THE CLASSIFICATION OF SURFACES WITH $p_g = q = 1$ ISOGENOUS TO A PRODUCT OF CURVES

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Abstract. A smooth, projective surface S is said to be *isogenous to a product* if there exist two smooth curves C, F and a finite group G acting freely on $C \times F$ so that $S = (C \times F)/G$. In this paper we classify all surfaces with $p_g = q = 1$ which are isogenous to a product.

0. Introduction

The classification of smooth, complex surfaces S of general type with small birational invariants is quite a natural problem in the framework of algebraic geometry. For instance, one may want to understand the case where the Euler characteristic $\chi(\mathcal{O}_S)$ is 1, that is, when the geometric genus $p_g(S)$ is equal to the irregularity $q(S)$. All surfaces of general type with these invariants satisfy $p_g \leq 4$. In addition, if $p_g = q = 4$ then the self-intersection K_S^2 of the canonical class of S is equal to 8 and S is the product of two genus 2 curves, whereas if $p_g = q = 3$ then $K_S^2 = 6$ or 8 and both cases are completely described ([CCML98], [HP02], [Pir02]). On the other hand, surfaces of general type with $p_g = q = 0, 1, 2$ are still far from being classified. We refer the reader to the survey paper [BaCaPi06] for a recent account on this topic and a comprehensive list of references.

A natural way of producing interesting examples of algebraic surfaces is to construct them as quotients of known ones by the action of a finite group. For instance Godeaux constructed in [Go31] the first example of surface of general type with vanishing geometric genus taking the quotient of a general quintic surface of $\mathbb{P}^3$ by a free action of $\mathbb{Z}_5$. In line with this Beauville proposed in [Be96, p. 118] the construction of a surface of general type with $p_g = q = 0$, $K_S^2 = 8$ as the quotient of a product of two curves C and F by the free action of a finite group G whose order is related to the genera $g(C)$ and $g(F)$ by the equality $|G| = (g(C)-1)(g(F)-1)$. Generalizing Beauville's example we say that a surface S is *isogenous to a product* if $S = (C \times F)/G$, for C and F smooth curves and G a finite group acting freely on $C \times F$. A systematic study of these surfaces has been carried out in [Ca00]. It is observed there that S is of general type if and only if both $g(C)$ and $g(F)$ are greater than or equal to 2 and that in this case S admits a unique minimal realization where they are as small as possible. From now on, we tacitly assume that such a realization is chosen, so that the genera of the curves and the group G are invariants of S. The action of G can be seen to respect the product structure on $C \times F$. This means that such actions fall in two cases: the *mixed* one, where there exists some element in G exchanging the two factors (in this situation C and F must be isomorphic) and the *unmixed* one, where G acts faithfully on both C and F and diagonally on their product.

After [Be96], examples of surfaces isogenous to a product with $p_g = q = 0$ appeared in [Par03] and [BaCa03], and their complete classification was obtained in [BaCaGr06].

The next natural step is therefore the analysis of the case $p_g = q = 1$. Surfaces of general type with these invariants are the irregular ones with the lowest geometric genus and for this reason it would be important to provide their complete description. So far, this has been obtained only in the cases $K_S^2 = 2, 3$ ([Ca81], [CaCi91], [CaCi93], [Pol05], [CaPi05]).

The goal of the present paper is to give the full list of surfaces with $p_g = q = 1$ that are isogenous to a product. Our work has to be seen as the sequel to the articles [Pol06] and [Pol07], which

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describe all unmixed cases with G abelian and some unmixed examples with G nonabelian. Apart from the complete list of the genera and groups occurring, our paper contains the first examples of irregular surfaces of mixed type. The mixed cases turn out to be much less frequent than the unmixed ones and, as when $p_g = q = 0$, they occur for only one value of the order of G. However, in contrast with what happens when $p_g = q = 0$, the mixed cases do not correspond to the maximum value of $|G|$ but appear for a rather small order, namely $|G| = 16$.
What is still lacking is a description of the connected components of the moduli space for our surfaces. We believe that it could be achieved by using the same methods of [BaCaGr06, Section 6] and [Pol07, Section 5], but we will not develop this point here.

Our classification procedure involves arguments from both geometry and combinatorial group theory. We will give here a brief account on how the result is achieved.
If S is any surface isogenous to a product and satisfying $p_g = q$ then $|G|$, $g(C)$, $g(F)$ are related as in Beauville's example and we have $K_S^2 = 8$. Besides, if $p_g = q = 1$ such surfaces are automatically minimal and of general type (Lemma 3.1).
If $S = (C \times F)/G$ is of unmixed type then the two projections $\pi_C \colon C \times F \longrightarrow C$, $\pi_F \colon C \times F \longrightarrow F$ induce two morphisms $\alpha \colon S \longrightarrow C/G$, $\beta \colon S \longrightarrow F/G$, whose smooth fibres are isomorphic to F and C, respectively. Moreover, the geometry of S is encoded in the geometry of the two coverings $h \colon C \longrightarrow C/G$, $f \colon F \longrightarrow F/G$ and its invariants impose strong restrictions on $g(C)$, $g(F)$ and $|G|$. Indeed we have $1 = q(S) = g(C/G) + g(F/G)$ so we may assume that $E := C/G$ is an elliptic curve and $F/G \cong \mathbb{P}^1$. Then $\alpha \colon S \longrightarrow E$ is the Albanese morphism of S and the genus g_{alb} of the general Albanese fibre equals $g(F)$. It is proven in [Pol07, Proposition 2.3] that $3 \leq g(F) \leq 5$ hence we can exploit the classification of finite groups of automorphisms acting on curves of low genus ([Br90], [Ki03], [KuKi90], [KuKu90]). A list of such groups for $g(F) = 3, 4, 5$ is given in Tables 1, 2, 3 of the Appendix. The covers f and h are determined by two suitable systems of generators for G, that we call $\mathcal{V}$ and $\mathcal{W}$, respectively. In order to obtain a free action of G on $C \times F$ and a quotient S with the desired invariants, $\mathcal{V}$ and $\mathcal{W}$ are subject to strict conditions of combinatorial nature (Proposition 3.3). The groups occurring in the Appendix are exactly those for which the existence of $\mathcal{V}$ is guaranteed, so we are left to verify the existence of $\mathcal{W}$ for each of them.
If $S = (C \times C)/G$ is of mixed type then the index two subgroup G° of G corresponding to transformations that do not exchange the coordinates in $C \times C$ acts faithfully on C. The quotient $E = C/G^{\circ}$ is isomorphic to the Albanese variety of S and $g_{\text{alb}} = g(C)$ (Proposition 3.6). Moreover $g(C)$ may only be 5, 7 or 9, hence $|G|$ is at most 64 (Proposition 3.10). The cover $h \colon C \longrightarrow E$ is determined by a suitable system of generators $\mathcal{V}$ for G° and since the action of G on $C \times C$ is required to be free, combinatorial restrictions involving the elements of $\mathcal{V}$ and those of $G \setminus G^{\circ}$ have to be imposed.(Proposition 3.8). Our classification is obtained by first listing those groups G° for which $\mathcal{V}$ exists and then by looking at the admissible extensions G of G°. We find that the only possibility occurring is for $g(C) = 5$ so that $|G|$ is necessarily 16 (Propositions 5.1, 5.2, 5.3).
In this way we proved the following

Theorem. *Let $S = (C \times F)/G$ be a surface with $p_g = q = 1$, isogenous to a product of curves. Then S is minimal of general type and $g(C)$, $g(F)$ and G are precisely those in the table below.*

$g(F) = g_{\mathrm{alb}}$	$g(C)$	G	`IdSmall` `Group`(G)	Type
3	3	$(\mathbb{Z}_2)^2$	$G(4,2)$	unmixed $(*)$
3	5	$(\mathbb{Z}_2)^3$	$G(8,5)$	unmixed $(*)$
3	5	$\mathbb{Z}_2 \times \mathbb{Z}_4$	$G(8,2)$	unmixed $(*)$
3	9	$\mathbb{Z}_2 \times \mathbb{Z}_8$	$G(16,5)$	unmixed $(*)$
3	5	D_4	$G(8,3)$	unmixed
3	7	D_6	$G(12,4)$	unmixed $(*)$
3	9	$\mathbb{Z}_2 \times D_4$	$G(16,11)$	unmixed
3	13	$D_{2,12,5}$	$G(24,5)$	unmixed
3	13	$\mathbb{Z}_2 \times A_4$	$G(24,13)$	unmixed
3	13	S_4	$G(24,12)$	unmixed
3	17	$\mathbb{Z}_2 \ltimes (\mathbb{Z}_2 \times \mathbb{Z}_8)$	$G(32,9)$	unmixed
3	25	$\mathbb{Z}_2 \times S_4$	$G(48,48)$	unmixed
4	3	S_3	$G(6,1)$	unmixed $(*)$
4	5	D_6	$G(12,4)$	unmixed
4	7	$\mathbb{Z}_3 \times S_3$	$G(18,3)$	unmixed
4	9	S_4	$G(24,12)$	unmixed $(*)$
4	13	$S_3 \times S_3$	$G(36,10)$	unmixed
4	13	$\mathbb{Z}_6 \times S_3$	$G(36,12)$	unmixed
4	13	$\mathbb{Z}_4 \ltimes (\mathbb{Z}_3)^2$	$G(36,9)$	unmixed
4	21	A_5	$G(60,5)$	unmixed $(*)$
4	25	$\mathbb{Z}_3 \times S_4$	$G(72,42)$	unmixed
4	41	S_5	$G(120,34)$	unmixed
5	3	D_4	$G(8,3)$	unmixed $(*)$
5	4	A_4	$G(12,3)$	unmixed $(*)$
5	5	$\mathbb{Z}_4 \ltimes (\mathbb{Z}_2)^2$	$G(16,3)$	unmixed
5	7	$\mathbb{Z}_2 \times A_4$	$G(24,13)$	unmixed
5	9	$\mathbb{Z}_8 \ltimes (\mathbb{Z}_2)^2$	$G(32,5)$	unmixed
5	9	$\mathbb{Z}_2 \ltimes D_{2,8,5}$	$G(32,7)$	unmixed
5	9	$\mathbb{Z}_4 \ltimes (\mathbb{Z}_4 \times \mathbb{Z}_2)$	$G(32,2)$	unmixed
5	9	$\mathbb{Z}_4 \ltimes (\mathbb{Z}_2)^3$	$G(32,6)$	unmixed
5	13	$(\mathbb{Z}_2)^2 \times A_4$	$G(48,49)$	unmixed
5	17	$\mathbb{Z}_4 \ltimes (\mathbb{Z}_2)^4$	$G(64,32)$	unmixed
5	21	$\mathbb{Z}_5 \ltimes (\mathbb{Z}_2)^4$	$G(80,49)$	unmixed
5	5	$D_{2,8,3}$	$G(16,8)$	mixed
5	5	$D_{2,8,5}$	$G(16,6)$	mixed
5	5	$\mathbb{Z}_4 \ltimes (\mathbb{Z}_2)^2$	$G(16,3)$	mixed

Here `IdSmallGroup`(G) denotes the label of the group G in the GAP4 database of small groups (see [GAP4]). The cases marked with $(*)$ already appeared in [Pol07] and have been included here for the sake of completeness.

This paper is organized as follows.
In Section 1 we present some preliminaries and we fix the algebraic set-up. In particular Proposition 1.2, which is essentially Riemann existence theorem, recalls how to translate the problem of finding Riemann surfaces with automorphisms into the algebraic problem of finding groups G with suitable systems of generators. Then we show that this problem has no solution for some of the groups listed in the Appendix.
In Section 2 we collect the basic facts about surfaces isogenous to a product, following the treatment given by Catanese in [Ca00].

In Section 3 we apply the structure theorems of Catanese to the case $p_g = q = 1$ and this leads to Propositions 3.3 and 3.8, that provide the translation of our classification problem from geometry to algebra. All these results are used in Sections 4 and 5, which are the core of the paper and give the complete classification of the unmixed and mixed cases, respectively.
When the amount of groups of a given order was much too high to provide a hand-made proof our computations were carried out by using the computer algebra program GAP4. More precisely, we exploited GAP4 in order to exclude the mixed case with $g(C) = 7, 9$ (see Subsections 5.2, 5.3). For the reader's convenience we included all the scripts.

