Q}_p \subset \mathbf{Biv}\,R$.*

Proof Let $(\xi_n)_{n\in\mathbb{N}}$ be a sequence in $W(R)[\frac{1}{p}]$ converging to ξ and fix j_0 as in *Lemma 4.2*. For any $n \in \mathbb{N}$, we can take the decomposition $\xi_n = \xi_n^{(<j_0)} + \xi_n^{(\geq j_0)}$, where $\xi_n^{(<j_0)} = (u_{n,j})_{j\in\mathbb{Z}} \in W(R)[\frac{1}{p}]$ (resp. $\xi_n^{(\geq j_0)} = (v_{n,j})_{j\in\mathbb{Z}}$) satisfies the conditions $u_{n,j} = x_{n,j}$ (resp. $v_{n,j} = 0$) for $j < j_0$ and $u_{n,j} = 0$ (resp. $v_{n,j} = x_{n,j}$) for $j \geq j_0$.

By *Lemma 4.2*, the sequence $(\xi_n^{(<j_0)})_{n\in\mathbb{N}}$ converges to zero and so $\xi = \lim\limits_{n\to\infty} \xi_n = \lim\limits_{n\to\infty} \xi_n^{(\geq j_0)}$. As $W(R)$ is complete for the topology of the valuation, this implies that $\xi \in W(R)[\frac{1}{p}]$ and the condition $F(\xi) = \xi$ means that all its components are roots of the polynomial $X^p - X$. This allows us to conclude that the components of ξ belong to the copy of $\mathbb{F}_p$ in R, so that $\xi \in biv\mathbb{F}_p \cong \mathbb{Q}_p$. **QED** ∎

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