Notations and conventions. All varieties, morphisms, etc. in this article are defined over the field $\mathbb{C}$ of the complex numbers. By "surface" we mean a projective, non-singular surface S, and for such a surface K_S denotes the canonical class, $p_g(S) = h^0(S,\ K_S)$ is the *geometric genus*, $q(S) = h^1(S,\ K_S)$ is the *irregularity* and $\chi(\mathcal{O}_S) = 1 - q(S) + p_g(S)$ is the *Euler characteristic.* Throughout the paper we use the following notation for groups:

- $\mathbb{Z}_n$: cyclic group of order n.
- $D_{p,q,r} = \mathbb{Z}_p \ltimes \mathbb{Z}_q = \langle x, y \mid x^p = y^q = 1,\ xyx^{-1} = y^r \rangle$: split metacyclic group of order pq. The group $D_{2,n,-1}$ is the dihedral group of order $2n$ and it will be denoted by D_n.
- $S_n,\ A_n$: symmetric, alternating group on n symbols. We write the composition of permutations from the right to the left; for instance, $(13)(12) = (123)$.
- $\mathrm{SL}_n(\mathbb{F}_q),\ \mathrm{PSL}_n(\mathbb{F}_q)$: special linear and projective special linear group of $n \times n$ matrices over a field with q elements.
- Whenever we give a presentation of a semi-direct product $H \ltimes N$, the first generators represent H and the last generators represent N. The action of H on N is specified by conjugation relations.
- The order of a finite group G is denoted by $|G|$. The index of a subgroup H in G is denoted by $[G : H]$.
- If $X = \{x_1, \dots, x_n\} \subset G$, the subgroup generated by X is denoted by $\langle x_1, \dots, x_n \rangle$. The derived subgroup of G is denoted by $[G, G]$.
- If $x \in G$, the order of x is denoted by $|x|$, its centralizer in G by $C_G(x)$ and the conjugacy class of x by $\mathrm{Cl}(x)$. If $x, y \in G$, their commutator is defined as $[x, y] = xyx^{-1}y^{-1}$.
- If $x \in G$ we denote by Int_x the inner automorphism of G defined as $\mathrm{Int}_x(g) = xgx^{-1}$.
- `IdSmallGroup`(G) indicates the label of the group G in the GAP4 database of small groups. For instance we have `IdSmallGroup`$(D_4) = G(8, 3)$ and this means that D_4 is the third in the list of groups of order 8.

1. Group-theoretic preliminaries

In this section we fix the algebraic set-up and we present some preliminary results of combinatorial type.

Definition 1.1. *Let G be a finite group and let*

$$\mathfrak{g}' \geq 0, \quad m_r \geq m_{r-1} \geq \ldots \geq m_1 \geq 2$$

be integers. A generating vector *for G of type $(\mathfrak{g}' \mid m_1, \dots, m_r)$ is a $(2\mathfrak{g}' + r)$-ple of elements*

$$\mathcal{V} = \{g_1, \dots, g_r;\ h_1, \dots, h_{2\mathfrak{g}'}\}$$

such that the following conditions are satisfied:

- *the set $\mathcal{V}$ generates G;*
- *$|g_i| = m_i$;*
- *$g_1 g_2 \cdots g_r \Pi_{i=1}^{\mathfrak{g}'} [h_i, h_{i+\mathfrak{g}'}] = 1$.*

*If such a $\mathcal{V}$ exists, then G is said to be $(\mathfrak{g}' \mid m_1, \dots, m_r)-$*generated.

For convenience we make abbreviations such as $(4 \mid 2^3, 3^2)$ for $(4 \mid 2, 2, 2, 3, 3)$ when we write down the type of the generating vector $\mathcal{V}$.
The following result is essentially a reformulation of Riemann's existence theorem.

Proposition 1.2. *A finite group G acts as a group of automorphisms of some compact Riemann surface X of genus $\mathfrak{g}$ if and only if there exist integers $\mathfrak{g}' \geq 0$ and $m_r \geq m_{r-1} \geq \ldots \geq m_1 \geq 2$ such that G is $(\mathfrak{g}' \mid m_1, \ldots, m_r)-$generated, with generating vector $\mathcal{V} = \{g_1, \ldots, g_r;\ h_1, \ldots, h_{2\mathfrak{g}'}\}$, and the following Riemann-Hurwitz relation holds:*

$$2\mathfrak{g} - 2 = |G| \left(2\mathfrak{g}' - 2 + \sum_{i=1}^{r} \left(1 - \frac{1}{m_i} \right) \right). \tag{1}$$

If this is the case, $\mathfrak{g}'$ is the genus of the quotient Riemann surface $Y := X/G$ and the $G-$cover $X \longrightarrow Y$ is branched in r points $P_1, \ldots, P_r$ with branching numbers $m_1, \ldots, m_r$, respectively. In addition, the subgroups $\langle g_i \rangle$ and their conjugates provide all the nontrivial stabilizers of the action of G on X.

We refer the reader to [Br90, Section 2], [Bre00, Chapter 3], [H71] and [Pol07, Section 1] for more details.
The rest of the section deals with the non existence of certain generating vectors for some of the groups listed in the Appendix. These results will be first applied in Section 4.

Lemma 1.3. *Let $\alpha \geq 2$ be a positive integer. Suppose that G contains a normal subgroup N such that*

(i) every element of order α in $[G, G]$ lies in N;
(ii) N is a proper subgroup of $[G, G]$.

Then G is not $(1 \mid \alpha)$-generated.

Proof. Let $h,\ k \in G$ with $|[h,\ k]| = \alpha$ and let $H = \langle h,\ k \rangle = \langle h, k, [h, k] \rangle$. We will show that $[H,\ H] \subset N \neq [G,\ G]$, hence $H \neq G$. By induction on m using (i) and the relation

$$[ab,\ c] = a[b,\ c]\, a^{-1}[a,\ c] \tag{2}$$

we see that $[h^m, k] \in N$ for every $m \geq 0$. Then by induction on n, using relation

$$[a,\ bc] = [a,\ b]\, b\, [a,\ c]\, b^{-1} \tag{3}$$

we have $[h^m, k^n] \in N$ for every $n \geq 0$. By (2) we have $[h^m k^n, k^t] \in N$ for every $m,\ n,\ t \geq 0$ and by (3) we have $[h^m, h^s k^t] = [h^s k^t, h^m]^{-1} \in N$ and $[h^m k^n, h^s k^t] \in N$ for every $m,\ n,\ s,\ t \geq 0$. Applying induction on r and (2) we see that $[h^{m_1} k^{n_1} \cdots h^{m_r} k^{n_r}, h^s k^t] \in N$ so by induction on v and (3) we have $[h^{m_1} k^{n_1} \cdots h^{m_r} k^{n_r}, h^{s_1} k^{t_1} \cdots h^{s_v} k^{t_v}] \in N$. $\square$

Lemma 1.4. *Referring to Table 2 in the Appendix, the groups in cases $(4m)$ and $(4ac)$ are not $(1 \mid 3)$-generated.*

Proof. We deal with the two cases separately.

• Case $(4m)$. $G = \mathbb{Z}_2 \ltimes (\mathbb{Z}_3)^2 = G(18, 4)$.
Let P be the 3-Sylow subgroup of G and let $h,\ k \in G$ be such that $|[h,\ k]| = 3$. If they generate G at least one of them, say h, would lie in the coset xP. Up to replacing k by hk we may assume that $k \in P$. Then we would have $|h| = 2$, $|k| = 3$ and $hkh^{-1} = k^{-1}$, that is, $\langle h,\ k \rangle \cong S_3 \neq G$.

• Case $(4ac)$. $G = D_4 \ltimes (\mathbb{Z}_3)^2 = G(72, 40)$.
We have $[G,\ G] = \langle y^2,\ w,\ z \rangle$ so we may apply Lemma 1.3 with $\alpha = 3$ and $N = \langle w,\ z \rangle$. $\square$

Lemma 1.5. *Referring to Table 3 in the Appendix, the groups in cases $(5j)$, $(5k)$, $(5m)$, $(5n)$, $(5o)$, $(5s)$, $(5t)$, $(5u)$, $(5x)$, $(5z)$, $(5aa)$, $(5ab)$, $(5ah)$, $(5ai)$, $(5aj)$, $(5al)$, $(5am)$, $(5ap)$, $(5aq)$, $(5ar)$, $(5as)$, $(5at)$ are not $(1 \mid 2)$-generated.*

Proof. We will prove the statement through a case-by-case analysis.

• Case $(5j)$. $G = D_8$.
We have $[G,\ G] = \langle y^2 \rangle$ so we may apply Lemma 1.3 with $\alpha = 2$ and $N = \langle y^4 \rangle$.

• Cases $(5k)$ and $(5n)$. $G = \mathbb{Z}_2 \times D_4$.
The group G is not generated by 2 elements, otherwise this would also be true for $G/\langle y^2 \rangle \cong (\mathbb{Z}_2)^3$.

• Case $(5m)$. $G = D_{2,8,3}$.
We have $[G,\, G] = \langle y^2 \rangle$ so Lemma 1.3 can be applied with $N = \langle y^4 \rangle$.

• Case $(5o)$. $G = \mathbb{Z}_2 \ltimes (\mathbb{Z}_2 \times \mathbb{Z}_4) = G(16, 13)$.
The group G cannot be generated by 2 elements, otherwise this would be true for $G/\langle z^2 \rangle \cong (\mathbb{Z}_2)^3$.

• Case $(5s)$. $G = \mathbb{Z}_2 \ltimes ((\mathbb{Z}_2)^2 \times \mathbb{Z}_3) = G(24, 8)$.
Here, $[G,\, G] = \langle y,\, w \rangle$ and we may use Lemma 1.3 taking $N = \langle y \rangle$.

• Case $(5t)$. $G = (\mathbb{Z}_2)^2 \times S_3$.
The order of $[G,\, G]$ is not even.

• Case $(5u)$. $G = S_4$.
The derived subgroup of G is A_4 and its subgroup $N = \langle (12)(34),\ (13)(24) \rangle$ is normal in G so Lemma 1.3 applies.

• Case $(5x)$. $G = D_{4,6,-1}$.
The order of $[G,\, G]$ is not even.

• Case $(5z)$. $G = \mathbb{Z}_2 \ltimes (\mathbb{Z}_2)^4 = G(32, 27)$.
We have $[G,\, G] = \langle w,\, t \rangle$. The group G cannot be generated by 2 elements, otherwise this would be true for $G/[G,\, G] \cong (\mathbb{Z}_2)^3$.

• Case $(5aa)$. $G = \mathbb{Z}_2 \ltimes (\mathbb{Z}_4 \times (\mathbb{Z}_2)^2) = G(32, 28)$.
We have $[G,\, G] = \langle w,\, y^2 \rangle$. Since $G/[G,\, G] \cong (\mathbb{Z}_2)^3$, the group G cannot be generated by 2 elements.

• Case $(5ab)$. $G = \mathbb{Z}_2 \ltimes (D_4 \times \mathbb{Z}_2) = G(32, 43)$.
We have $[G,\, G] = \langle z \rangle$ and $G/[G,\, G] \cong (\mathbb{Z}_2)^3$ so G cannot have only 2 generators.

• Case $(5ah)$. $G = \mathbb{Z}_2 \times S_4$.
The derived subgroup of G is A_4, so this case is excluded using Lemma 1.3 with $N = \langle (12)(34),\ (13)(24) \rangle$.

• Case $(5ai)$. $G = \mathbb{Z}_2 \ltimes (\mathbb{Z}_{12} \times \mathbb{Z}_2) = G(48, 14)$.
We may use Lemma 1.3 because $[G,\, G] = \langle y^4 z \rangle \cong \mathbb{Z}_6$ and $N = \langle z \rangle$ is normal in G.

• Case $(5aj)$. $G = \mathbb{Z}_4 \ltimes A_4 = G(48, 30)$.
We have $[G,\, G] = \langle (12)(34),\ (123) \rangle \cong A_4$ so we apply Lemma 1.3 with $N = \langle (12)(34),\ (14)(23) \rangle$.

• Case $(5al)$. $G = A_5$.
By using [Is94, Problem 3.10 (a) p.45] we see that if $|[h_1, h_2]| = 2$ then h_1 and h_2 have order either 2 or 3. Then straightforward computations show that $\langle h_1,\, h_2 \rangle$ is necessarily a proper subgroup of A_5.

• Case $(5am)$. $G = \mathbb{Z}_4 \ltimes (D_4 \times \mathbb{Z}_2)$.
We have $[G,\, G] = \langle z,\, w \rangle \cong \mathbb{Z}_4 \times \mathbb{Z}_2$. Thus we can use Lemma 1.3 with $N = \langle w,\, z^2 \rangle$.

• Case ($5ap$). $G = G(96, 3)$
In this case $[G, G] = \langle y, z, u, v, w\rangle = \langle y, z\rangle$. The subgroup $N = \langle u, v, w\rangle \cong (\mathbb{Z}_2)^3$ is central in $[G, G]$; this implies that the elements of order 2 in $[G, G]$ may not lie in the cosets yN, zN or yzN. Therefore we may apply once more Lemma 1.3 because N is normal in G.

• Case ($5aq$). $G = S_3 \ltimes (\mathbb{Z}_2)^4 = G(96, 195)$.
The derived subgroup of G is generated by y, w, u, v and its unique 2-Sylow subgroup is $N = \langle w, u, v\rangle$, which is therefore normal in G. Thus we may apply Lemma 1.3.

• Case ($5ar$). $G = A_5 \times \mathbb{Z}_2$.
This case may not occur, otherwise case ($5al$) would also occur.

• Case ($5as$). $G = D_5 \ltimes (\mathbb{Z}_2)^4 = G(160, 234)$.
In this case $[G, G] = \langle y, z, u, v, w\rangle$ and the 2-Sylow subgroup $N = \langle z, w, v, u\rangle$ of $[G, G]$ satisfies the hypotheses of Lemma 1.3.

• Case ($5at$). $G = G(192, 181)$.
In this case $[G, G] = \langle y, z, w, v, u, t\rangle$. There is only one 2-Sylow subgroup in $[G, G]$, namely $N = \langle z, w, v, u, t\rangle$, which has the necessary properties allowing application of Lemma 1.3. $\square$

2. Basic on surfaces isogenous to a product

In this section we collect for the reader's convenience some foundational results on surfaces isogenous to a product of curves, referring to [Ca00] for further details.

Definition 2.1. *A complex surface S of general type is said to be* isogenous to a product *if there exist two smooth curves C, F and a finite group G acting freely on $C \times F$ so that $S = (C \times F)/G$.*

There are two cases: the *unmixed* one, where G acts diagonally, and the *mixed* one, where there exist elements of G exchanging the two factors (and then C, F are isomorphic).

In both cases, since the action of G on $C \times F$ is free, we have

$$\begin{aligned} K_S^2 &= \frac{K_{C\times F}^2}{|G|} = \frac{8(g(C)-1)(g(F)-1)}{|G|} \\ \chi(\mathcal{O}_S) &= \frac{\chi(\mathcal{O}_{C\times F})}{|G|} = \frac{(g(C)-1)(g(F)-1)}{|G|}, \end{aligned} \tag{4}$$

hence $K_S^2 = 8\chi(\mathcal{O}_S)$.
Let C, F be curves of genus ≥ 2. Then the inclusion $\mathrm{Aut}(C \times F) \supset \mathrm{Aut}(C) \times \mathrm{Aut}(F)$ is an equality if C and F are not isomorphic, whereas $\mathrm{Aut}(C \times C) = \mathbb{Z}_2 \ltimes (\mathrm{Aut}(C) \times \mathrm{Aut}(C))$, the $\mathbb{Z}_2$ being generated by the involution exchanging the two coordinates. If $S = (C \times F)/G$ is a surface isogenous to a product, we will always consider its unique *minimal realization*. This means that

- in the unmixed case, we have $G \subset \mathrm{Aut}(C)$ and $G \subset \mathrm{Aut}(F)$ (i.e. G acts faithfully on both C and F);
- in the mixed case, where $C \cong F$, we have $G^\circ \subset \mathrm{Aut}(C)$, for $G^\circ := G \cap (\mathrm{Aut}(C) \times \mathrm{Aut}(C))$.

(See [Ca00, Corollary 3.9 and Remark 3.10]).

Let us consider first the unmixed case. The group G acts on both C and F, and diagonally on $C \times F$, i.e. $g(x, y) = (gx, gy)$. Hence we obtain

Proposition 2.2. *Let G be a finite group, acting faithfully on two smooth, projective curves C, F. Let Σ_C and Σ_F be the set of elements of G having some fixed points on C and F, respectively. Assume moreover that condition*

(u) $\Sigma_C \cap \Sigma_F = \{1_G\}$

is satisfied. Then the diagonal action of G on $C \times F$ is free, hence $S = (C \times F)/G$ is a surface isogenous to an unmixed product. Conversely, every surface isogenous to an unmixed product arises in this way.

Now let us consider the mixed case. The following is [Ca00, Proposition 3.16].

Proposition 2.3. *Assume that G° is a finite group which satisfies the following properties:*

- *G° acts faithfully on a smooth curve C of genus $g(C) \geq 2$;*
- *there exists a nonsplit extension*

$$1 \longrightarrow G^\circ \longrightarrow G \longrightarrow \mathbb{Z}_2 \longrightarrow 1. \tag{5}$$

Let us fix a lift $\tau' \in G$ of the generator of $\mathbb{Z}_2$. Conjugation by τ' defines an element $[\varphi]$ of order ≤ 2 in $\mathrm{Out}(G^\circ) := \mathrm{Aut}(G^\circ)/\mathrm{Inn}(G^\circ)$.

Let us choose a representative $\varphi \in \mathrm{Aut}(G^\circ)$ and let $\tau \in G^\circ$ be such that $\varphi^2 = \mathrm{Int}_\tau$. Denote by Σ_C the set of elements in G° fixing some point on C and assume that both conditions $(m1)$ and $(m2)$ below are satisfied:

- $(m1)$ $\Sigma_C \cap \varphi(\Sigma_C) = \{1_{G^\circ}\}$
- $(m2)$ *for all $\gamma \in G^\circ$ we have $\varphi(\gamma)\tau\gamma \notin \Sigma_C$.*

Then there exists a free, mixed action of G on $C \times C$, hence $S = (C \times C)/G$ is a surface of general type isogenous to a mixed product. More precisely, we have

$$\begin{aligned} g(x, y) &= (gx, (\varphi g)y) \quad \text{for } g \in G^\circ \\ \tau'(x, y) &= (y, \tau x). \end{aligned}$$

Conversely, every surface of general type isogenous to a mixed product arises in this way.

Remark 2.4. If $\Sigma_C \neq \{1_{G^\circ}\}$ then the element $[\varphi] \in \mathrm{Out}(G^\circ)$ has order exactly 2. In fact, if $\varphi = \mathrm{Int}_g$ for some $g \in G^\circ$, then for any $h \in \Sigma_C$ we would have $\varphi h \in \Sigma_C \cap \varphi(\Sigma_C)$, contradicting $(m1)$. In particular this implies that the group G cannot be abelian.

Remark 2.5. The exact sequence (5) is non split if and only if the number of elements of order 2 in G equals the number of elements of order 2 in G°.

Remark 2.6. Condition (u) in Proposition 2.2 and conditions (m_1), (m_2) in Proposition 2.3 guarantee that G acts freely on $C \times F$.

3. The case $p_g = q = 1$. Building data

Our goal is to classify all surfaces with $p_g = q = 1$ isogenous to a product. In this section we apply the previous results in order to translate the classification problem from geometry to algebra. This leads to Proposition 3.3 in the unmixed case and to Proposition 3.8 in the mixed one.

Lemma 3.1. *Let $S = (C \times F)/G$ be a surface isogenous to a product with $p_g = q = 1$. Then*

- (i) $K_S^2 = 8$.
- (ii) $|G| = (g(C) - 1)(g(F) - 1)$.
- (iii) *S is a minimal surface of general type.*

Proof. Claims (i) and (ii) follow from (4). Now let us consider (iii). Since $C \times F$ is minimal and the cover $C \times F \longrightarrow S$ is étale, S is minimal as well. Moreover (ii) implies either $g(C) = g(F) = 0$ or $g(C) \geq 2$, $g(F) \geq 2$. The first case is impossible otherwise $S = \mathbb{P}^1 \times \mathbb{P}^1$ and $p_g = q = 0$; thus the second case occurs and S is of general type ([Ca00, Remark 3.2]). $\square$

3.1. **Unmixed case.** We will make use of [Pol07, Propositions 2.2, 2.3, 3.1], that we recall in Propositions 3.2, 3.3, 3.4 for the reader's convenience. Notice that Proposition 3.3 is a direct consequence of Proposition 2.2, Proposition 3.2 and Lemma 3.1.

Proposition 3.2. *Let $S = (C \times F)/G$ be a surface with $p_g = q = 1$, isogenous to an unmixed product. Then $g(C) \geq 3$ and $g(F) \geq 3$. Moreover, up to exchanging F and C we may assume $F/G \cong \mathbb{P}^1$ and $C/G \cong E$, where E is an elliptic curve isomorphic to the Albanese variety of S. In particular, $g_{\mathrm{alb}} = g(F)$.*

Proposition 3.3. *Let G be a finite group which is both $(0 \mid m_1, \ldots, m_r)$ and $(1 \mid n_1, \ldots, n_s)$-generated, with generating vectors $\mathcal{V} = \{g_1, \ldots, g_r\}$ and $\mathcal{W} = \{\ell_1, \ldots, \ell_s;\ h_1, h_2\}$, respectively. Let $g(F)$, $g(C)$ be the positive integers defined by the Riemann-Hurwitz relations*

$$\begin{aligned} 2g(F) - 2 &= |G|\Big(-2 + \sum_{i=1}^{r}\Big(1 - \frac{1}{m_i}\Big)\Big) \\ 2g(C) - 2 &= |G|\sum_{j=1}^{s}\Big(1 - \frac{1}{n_j}\Big). \end{aligned} \tag{6}$$

Assume moreover that

- *$g(C) \geq 3$, $g(F) \geq 3$;*
- *$|G| = (g(C) - 1)(g(F) - 1)$;*
- *condition*

$$\left(\bigcup_{\sigma \in G}\bigcup_{i=1}^{r}\langle \sigma g_i \sigma^{-1}\rangle\right) \cap \left(\bigcup_{\sigma \in G}\bigcup_{j=1}^{s}\langle \sigma \ell_j \sigma^{-1}\rangle\right) = \{1_G\} \tag{U}$$

  is satisfied.

Then there is a free, diagonal action of G on $C \times F$ such that the quotient $S = (C \times F)/G$ is a minimal surface of general type with $p_g = q = 1$, $K_S^2 = 8$. Conversely, any surface with $p_g = q = 1$, isogenous to an unmixed product, arises in this way.

Proposition 3.4. *In the unmixed case, the only possibilities for $g(F)$ and $\mathbf{n} = (n_1, \ldots, n_s)$ are*

(*a*) $g(F) = 3$, $\mathbf{n} = (2^2)$
(*b*) $g(F) = 4$, $\mathbf{n} = (3)$
(*c*) $g(F) = 5$, $\mathbf{n} = (2)$.

The following easy lemma will be useful in the sequel.

Lemma 3.5. *Let $S = (C \times F)/G$ be a surface of general type with $p_g = q = 1$, isogenous to an unmixed product and let $\mathbf{m} = (m_1, \ldots, m_r)$ be as in Proposition 3.3. Then every m_i divides $\frac{|G|}{(g(F)-1)}$.*

Proof. Since $\langle g_i \rangle$ is a stabilizer for the G-action on F and since G acts freely on $(C \times F)$, the subgroup $\langle g_i \rangle \cong \mathbb{Z}_{m_i}$ acts freely on C. By Riemann-Hurwitz formula applied to the cover $C \longrightarrow C/\langle g_i \rangle$ we have $g(C) - 1 = m_i(g(C/\langle g_i \rangle) - 1)$. Thus m_i divides $g(C) - 1 = \frac{|G|}{(g(F)-1)}$. $\square$

3.2. **Mixed case.**

Proposition 3.6. *Let $S = (C \times C)/G$ be a surface with $p_g = q = 1$ isogenous to a mixed product. Then $E := C/G^\circ$ is an elliptic curve isomorphic to the Albanese variety of S.*

Proof. We have (see [Ca00, Proposition 3.15])

$$\begin{aligned} \mathbb{C} = H^0(\Omega_S^1) &= (H^0(\Omega_C^1) \oplus H^0(\Omega_C^1))^G = (H^0(\Omega_C^1)^{G^\circ} \oplus H^0(\Omega_C^1)^{G^\circ})^{G/G^\circ} \\ &= (H^0(\Omega_E^1) \oplus H^0(\Omega_E^1))^{G/G^\circ}. \end{aligned}$$

Since S is of mixed type, the quotient $\mathbb{Z}_2 = G/G^\circ$ exchanges the two last summands, whence $h^0(\Omega^1_E) = 1$. Thus E is an elliptic curve and there is a commutative diagram

$$\begin{tikzcd} C \times C \arrow[r, "\rho"] \arrow[d, "\pi"] & E \times E \arrow[d, "\varepsilon"] \\ S \arrow[r, "\hat{\rho}"] \arrow[dr, "\alpha"'] & E^{(2)} \arrow[d, "\hat{\alpha}"] \\ & E \end{tikzcd} \tag{7}$$

showing that the Albanese morphism α of S factors through the Abel-Jacobi map $\hat{\alpha}$ of the double symmetric product $E^{(2)}$ of E. □

Remark 3.7. The surface S is not covered by elliptic curves because it is of general type (Lemma 3.1), so Proposition 3.6 yields $\Sigma_C \neq \{1_{G^0}\}$. Therefore G is never abelian by Remark 2.4.

Notice that by Lemma 3.1 we have $|G| = (g(C) - 1)^2$. Therefore Proposition 2.3 becomes

Proposition 3.8. *Assume that G° is a finite group verifying the following properties:*

- *G° is $(1 \mid n_1, \ldots, n_s)$-generated, with generating vector $\mathcal{V} = \{\ell_1, \ldots, \ell_s;\ h_1, h_2\}$;*
- *there is a nonsplit extension*

$$1 \longrightarrow G^\circ \longrightarrow G \longrightarrow \mathbb{Z}_2 \longrightarrow 1 \tag{8}$$

  which defines an element $[\varphi]$ of order 2 in $\mathrm{Out}(G^\circ)$.

Let $g(C) \in \mathbb{Z}_{\geq 0}$ be defined by the Riemann-Hurwitz relation $2g(C) - 2 = |G^\circ| \sum_{j=1}^{s} \left(1 - \frac{1}{n_j}\right)$. Assume, in addition, that $|G| = (g(C) - 1)^2$ and that both conditions $(M1)$ and $(M2)$ below are satisfied:

$(M1)$ *for all $g \in G \setminus G^\circ$ we have*

$$\{\ell_1, \ldots, \ell_s\} \cap \{g\ell_1 g^{-1}, \ldots, g\ell_s g^{-1}\} = \emptyset;$$

$(M2)$ *for all $g \in G \setminus G^\circ$ we have*

$$g^2 \notin \bigcup_{j=1}^{s} \bigcup_{\sigma \in G^\circ} \langle \sigma \ell_i \sigma^{-1} \rangle.$$

Then there is a free, mixed action of G on $C \times C$ such that the quotient $S = (C \times C)/G$ is a minimal surface of general type with $p_g = q = 1$, $K_S^2 = 8$.
Conversely, every surface S with $p_g = q = 1$, isogenous to a mixed product, arises in this way.

In the unmixed case we know that $g_{\mathrm{alb}} = g(F)$ (Proposition 3.2). Here we have

Proposition 3.9. *Let $S = (C \times C)/G$ be a surface with $p_g = q = 1$, isogenous to a mixed product. Then $g_{\mathrm{alb}} = g(C)$.*

Proof. Let us look at diagram (7). The Abel-Jacobi map $\hat{\alpha}$ gives to $E^{(2)}$ the structure of a $\mathbb{P}^1$-bundle over E ([CaCi93]); let $\mathfrak{f}$ be the generic fibre of this bundle and $F^* := \rho^* \varepsilon^*(\mathfrak{f})$. If F_{alb} is the generic Albanese fibre of S we have $F_{\mathrm{alb}} = \pi(F^*)$. Let $\mathbf{n} = (n_1, \ldots, n_s)$ be such that G° is $(1 \,|\, n_1, \ldots n_s)$-generated and $2g(C) - 2 = |G^\circ| \sum_{j=1}^{s} \left(1 - \frac{1}{n_j}\right)$. The $(G^\circ \times G^\circ)$-cover ρ is branched exactly along the union of s "horizontal" copies of E and s "vertical" copies of E; moreover for each i there are one horizontal copy and one vertical copy whose branching number is n_i. Since $\varepsilon^*(\mathfrak{f})$ is an elliptic curve that intersects all these copies of E transversally in one point, by Riemann-Hurwitz formula applied to $F^* \longrightarrow \varepsilon^*(\mathfrak{f})$ we obtain

$$2g(F^*) - 2 = |G^\circ|^2 \cdot \sum_{j=1}^{s} 2 \left(1 - \frac{1}{n_j}\right).$$

On the other hand the G-cover π is étale, so we have

$$2g(F_{\text{alb}}) - 2 = \frac{1}{|G|}(2g(F^*) - 2) = |G^\circ| \sum_{j=1}^{s} \left(1 - \frac{1}{n_j}\right)$$
$$= 2g(C) - 2,$$

whence $g_{\text{alb}} = g(C)$. $\square$

In the mixed case the analogous of Proposition 3.4 is

Proposition 3.10. *Let $S = (C \times C)/G$ be a surface with $p_g = q = 1$, isogenous to a mixed product. Then there are at most the following possibilities:*

- $g(C) = 5, \quad \mathbf{n} = (2^2), \quad |G| = 16$;
- $g(C) = 7, \quad \mathbf{n} = (3), \quad |G| = 36$;
- $g(C) = 9, \quad \mathbf{n} = (2), \quad |G| = 64$.

Proof. By Proposition 3.8 we have $2g(C) - 2 = |G^\circ| \sum_{j=1}^{s} \left(1 - \frac{1}{n_j}\right)$ and $|G^\circ| = \frac{1}{2}(g(C) - 1)^2$, so $g(C)$ must be odd and we obtain

$$4 = (g(C) - 1) \sum_{j=1}^{s} \left(1 - \frac{1}{n_j}\right).$$

Therefore $4 \geq \frac{1}{2}(g(C) - 1)$, which implies $g(C) \leq 9$. Thus the only possibilities are $g(C) = 3, 5, 7, 9$.
Possibility $g(C) = 3$ is ruled out by Remark 3.7.
If $g(C) = 5$ then $\sum_{j=1}^{s} \left(1 - \frac{1}{n_j}\right) = 1$, so $\mathbf{n} = (2^2)$ and $|G| = 16$.
If $g(C) = 7$ then $\sum_{j=1}^{s} \left(1 - \frac{1}{n_j}\right) = \frac{2}{3}$, so $\mathbf{n} = (3)$ and $|G| = 36$.
If $g(C) = 9$ then $\sum_{j=1}^{s} \left(1 - \frac{1}{n_j}\right) = \frac{1}{2}$, so $\mathbf{n} = (2)$ and $|G| = 64$. $\square$

We will see in Section 3.10 that only the case $g(C) = 5$ occurs.

4. The classification in the unmixed case

The classification of surfaces of general type with $p_g = q = 1$ isogenous to an unmixed product is carried out in [Pol07] when the group G is abelian. Therefore in this section we assume that G is nonabelian. By Proposition 3.4 we have $g(F) = 3$, 4 or 5. Let us analyze the three cases separately.

4.1. The case $g(F) = 3$.

Proposition 4.1. *If $g(F) = 3$ we have precisely the following possibilities.*

G	`IdSmall Group`(G)	$\mathbf{m}$
D_4	$G(8,3)$	$(2^2, 4^2)$
D_6	$G(12,4)$	$(2^3, 6)$
$\mathbb{Z}_2 \times D_4$	$G(16,11)$	$(2^3, 4)$
$D_{2,12,5}$	$G(24,5)$	$(2, 4, 12)$
$\mathbb{Z}_2 \times A_4$	$G(24,13)$	$(2, 6^2)$
S_4	$G(24,12)$	$(3, 4^2)$
$\mathbb{Z}_2 \ltimes (\mathbb{Z}_2 \times \mathbb{Z}_8)$	$G(32,9)$	$(2, 4, 8)$
$\mathbb{Z}_2 \times S_4$	$G(48,48)$	$(2, 4, 6)$

Proof. The group G acts faithfully on the genus 3 curve F so that $F/G \cong \mathbb{P}^1$. In addition, since $\mathbf{n} = (2^2)$, it follows that G is $(1 \mid 2^2)$-generated and by the second relation in (6) we have $|G| = 2(g(C) - 1)$. Now let us look at Table 1 of the Appendix.
Case $(3k)$ cannot occur because $|G|$ is odd. Cases $(3a)$ and $(3d)$ are excluded by Lemma 3.5. If G is as in case $(3f)$, $(3n)$, $(3u)$ or $(3w)$ all elements of order 2 lie in the same conjugacy class, so condition (U) in Proposition 3.3 cannot be satisfied.

• Case $(3c)$. $G = D_4$, $\mathbf{m} = (2^5)$.
Let $\mathcal{V} = \{g_1, g_2, g_3, g_4, g_5\}$ be any generating vector of type $(0 \mid 2^5)$ for G and let $Y = \langle y \rangle$. At least one generator, say g_1, has to lie in the coset xY. Hence, g_1 will be conjugate either to x or to xy. Since $g_1g_2g_3g_4g_5 = 1$, the number of generators lying in xY has to be even, so one of the generators has to be y^2, which is central. If all g_i's would be either conjugate to g_1 or equal to y^2, they would generate an abelian subgroup. Therefore at least one of them has to be conjugate to $g_1 y$, exhausting all conjugacy classes of involutions in G. Thus, condition (U) cannot be satisfied and this case cannot occur.

• Case $(3g)$. $G = D_{2,8,5}$, $\mathbf{m} = (2, 8^2)$.
Let $\mathcal{V} = \{g_1, g_2, g_3\}$ be any generating vector of type $(0 \mid 2, 8^2)$ for G and let $Y = \langle y \rangle$. The elements of order 2 in G are x and xy^4, which are conjugate, and the central element y^4. If g_1 were equal to y^4 then g_1, g_2 and $g_3 = g_2^{-1}g_1^{-1}$ would generate an abelian group. Therefore g_1 must be conjugate to x. On the other hand, since $[G : Y] = 2$, any choice of g_2 would give $g_2^2 \in Y$, so $g_2^4 = y^4$. Then condition (U) cannot be satisfied and this case cannot occur.

• Case $(3h)$. $G = D_{4,4,-1}$, $\mathbf{m} = (4^3)$.
Let $\mathcal{V} = \{g_1, g_2, g_3\}$ be any generating vector of type $(0 \mid 4^3)$ for G and let $Y = \langle y \rangle$. The elements of order 2 are x^2, y^2 and x^2y^2, which are all central. Since the cosets g_1Y, g_2Y and g_3Y generate $G/Y \cong \mathbb{Z}_4$, at least one generator, say g_1, has to lie in xY or in x^3Y so $g_1^2 \in x^2Y$. Let now $\mathcal{W} = \{\ell_1, \ell_2;\ h_1, h_2\}$ be a generating vector of type $(1 \mid 2^2)$ for G. Since $[G, G] = \langle y^2 \rangle$ we have the possibilities $\ell_1\ell_2 = 1$ or $\ell_1\ell_2 = y^2$. The former would imply that $\langle h_1, h_2, \ell_1, \ell_2 \rangle = \langle h_1, h_2, \ell_1 \rangle$ is abelian because ℓ_1 is central. The latter would imply either $\ell_1 = g_1^2$ or $\ell_2 = g_1^2$, so condition (U) cannot hold.

• Case $(3j)$. $G = \mathbb{Z}_2 \ltimes (\mathbb{Z}_2 \times \mathbb{Z}_4) = G(16, 13)$, $\mathbf{m} = (2^3, 4)$.
Let $\mathcal{V} = \{g_1, g_2, g_3, g_4\}$ be a generating vector of type $(0 \mid 2^3, 4)$ for G. We claim that the conjugacy classes of g_i for $i = 1, 2, 3$ and g_4^2 exhaust all conjugacy classes of elements of order 2. These are: $\mathrm{Cl}(x) = \{x, xz^2\}$; $\mathrm{Cl}(z^2) = \{z^2\}$; $\mathrm{Cl}(y) = \{y, yz^2\}$ and $\mathrm{Cl}(xyz) = \{xyz, xyz^2\}$. Since $[G, G] = \langle z^2 \rangle$ and $G/[G, G] \cong (\mathbb{Z}_2)^3$ we have $g_4^2 = z^2$. If some of the g_i, say g_1, were conjugate to g_2 or g_3 or g_4^2, by relation $g_1g_2g_3g_4 = 1$ we would have $G = \langle g_1, g_2, g_3 \rangle \subseteq \langle z^2, g_2, g_3 \rangle$. Then the quotient $G/[G, G]$ could be generated by 2 elements, a contradiction. Thus, the claim is proved and condition (U) cannot be verified.

• Case $(3p)$. $G = S_4$, $\mathbf{m} = (2^3, 3)$.
Let $\mathcal{V} = \{g_1, g_2, g_3, g_4\}$ be any generating vector of type $(0 \mid 2^3, 3)$ for G. At least one g_i has to be a transposition, otherwise all generators would lie in A_4. Since $g_1g_2g_3 = g_4^{-1}$ is an even permutation, exactly two generators will be conjugate to (12). Therefore one generator will be conjugate to $(12)(34)$. Hence, condition (U) cannot be satisfied.

• Case $(3r)$. $G = \mathbb{Z}_2 \ltimes D_{2,8,5} = G(32, 11)$, $\mathbf{m} = (2, 4, 8)$.
Let $\mathcal{V} = \{g_1, g_2, g_3\}$ be any generating vector of type $(0 \mid 2, 4, 8)$ for G. The elements of order 2 in G are parted in 3 conjugacy classes: the class of the central element z^4, the class of y and yz^4, and the class of x and xz^4. Since $[G, G] = \langle yz^2 \rangle$ and $G/[G, G] \cong \mathbb{Z}_2 \times \mathbb{Z}_4$ we have $g_3^4 \in \langle yz^2 \rangle$ so $g_3^4 = z^4$. Since for $N = \langle y, z \rangle$ we have $[G : N] = 2$ the element g_2^2 lies in N. If $g_2^2 = z^4$ then we would have $g_1^2 = (g_3^{-1}g_2^{-1})^2 = 1$ so that $(g_2)^2g_3g_2g_3 = g_2$, that is, $g_2g_3^{-1}g_2^{-1} = z^4g_3 = g_3^5$. Then

G would be a quotient of $D_{4,8,3}$ with nontrivial kernel because $g_2^2 = z^4 = g_3^4$, a contradiction. Therefore $g_2^2 \in \mathrm{Cl}(y)$. The generator g_1 cannot be central and it cannot lie in $\mathrm{Cl}(y)$, otherwise g_2^2 would be central in $G = \langle g_1, g_2 \rangle$. Thus $g_1 \in \mathrm{Cl}(x)$, $g_2^2 \in \mathrm{Cl}(y)$ and $g_3^4 \in \mathrm{Cl}(z^4)$, contradicting condition (U).

• Case $(3t)$. $G = G(48, 33)$, $\quad \mathbf{m} = (2, 3, 12)$.
We have $G = Y \ltimes N$ where $Y = \langle y \rangle$ and $N = \langle x, w, z \rangle$. Therefore all elements of order 2 lie in N. They are t, which is central, and xz, xw, xzt, xwt, xwz, $xwzt$, which are all conjugate. If $\mathcal{V} = \{g_1, g_2, g_3\}$ is any generating vector of type $(0 \,|\, 2, 3, 12)$ for the nonabelian group G then g_1 has to lie in $\mathrm{Cl}(xz)$. On the other hand, since $[G, G] = \langle w, z \rangle$ and $G/[G, G] \cong \mathbb{Z}_6$, for any choice of g_3 we have $g_3^6 \in \langle w, z \rangle$ so $g_3^6 = t$. Thus condition (U) cannot be satisfied.

• Case $(3v)$. $G = S_4 \ltimes (\mathbb{Z}_4)^2 = G(96, 64)$, $\quad \mathbf{m} = (2, 3, 8)$.
The elements of order 2 lie either in $N = (\mathbb{Z}_4)^2$, and they are all conjugate to z^2, or in the coset xN, and they are all conjugate to x. The elements of order 3 lie in the coset yN or in the coset y^2N. Let $\mathcal{V} = \{g_1, g_2, g_3\}$ be any generating vector of type $(0 \,|\, 2, 3, 8)$ for G. If g_1 lied in the abelian subgroup N, we would have $\langle g_1, g_2 \rangle \cong \mathbb{Z}_3 \ltimes (\mathbb{Z}_2)^2 \neq G$. Therefore $g_1 \in \mathrm{Cl}(x)$. On the other hand, since $[G, G] = \langle y, w, z \rangle$ and $G/[G, G] \cong \mathbb{Z}_2$, for every choice of g_3 we have $g_3^2 \in \langle y, w, z \rangle$, so $g_3^4 \in \mathrm{Cl}(z^2)$. Thus condition (U) cannot be satisfied.

Now we show that all the remaining possibilities do occur. In each case, we exhibit a pair $\mathcal{V}$, $\mathcal{W}$ of generating vectors satisfying the assumptions of Proposition 3.3.

• Case $(3b)$. $G = D_4$, $\quad \mathbf{m} = (2^2, 4^2)$, $\quad g(C) = 5$.
Set

$$g_1 = x, \;\; g_2 = x, \;\; g_3 = y^{-1}, \;\; g_4 = y$$
$$\ell_1 = xy, \;\; \ell_2 = xy, \;\; h_1 = y^2, \;\; h_2 = x.$$

• Case $(3e)$. $G = D_6$, $\quad \mathbf{m} = (2^3, 6)$, $\quad g(C) = 7$.
Set

$$g_1 = x, \;\; g_2 = xy^2, \;\; g_3 = y^3, \;\; g_4 = y$$
$$\ell_1 = xy, \;\; \ell_2 = xy, \;\; h_1 = x, \;\; h_2 = y^3.$$

• Case $(3i)$. $G = \mathbb{Z}_2 \times D_4$, $\quad \mathbf{m} = (2^3, 4)$, $\quad g(C) = 9$.
Set

$$g_1 = z, \;\; g_2 = x, \;\; g_3 = zxy^{-1}, \;\; g_4 = y$$
$$\ell_1 = zx, \;\; \ell_2 = zxy^2, \;\; h_1 = x, \;\; h_2 = zy.$$

• Case $(3l)$. $G = D_{2,12,5}$, $\quad \mathbf{m} = (2, 4, 12)$, $\quad g(C) = 13$.
Set

$$g_1 = x, \;\; g_2 = xy^{-1}, \;\; g_3 = y$$
$$\ell_1 = xy^2, \;\; \ell_2 = xy^2, \;\; h_1 = y, \;\; h_2 = y.$$

• Case $(3m)$. $G = \mathbb{Z}_2 \times A_4$, $\quad \mathbf{m} = (2, 6^2)$, $\quad g(C) = 13$.
Set

$$g_1 = (12)(34), \;\; g_2 = z(123), \;\; g_3 = z(234)$$
$$\ell_1 = z(14)(23), \;\; \ell_2 = z(13)(24), \;\; h_1 = (123), \;\; h_2 = (124).$$

• Case $(3o)$. $G = S_4$, $\quad \mathbf{m} = (3, 4^2)$, $\quad g(C) = 13$.
Set

$$g_1 = (234), \;\; g_2 = (1234), \;\; g_3 = (1423)$$
$$\ell_1 = (12), \;\; \ell_2 = (13), \;\; h_1 = (12), \;\; h_2 = (1234).$$

• Case $(3q)$. $G = \mathbb{Z}_2 \ltimes (\mathbb{Z}_2 \times \mathbb{Z}_8) = G(32, 9)$, $\quad \mathbf{m} = (2, 4, 8)$, $\quad g(C) = 17$.
Set

$$g_1 = x, \;\; g_2 = xz, \;\; g_3 = z^{-1}$$
$$\ell_1 = xz^2, \;\; \ell_2 = xy, \;\; h_1 = x, \;\; h_2 = z.$$

• Case $(3s)$. $G = \mathbb{Z}_2 \times S_4$, $\quad \mathbf{m} = (2, 4, 6)$, $\quad g(C) = 25$.
Set

$$g_1 = z(12), \;\; g_2 = (1243), \;\; g_3 = z(234)$$
$$\ell_1 = (12), \;\; \ell_2 = (34), \;\; h_1 = z(123), \;\; h_2 = z(124).$$

□

4.2. **The case $g(F) = 4$.**

Proposition 4.2. *If $g(F) = 4$ we have precisely the following possibilities.*

G	IdSmall Group(G)	$\mathbf{m}$
S_3	$G(6,1)$	(2^6)
D_6	$G(12,4)$	(2^5)
$\mathbb{Z}_3 \times S_3$	$G(18,3)$	$(2^2, 3^2)$
$\mathbb{Z}_3 \times S_3$	$G(18,3)$	$(3, 6^2)$
S_4	$G(24,12)$	$(2^3, 4)$
$S_3 \times S_3$	$G(36,10)$	$(2, 6^2)$
$\mathbb{Z}_6 \times S_3$	$G(36,12)$	$(2, 6^2)$
$\mathbb{Z}_4 \ltimes (\mathbb{Z}_3)^2$	$G(36,9)$	$(3, 4^2)$
A_5	$G(60,5)$	$(2, 5^2)$
$\mathbb{Z}_3 \times S_4$	$G(72,42)$	$(2, 3, 12)$
S_5	$G(120,34)$	$(2, 4, 5)$

Proof. In this case $\mathbf{n} = (3)$ so G is $(1 \mid 3)$-generated. Moreover the second relation in (6) implies $|G| = 3(g(C) - 1)$.
Let us look at Table 2 of the Appendix. Cases $(4c)$, $(4d)$, $(4e)$, $(4i)$, $(4j)$, $(4n)$, $(4o)$, $(4p)$, $(4t)$ and $(4z)$ are excluded because $|G|$ is not divisible by 3. Case $(4x)$ is ruled out because 3 does not divide the order of $[G, G]$. Cases $(4b)$, $(4f)$, $(4h)$, $(4r)$ and $(4s)$ may not occur by Lemma 3.5. Case $(4m)$ and case $(4ac)$ are excluded by Lemma 1.4.
Let us rule out case $(4u)$, $G = S_3 \times S_3$, $\quad \mathbf{m} = (2^3, 3)$.
Let $\mathcal{V} = \{g_1, g_2, g_3, g_4\}$ and $\mathcal{W} = \{\ell_1;\, h_1, h_2\}$ be generating vectors for G of type $(2^3, 3)$ and $(1 \mid 3)$, respectively. We will show that g_4 and ℓ_1 are necessarily conjugated. There are three conjugacy classes of elements of order 3 in G: the class of 3-cycles in $N_1 = \langle (123) \rangle$, the class of 3-cycles in $N_2 = \langle (456) \rangle$ and the class of the remaining nontrivial elements in $N_1 N_2$. The element g_4 cannot lie in N_1. Conversely, $G/N_1 \cong \mathbb{Z}_2 \times S_3 \cong D_6$ would be generated by $g_1 N_1$, $g_2 N_1$ and $g_3 N_1$ subject to the relations $g_1 N_1 g_2 N_1 = g_3 N_1$ and $g_1^2 N_1 = g_2^2 N_1 = g_3^2 N_1 = N_1$, so it would be abelian. Similarly, g_4 cannot lie in N_2. On the other hand the element ℓ_1 cannot lie in N_1, otherwise G/N_1 would be generated by the commuting elements $h_1 N_1$ and $h_2 N_1$. Similarly, ℓ_1 cannot lie in N_2 so g_4 and ℓ_1 are conjugate and condition (U) is not satisfied.

Now we show that all remaining possibilities do occur.

• Case $(4a)$. $G = S_3$, $\quad \mathbf{m} = (2^6)$, $\quad g(C) = 3$.
Set

$$g_1 = g_2 = (12), \;\; g_3 = g_4 = (23), \;\; g_5 = g_6 = (13)$$
$$\ell_1 = (123), \;\; h_1 = (123), \;\; h_2 = (12).$$

• Case $(4g)$. $G = D_6, \quad \mathbf{m} = (2^5), \quad g(C) = 5.$
Set

$$g_1 = x, \;\; g_2 = x, \;\; g_3 = xy, \;\; g_4 = xy^4, \;\; g_5 = y^3$$
$$\ell_1 = y^2, \;\; h_1 = xy, \;\; h_2 = x.$$

• Case $(4k)$. $G = \mathbb{Z}_3 \times S_3, \quad \mathbf{m} = (2^2, 3^2), \quad g(C) = 7.$
Set

$$g_1 = (23), \;\; g_2 = (12), \;\; g_3 = z(123), \;\; g_4 = z^2$$
$$\ell_1 = (132), \;\; h_1 = z(12), \;\; h_2 = (123).$$

• Case $(4l)$. $G = \mathbb{Z}_3 \times S_3, \quad \mathbf{m} = (3, 6^2), \quad g(C) = 7.$
Set

$$g_1 = z(123), \;\; g_2 = z(12), \;\; g_3 = z(13)$$
$$\ell_1 = (123), \;\; h_1 = (12), \;\; h_2 = z(23).$$

• Case $(4q)$. $G = S_4, \quad \mathbf{m} = (2^3, 4), \quad g(C) = 9.$
Set

$$g_1 = (23), \;\; g_2 = (24), \;\; g_3 = (12), \;\; g_4 = (1234)$$
$$\ell_1 = (132), \;\; h_1 = (12), \;\; h_2 = (1234).$$

• Case $(4v)$. $G = S_3 \times S_3 = G(36, 10), \quad \mathbf{m} = (2, 6^2), \quad g(C) = 13.$
Set

$$g_1 = (12)(45), \;\; g_2 = (132)(56), \;\; g_3 = (13)(465)$$
$$\ell_1 = (123)(465), \;\; h_1 = (123)(45), \;\; h_2 = (12)(456).$$

• Case $(4w)$. $G = \mathbb{Z}_6 \times S_3, \quad \mathbf{m} = (2, 6^2), \quad g(C) = 13.$
Set

$$g_1 = (12), \;\; g_2 = z(132), \;\; g_3 = z^5(13)$$
$$\ell_1 = (123), \;\; h_1 = z^3(123), \;\; h_2 = z^4(12).$$

• Case $(4y)$. $G = \mathbb{Z}_4 \ltimes (\mathbb{Z}_3)^2 = G(36, 9), \quad \mathbf{m} = (3, 4^2), \quad g(C) = 13.$
Set

$$g_1 = y, \;\; g_2 = y^2x^3, \;\; g_3 = x$$
$$\ell_1 = y^2z^2, \;\; h_1 = y^2z, \;\; h_2 = yx.$$

• Case $(4aa)$. $G = A_5, \quad \mathbf{m} = (2, 5^2), \quad g(C) = 21.$
Set

$$g_1 = (12)(34), \;\; g_2 = (14235), \;\; g_3 = (14253)$$
$$\ell_1 = (124), \;\; h_1 = (12345), \;\; h_2 = (123).$$

• Case $(4ab)$. $G = \mathbb{Z}_3 \times S_4, \quad \mathbf{m} = (2, 3, 12), \quad g(C) = 25.$
Set

$$g_1 = (12), \;\; g_2 = z(234), \;\; g_3 = z^2(1432)$$
$$\ell_1 = (123), \;\; h_1 = (142), \;\; h_2 = z(23).$$

• Case $(4ad)$. $G = S_5, \quad \mathbf{m} = (2, 4, 5), \quad g(C) = 41.$
Set

$$g_1 = (12), \;\; g_2 = (1543), \;\; g_3 = (12345)$$
$$\ell_1 = (132), \;\; h_1 = (12), \;\; h_2 = (12345).$$

$\square$

4.3. **The case** $g(F) = 5$**.**

Proposition 4.3. *If* $g(F) = 5$ *we have precisely the following possibilities.*

G	IdSmall Group(G)	$\mathbf{m}$
D_4	$G(8,3)$	(2^6)
A_4	$G(12,3)$	(3^4)
$\mathbb{Z}_4 \ltimes (\mathbb{Z}_2)^2$	$G(16,3)$	$(2^2, 4^2)$
$\mathbb{Z}_2 \times A_4$	$G(24,13)$	$(2^2, 3^2)$
$\mathbb{Z}_2 \times A_4$	$G(24,13)$	$(3, 6^2)$
$\mathbb{Z}_8 \ltimes (\mathbb{Z}_2)^2$	$G(32,5)$	$(2, 8^2)$
$\mathbb{Z}_2 \ltimes D_{2,8,5}$	$G(32,7)$	$(2, 8^2)$
$\mathbb{Z}_4 \ltimes (\mathbb{Z}_4 \times \mathbb{Z}_2)$	$G(32,2)$	(4^3)
$\mathbb{Z}_4 \ltimes (\mathbb{Z}_2)^3$	$G(32,6)$	(4^3)
$(\mathbb{Z}_2)^2 \times A_4$	$G(48,49)$	$(2, 6^2)$
$\mathbb{Z}_4 \ltimes (\mathbb{Z}_2)^4$	$G(64,32)$	$(2,4,8)$
$\mathbb{Z}_5 \ltimes (\mathbb{Z}_2)^4$	$G(80,49)$	$(2, 5^2)$

Proof. In this case $\mathbf{n} = (2)$, so G must be $(1 \mid 2)$-generated. Moreover the second relation in (6) yields $|G| = 4(g(C) - 1)$. Now let us look at Table 3 of the Appendix. Cases $(5a)$, $(5e)$ and $(5y)$ must be excluded because $|G|$ is not divisible by 4. Cases $(5j)$, $(5k)$, $(5m)$, $(5n)$, $(5o)$, $(5s)$, $(5t)$, $(5u)$, $(5x)$, $(5z)$, $(5aa)$, $(5ab)$, $(5ah)$, $(5ai)$, $(5aj)$, $(5al)$, $(5am)$, $(5ap)$, $(5aq)$, $(5ar)$, $(5as)$, $(5at)$ cannot occur by Lemma 1.5. Finally, cases $(5c)$, $(5d)$, $(5f)$, $(5g)$, $(5h)$, $(5p)$, $(5q)$, $(5r)$, $(5ag)$ are ruled out by Lemma 3.5.
Now we show that all remaining possibilities do occur.

- Case $(5b)$. $G = D_4$, $\mathbf{m} = (2^6)$, $g(C) = 3$.

Set

$$g_1 = g_2 = g_3 = g_4 = x, \;\; g_5 = g_6 = xy$$
$$\ell_1 = y^2, \;\; h_1 = x, \;\; h_2 = y.$$

- Case $(5i)$. $G = A_4$, $\mathbf{m} = (3^4)$, $g(C) = 4$.

Set

$$g_1 = (123), \;\; g_2 = (132), \;\; g_3 = (124), \;\; g_4 = (142)$$
$$\ell_1 = (12)(34), \;\; h_1 = (123), \;\; h_2 = (124).$$

- Case $(5l)$. $G = \mathbb{Z}_4 \ltimes (\mathbb{Z}_2)^2 = G(16,3)$, $\mathbf{m} = (2^2, 4^2)$, $g(C) = 5$.

Set

$$g_1 = y, \;\; g_2 = y, \;\; g_3 = x, \;\; g_4 = x^3$$
$$\ell_1 = z, \;\; h_1 = x, \;\; h_2 = y.$$

- Case $(5v)$. $G = \mathbb{Z}_2 \times A_4$, $\mathbf{m} = (2^2, 3^2)$, $g(C) = 7$.

Set

$$g_1 = z, \;\; g_2 = z(12)(34), \;\; g_3 = (123), \;\; g_4 = (234)$$
$$\ell_1 = (13)(24), \;\; h_1 = z(123), \;\; h_2 = (12)(34).$$

- Case $(5w)$. $G = \mathbb{Z}_2 \times A_4$, $\mathbf{m} = (3, 6^2)$, $g(C) = 7$.

Set

$$g_1 = (123), \;\; g_2 = z(243), \;\; g_3 = z(142)$$
$$\ell_1 = (13)(24), \;\; h_1 = z(123), \;\; h_2 = (12)(34).$$

• Case $(5ac)$. $G = \mathbb{Z}_8 \ltimes (\mathbb{Z}_2)^2 = G(32,5)$, $\quad \mathbf{m} = (2, 8^2)$, $\quad g(C) = 9$.
Set

$$g_1 = y, \ \ g_2 = (xy)^{-1}, \ \ g_3 = x$$
$$\ell_1 = z, \ \ h_1 = x, \ \ h_2 = y.$$

• Case $(5ad)$. $G = \mathbb{Z}_2 \ltimes D_{2,8,5} = G(32,7)$, $\quad \mathbf{m} = (2, 8^2)$, $\quad g(C) = 9$.
Set

$$g_1 = x, \ \ g_2 = z^5, \ \ g_3 = xyz^7$$
$$\ell_1 = y, \ \ h_1 = z, \ \ h_2 = x.$$

• Case $(5ae)$. $G = \mathbb{Z}_4 \ltimes (\mathbb{Z}_4 \times \mathbb{Z}_2) = G(32,2)$, $\quad \mathbf{m} = (4^3)$, $\quad g(C) = 9$.
Set

$$g_1 = y, \ \ g_2 = xy^2, \ \ g_3 = x^3y$$
$$\ell_1 = z, \ \ h_1 = x, \ \ h_2 = y.$$

• Case $(5af)$. $G = \mathbb{Z}_4 \ltimes (\mathbb{Z}_2)^3 = G(32,6)$, $\quad \mathbf{m} = (4^3)$, $\quad g(C) = 9$.
Set

$$g_1 = x, \ \ g_2 = xy, \ \ g_3 = yx^2$$
$$\ell_1 = z, \ \ h_1 = xw, \ \ h_2 = y.$$

• Case $(5ak)$. $G = (\mathbb{Z}_2)^2 \times A_4$, $\quad \mathbf{m} = (2, 6^2)$, $\quad g(C) = 13$.
Set

$$g_1 = z(12)(34), \ \ g_2 = w(123), \ \ g_3 = wz(234)$$
$$\ell_1 = (12)(34), \ \ h_1 = z(14)(23), \ \ h_2 = w(123).$$

• Case $(5an)$. $G = \mathbb{Z}_4 \ltimes (\mathbb{Z}_2)^4 = G(64,32)$, $\quad \mathbf{m} = (2,4,8)$, $\quad g(C) = 17$.
Set

$$g_1 = y, \ \ g_2 = x, \ \ g_3 = x^3y$$
$$\ell_1 = z, \ \ h_1 = y, \ \ h_2 = x.$$

• Case $(5ao)$. $G = \mathbb{Z}_5 \ltimes (\mathbb{Z}_2)^4 = G(80,49)$, $\quad \mathbf{m} = (2, 5^2)$, $\quad g(C) = 21$.
Set

$$g_1 = y, \ \ g_2 = xyw, \ \ g_3 = wyx^4y$$
$$\ell_1 = yv, \ \ h_1 = y, \ \ h_2 = xyw.$$

□

5. The classification in the mixed case

In this section we use Proposition 3.8 in order to classify the surfaces with $p_g = q = 1$ isogenous to a mixed product. By Proposition 3.10 we have $g(C) = 5$, 7 or 9. Let us consider the three cases separately.

5.1. The case $g(C) = 5$, $|G| = 16$.

Proposition 5.1. *If $g(C) = 5$, $|G| = 16$ we have precisely the following possibilities.*

G°	`IdSmallGroup`(G°)	G	`IdSmallGroup`(G)
D_4	$G(8,3)$	$D_{2,8,3}$	$G(16,8)$
$\mathbb{Z}_2 \times \mathbb{Z}_4$	$G(8,2)$	$D_{2,8,5}$	$G(16,6)$
$(\mathbb{Z}_2)^3$	$G(8,5)$	$\mathbb{Z}_4 \ltimes (\mathbb{Z}_2)^2$	$G(16,3)$

Proof. In this case $\mathbf{n} = (2^2)$, so our first task is to find all nonsplit sequences of type (5) for which G° is a $(1 \mid 2^2)$-generated group of order 8. The three abelian groups of order 8 and D_4 are $(1 \mid 2^2)$-generated whereas the quaternion group Q_8 is not. Indeed, if $\{\ell_1, \ell_2; h_1, h_2\}$ were a generating vector of type $(1 \mid 2^2)$ for Q_8 then necessarily $\ell_1 = \ell_2 = -1$ would be central, $[h_1, h_2] = 1$ and $Q_8 = \langle h_1, h_2, \ell_1\rangle$ would be abelian, a contradiction.
Since $\mathbb{Z}_8$ has only one element ℓ of order 2, condition $(M1)$ in Proposition 3.8 cannot be satisfied for any choice of $\mathcal{V}$. Thus, by Remark 3.7 we are left to analyze the possible embeddings of $\mathbb{Z}_2 \times \mathbb{Z}_4$, D_4 and $(\mathbb{Z}_2)^3$ in nonabelian groups of order 16. The groups $\mathbb{Z}_2 \times \mathbb{Z}_4$, D_4 and $(\mathbb{Z}_2)^3$ have 3, 5 and 7 elements of order 2, respectively. Therefore if n_2 denotes the number of elements of order 2 in G, by Remark 2.5 we must consider only those groups G of order 16 with $n_2 \in \{3, 5, 7\}$. The nonabelian groups of order 16 with $n_2 = 3$ are $D_{2,8,5}$, $\mathbb{Z}_2 \times Q_8$ and $D_{4,4,-1}$ and they all contain a copy of $\mathbb{Z}_2 \times \mathbb{Z}_4$. The only nonabelian group of order 16 with $n_2 = 5$ is $D_{2,8,3}$ and it contains a subgroup isomorphic to D_4. The nonabelian groups of order 16 with $n_2 = 7$ are $\mathbb{Z}_4 \ltimes (\mathbb{Z}_2)^2 = G(16, 3)$ and $\mathbb{Z}_2 \ltimes Q_8$, and only the former contains a subgroup isomorphic to $(\mathbb{Z}_2)^3$ (cfr. [Wi05]).
Summarizing, we are left with the following cases:

G°	G
D_4	$D_{2,8,3}$
$\mathbb{Z}_2 \times \mathbb{Z}_4$	$D_{2,8,5}$
$\mathbb{Z}_2 \times \mathbb{Z}_4$	$\mathbb{Z}_2 \times Q_8$
$\mathbb{Z}_2 \times \mathbb{Z}_4$	$D_{4,4,-1}$
$(\mathbb{Z}_2)^3$	$\mathbb{Z}_4 \ltimes (\mathbb{Z}_2)^2$

Let us analyze them separately.

• $G^\circ = D_4, \;\; G = D_{2,8,3}$.
Referring to the presentation for G in the Appendix, Table 3, case $(5m)$, we consider the subgroup $G^\circ := \langle x, y^2\rangle \cong D_4$. Set

$$\ell_1 = \ell_2 = x, \;\; h_1 = h_2 = y^2.$$

Condition $(M1)$ holds because $C_G(x) = \langle x, y^4\rangle \subset G^\circ$. Condition $(M2)$ is satisfied because the conjugacy class of x in G° is contained in the coset $x\langle y^2\rangle$ while for every $g \in yG^\circ$ we have $g^2 \in \langle y\rangle$. Therefore this case occurs by Proposition 3.8.

• $G^\circ = \mathbb{Z}_2 \times \mathbb{Z}_4, \;\; G = D_{2,8,5}$.
Referring to the presentation for G in the Appendix, Table 1, case $(3g)$, we consider the subgroup $G^\circ := \langle x, y^2\rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_4$. Set

$$\ell_1 = \ell_2 = x, \; h_1 = h_2 = y^2.$$

Conditions $(M1)$ and $(M2)$ are verified as in the previous case, so this possibility occurs.

• $G^\circ = \mathbb{Z}_2 \times \mathbb{Z}_4, \;\; G = \mathbb{Z}_2 \times Q_8$.
All elements of order 2 in G are central so condition $(M1)$ cannot be satisfied and this case does not occur.

• $G^\circ = \mathbb{Z}_2 \times \mathbb{Z}_4, \;\; G = D_{4,4,-1}$.
Condition $(M1)$ cannot hold because all elements of order 2 in G are central.

• $G^\circ = (\mathbb{Z}_2)^3, \;\; G = \mathbb{Z}_4 \ltimes (\mathbb{Z}_2)^2 = G(16, 3)$.
Referring to the presentation of G given in the Appendix, Table 3, case $(5l)$, we consider the subgroup $G^\circ := \langle y, z, x^2\rangle \cong (\mathbb{Z}_2)^3$. Set

$$\ell_1 = \ell_2 = y, \; h_1 = z, \;\; h_2 = x^2.$$

Condition $(M1)$ holds because G° is abelian and $[x, y] \neq 1$. Condition $(M2)$ is satisfied because if $g \in xG^\circ$ then $g^2 \in \langle z,\, x^2\rangle$. Therefore this case occurs.

$\square$

5.2. **The case** $g(C) = 7,\ |G| = 36$**.**

Proposition 5.2. *The case* $g(C) = 7,\ |G| = 36$ *does not occur.*

Proof. In this case $\mathbf{n} = (3)$, so G° is a group of order 18 which is $(1 \mid 3)$-generated. There are five groups of order 18 up to isomorphism, namely

$$\mathbb{Z}_{18}, \quad \mathbb{Z}_2 \times (\mathbb{Z}_3)^2, \quad D_9, \quad \mathbb{Z}_3 \times S_3, \quad G(18, 4),$$

where $G(18, 4)$ is as in the Appendix, Table 2, case $(4m)$. Among these, the only one which is $(1 \mid 3)$-generated is $\mathbb{Z}_3 \times S_3 = G(18, 3)$. Thus G would fit into a short exact sequence

$$1 \longrightarrow \mathbb{Z}_3 \times S_3 \longrightarrow G \longrightarrow \mathbb{Z}_2 \longrightarrow 1. \tag{9}$$

The GAP4 script below shows that the only groups of order 36 containing a subgroup isomorphic to $\mathbb{Z}_3 \times S_3$ are $G(36, 10)$ (see Appendix, Table 2, case $(4v)$) and $G(36, 12) = \mathbb{Z}_6 \times S_3$.

```
gap> s:=NumberSmallGroups(36);
14
gap> set:=[1..s];
[1..14]
gap> for t in set do
> c:=0;
> G:=SmallGroup(36,t);
> N:=NormalSubgroups(G);
> for G0 in N do
> if IdSmallGroup(G0)=[18,3] then
> c:=c+1; fi;
> if IdSmallGroup(G0)=[18,3] and c=1 then
> Print(IdSmallGroup(G), " ");
> fi; od; od; Print("\n");
[36,10] [36,12]
gap>
```

The groups $G(36, 10)$ and $\mathbb{Z}_6 \times S_3$ contain 7 and 15 elements of order 2, respectively. On the other hand $\mathbb{Z}_3 \times S_3$ contains 3 elements of order 2, so by Remark 2.5 all possible extensions of the form (9) are split and this case cannot occur. $\square$

5.3. **The case** $g(C) = 9,\ |G| = 64$**.**

Proposition 5.3. *The case* $g(C) = 9,\ |G| = 64$ *does not occur.*

The proof will be the consequence of the results below. First notice that, since $\mathbf{n} = (2)$, the group G° must be $(1\,|\,2)$-generated.

Computational Fact 5.4. *There exist precisely* 8 *groups of order* 32 *which are* $(1 \mid 2)$*-generated. They are* $G(32, t)$ *for* $t \in \{2, 4, 5, 6, 7, 8, 12, 17\}$*. Moreover the number* n_2 *of their elements of order* 2 *is as follows:*

t	2	4	5	6	7	8	12	17
$n_2(G(32, t))$	7	3	7	11	11	3	3	3

Proof. First we use the following GAP4 script that finds the groups of order 32 which are $(1 \mid 2)$-generated.

```
gap> s:=NumberSmallGroups(32);
51
gap> set:=[1..s];
[1..51]
gap> for t in set do
> c:=0;
> G0:=SmallGroup(32,t);
> for h1 in G0 do
> for h2 in G0 do
> H:=Subgroup(G0, [h1,h2]);
> if Order(h1*h2*h1^-1*h2^-1)=2 and
> Order(H)=32 then
> c:=c+1; fi;
> if Order(h1*h2*h1^-1*h2^-1)=2 and
> Order(H)=32 and c=1 then
> Print(IdSmallGroup(G0), " ");
> fi; od; od; od; Print("\n");
[32,2] [32,4] [32,5] [32,6] [32,7] [32,8] [32,12]
[32,17]
gap>
```

Now we use another GAP4 script, that computes the number of elements of order 2 in each case.

```
gap> set:=[2,4,5,6,7,8,12,17];
 [ 2, 4, 5, 6, 7, 8, 12, 17 ]
gap> for t in set do
> n2:=0;
> G0:=SmallGroup(32,t);
> for g in G0 do
> if Order(g)=2 then
> n2:=n2+1; fi; od;
> Print(IdSmallGroup(G0), " "); Print(n2, "\n");
> od;
[32,2] 7
[32,4] 3
[32,5] 7
[32,6] 11
[32,7] 11
[32,8] 3
[32,12] 3
[32,17] 3
gap>
```

$\square$

Computational Fact 5.5. *Let $t \in \{2, 4, 5, 6, 7, 8, 12, 17\}$. A nonsplit extension of the form*

$$1 \longrightarrow G(32,t) \longrightarrow G(64,s) \longrightarrow \mathbb{Z}_2 \longrightarrow 1 \tag{10}$$

exists if and only if the pair (t,s) is one of the following:

$(2,9)$, $(2,57)$, $(2,59)$, $(2,63)$, $(2,64)$, $(2,68)$, $(2,70)$, $(2,72)$, $(2,76)$, $(2,79)$, $(2,81)$, $(2,82)$,
$(4,11)$, $(4,28)$, $(4,122)$, $(4,127)$, $(4,172)$, $(4,182)$,
$(5,5)$, $(5,9)$, $(5,112)$, $(5,113)$, $(5,114)$, $(5,132)$, $(5,164)$, $(5,165)$, $(5,166)$,
$(6,33)$, $(6,35)$,

$(7,33)$,
$(8,37)$,
$(12,7)$, $(12,13)$, $(12,14)$, $(12,15)$, $(12,16)$, $(12,126)$, $(12,127)$, $(12,143)$, $(12,156)$,
$(12,158)$, $(12,160)$,
$(17,28)$, $(17,43)$, $(17,45)$, $(17,46)$.

Proof. Assume $t = 2$. The following GAP4 script finds the groups of order 64 containing a subgroup isomorphic to $G(32,2)$.

```
gap> s:=NumberSmallGroups(64);
267
gap> set:=[1..s];
[1..267]
gap> for t in set do
> c:=0;
> G:=SmallGroup(64,t);
> N:=NormalSubgroups(G);
> for G0 in N do
> if IdSmallGroup(G0)=[32,2] then
> c:=c+1; fi;
> if IdSmallGroup(G0)=[32,2] and c=1 then
> Print(IdSmallGroup(G), " ");
> fi; od; od; Print("\n");
[64,8] [64,9] [64,56] [64,57] [64,58] [64,59] [64,61] [64,62] [64,63] [64,64]
[64,66] [64,67] [64,68] [64,69] [64,70] [64,72] [64,73] [64,74] [64,75] [64,76]
[64,77] [64,78] [64,79] [64,80] [64,81] [64,82]
gap>
```

By Remark 2.5 and Computational Fact 5.4, in order to detect all the groups $G(64, s)$ fitting in some nonsplit extension of type (10) with $t = 2$, it is sufficient to select from the previous list the groups containing exactly $n_2 = 7$ elements of order 2. This is achieved by using the GAP4 script below:

```
gap> set:=[8,9,56,57,58,59,61,62,63,64,66,67,68,69,70,
>72,73,74,75,76,77,78,79,80,81,82];
[8,9,56,57,58,59,61,62,63,64,66,67,68,69,70,72,73,74,75,
 76,77,78,79,80,81,82]
gap> for t in set do
> n2:=0;
> G:=SmallGroup(64,t);
> for g in G do
> if Order(g)=2 then n2:=n2+1;
> fi; od;
> if n2=7 then
> Print(IdSmallGroup(G), " ");
> fi; od; Print("\n");
[64,9] [64,57] [64,59] [64,63] [64,64] [64,68] [64,70]
[64,72] [64,76] [64,79] [64,81] [64,82]
gap>
```

This proves the claim in the case $t = 2$. The proof for the other values of t may be carried out exactly in the same way. $\square$

Let us denote by $[G, G]_2$ and $[G^\circ, G^\circ]_2$ the subsets of elements of order 2 in $[G, G]$ and $[G^\circ, G^\circ]$, respectively.

Lemma 5.6. *Assume $g(C) = 9$ and that one of the following situations occur:*

- $[G,G]_2 \subseteq Z(G)$;
- *there exists some element $y \in G \setminus G^\circ$ commuting with all elements in $[G^\circ,\, G^\circ]_2$.*

Then given any generating vector $\mathcal{V} = \{\ell_1;\ h_1, h_2\}$ of type $(1\,|\,2)$ for G°, condition $(M1)$ in Proposition 3.8 cannot be satisfied.

Proof. Since $\ell_1 \in [G^\circ,\, G^\circ]_2 \subseteq [G,\, G]_2$, in any of the above situations $C_G(\ell_1)$ is not contained in G°, so $(M1)$ cannot hold. □

Computational Fact 5.7. *Let $G = G(64, s)$ be one of the groups appearing in the list of Computational Fact 5.5. Then $[G,G]_2$ is not contained in $Z(G)$ if and only if $s = 5$, 33, 35, 37.*

Proof. The following GAP4 script selects from the aforementioned list the groups having a noncentral element of order 2 in their derived subgroup.

```
gap> set:=[5,7,9,11,13,,14,15,16,28,33,35,37,43,45,46,
>57,59,63,64,68,70,72,76,79,81,82,112,113,114, 122,126,
>127,132,143,156,158,160,164,165,166,172,182];
[5,7,9,11,13,,14,15,16,28,33,35,37,43,45,46,
 57,59,63,64,68,70,72,76,79,81,82,112,113,114, 122,126,
 127,132,143,156,158,160,164,165,166,172,182]
gap> for t in set do
> c:=0;
> G:=SmallGroup(64,t);
> D:=DerivedSubgroup(G);
> for d in D do
> B:=d in Center(G);
> if Order(d)=2 and B=false then
> c:=c+1; fi;
> if Order(d)=2 and B=false and c=1 then
> Print(IdSmallGroup(G), " ");
> fi; od; od; Print("\n");
[64,5] [64,33] [64,35] [64,37]
gap>
```

□

Computational facts 5.5, 5.7 and Lemma 5.6 imply that we only need to analyze the following pairs $(G^\circ,\, G)$:

G°	G
$G(32,5)$	$G(64,5)$
$G(32,6)$	$G(64,33)$
$G(32,7)$	$G(64,33)$
$G(32,6)$	$G(64,35)$
$G(32,8)$	$G(64,37)$

Proposition 5.8. *The case $G^\circ = G(32,5)$ does not occur.*

Proof. A presentation for the group G° can be found in the Appendix, Table 3, case $(5ac)$. Its derived subgroup contains exactly one element of order 2, namely z. It follows that if $\{\ell_1;\, h_1, h_2\}$ is any generating vector of type $(1\,|\,2)$ for G°, then $\ell_1 = z$. Since $[G^\circ,\, G^\circ]$ is characteristic in G°, condition $(M1)$ cannot be satisfied for any embedding of G° into G. □

The groups $G(64,33)$, $G(64,35)$ and $G(64,37)$ can be presented as follows.

$$\begin{aligned} G(64,33) = \langle x,\, y,\, z,\, w,\, v,\, u \mid\ & z^2 = w^2 = v^2 = u^2 = 1,\ x^2 = w,\ y^2 = u, \\ & [x,\, zy] = z,\ [x,\, vz] = v,\ [x,\, vu] = u, \\ & [y,\, z] = [y,\, v] = [z,\, v] = [w,\, v] = [x,\, u] = 1\rangle \end{aligned} \tag{11}$$

$$
\begin{aligned}
(12) \qquad G(64,35) = \langle x,\, y,\, z,\, w,\, v,\, u \mid\, & w^2 = v^2 = u^2 = 1,\ z^2 = y^2 = u,\ x^2 = w, \\
& [y,\, z] = [z,\, w] = u,\, [x, yz] = z,\, [x,\, z] = uv, \\
& [y,\, v] = [z,\, v] = [w,\, v] = [x,\, u] = 1\rangle
\end{aligned}
$$

$$
\begin{aligned}
(13) \qquad G(64,37) = \langle x,\, y,\, z,\, w,\, v,\, u \mid\, & v^2 = u^2 = 1,\ w^2 = z^2 = y^2 = u,\ x^2 = w, \\
& [y,\, z] = [z,\, w] = u,\, [x,\, yz] = z,\, [x,\, z] = uv, \\
& [y,\, v] = [z,\, v] = [w,\, v] = 1\rangle.
\end{aligned}
$$

This can be verified by using the following GAP4 script. We only consider the case $G = G(64,33)$, the others two being completely similar.

```
gap> G:=SmallGroup(64,33);
< pc group of size 64 with 6 generators >
gap> P:=PresentationViaCosetTable(G);
< presentation with 6 gens and 14 rels of total length 49 >
gap> TzPrintRelators(P);
#I  1. f3^2
#I  2. f4^2
#I  3. f5^2
#I  4. f6^2
#I  5. f1^2*f4
#I  6. f2^2*f6
#I  7. f2*f3*f2^-1*f3
#I  8. f2*f5*f2^-1*f5
#I  9. f3*f5*f3*f5
#I  10. f4*f5*f4*f5
#I  11. f1*f6*f1^-1*f6
#I  12. f1*f3*f2*f1^-1*f2^-1
#I  13. f1*f5*f3*f1^-1*f3
#I  14. f1*f5*f6*f1^-1*f5
gap>
```

Setting

$$x := \mathtt{f1},\ y := \mathtt{f2},\ z := \mathtt{f3},\ w := \mathtt{f4},\ v := \mathtt{f5},\ u := \mathtt{f6}$$

we obtain presentation (11).

Computational Fact 5.9. *Referring to presentations* (11), (12) *and* (13)*, we have the following facts.*

- *The group $G(64,33)$ contains exactly one subgroup N_1 isomorphic to $G(32,6)$ and one subgroup N_2 isomorphic to $G(32,7)$, namely*
$$N_1 := \langle x,\, z,\, w,\, v,\, u\rangle, \quad N_2 := \langle xy,\, z,\, w,\, v,\, u\rangle.$$
- *The group $G(64,35)$ contains exactly two subgroups N_3, N_4 isomorphic to $G(32,6)$, namely*
$$N_3 := \langle x,\, z,\, w,\, v,\, u\rangle, \quad N_4 := \langle xy,\, z,\, w,\, v,\, u\rangle.$$
- *The group $G(64,37)$ contains exactly two subgroups N_5, N_6 isomorphic to $G(32,8)$, namely*
$$N_5 := \langle x,\, z,\, w,\, v,\, u\rangle, \quad N_6 := \langle xy,\, z,\, w,\, v,\, u\rangle.$$

In addition, for every $i \in \{1, \ldots, 6\}$ we have

(*a*) $[N_i,\, N_i] = \langle v, u\rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

(*b*) $y \notin N_i$ *and y commutes with all elements in $[N_i,\, N_i]$.*

Proof. We use GAP4 as follows.

```
gap> G:=SmallGroup(64,33);
<pc group of size 64 with 6 generators>
gap> for N in NormalSubgroups(G) do
> if IdSmallGroup(N)=[32,6] or
> IdSmallGroup(N)=[32,7] then
> Print(N, " "); Print(IdSmallGroup(N), " ");
> Print(DerivedSubgroup(N), "\n");
> fi; od;
Group( [ f1*f2, f3, f4, f5, f6 ] )  [32,7]  Group( [ f5, f6 ] )
Group( [ f1, f3, f4, f5, f6 ] )  [32,6]  Group( [ f5, f6 ] )
```

```
gap> G:=SmallGroup(64,35);
<pc group of size 64 with 6 generators>
gap> for N in NormalSubgroups(G) do
> if IdSmallGroup(N)=[32,6] then
> Print(N, " "); Print(IdSmallGroup(N), " ");
> Print(DerivedSubgroup(N), "\n");
> fi; od;
Group( [ f1*f2, f3, f4, f5, f6 ] )  [32,6]  Group( [ f5, f6 ] )
Group( [ f1, f3, f4, f5, f6 ] )  [32,6]  Group( [ f5, f6 ] )
```

```
gap> G:=SmallGroup(64,37);
<pc group of size 64 with 6 generators>
gap> for N in NormalSubgroups(G) do
> if IdSmallGroup(N)=[32,8] then
> Print(N, " "); Print(IdSmallGroup(N), " ");
> Print(DerivedSubgroup(N), "\n");
> fi; od;
Group( [ f1*f2, f3, f4, f5, f6 ] )  [32,8]  Group( [ f5, f6 ] )
Group( [ f1, f3, f4, f5, f6 ] )  [32,8]  Group( [ f5, f6 ] )
gap>
```

□

Proposition 5.10. *The cases* $G^\circ = G(32,6)$, $G(32,7)$, $G(32,8)$ *do not occur.*

Proof. By Lemma 5.6 and Computational Fact 5.9 it follows that, given any nonsplit extension of type (10) with G° as above, condition $(M1)$ in Proposition 3.8 cannot be satisfied. □

Summing up, we finally obtain

Proof of Proposition 5.3. It follows from Propositions 5.8 and 5.10.

□

Appendix

This Appendix includes the classification of automorphism groups G acting on Riemann surfaces of genus 3, 4 and 5. We only consider the cases where G is nonabelian and the quotient is isomorphic to $\mathbb{P}^1$.
Table 1 is adapted from [Br90, pages 254, 255], Table 2 from [Ki03, Theorem 1] and [KuKu90], Table 3 from [KuKi90].
Whenever a presentation of G was not available in the quoted papers we used GAP4 in order to find one. Moreover for every group we provide the branching data and the `IdSmallGroup`(G).
The authors wish to thank S. A. Broughton for communicating that the group $G(48,33)$ (Table 1, case $(3t)$) was missing in [Br90].

Case	G	`IdSmallGroup`(G)	$\mathbf{m}$	Presentation
$(3a)$	S_3	$G(6,1)$	$(2^4,3)$	$\langle x,y \mid x=(12),\ y=(123)\rangle$
$(3b)$	D_4	$G(8,3)$	$(2^2,4^2)$	$\langle x,y \mid x^2=y^4=1,\ xyx^{-1}=y^{-1}\rangle$
$(3c)$	D_4	$G(8,3)$	(2^5)	$\langle x,y \mid x^2=y^4=1,\ xyx^{-1}=y^{-1}\rangle$
$(3d)$	$D_{4,3,-1}$	$G(12,1)$	$(4^2,6)$	$\langle x,y \mid x^4=y^3=1,\ xyx^{-1}=y^{-1}\rangle$
$(3e)$	D_6	$G(12,4)$	$(2^3,6)$	$\langle x,y \mid x^2=y^6=1,\ xyx^{-1}=y^{-1}\rangle$
$(3f)$	A_4	$G(12,3)$	$(2^2,3^2)$	$\langle x,y \mid x=(12)(34),\ y=(123)\rangle$
$(3g)$	$D_{2,8,5}$	$G(16,6)$	$(2,8^2)$	$\langle x,y \mid x^2=y^8=1,\ xyx^{-1}=y^5\rangle$
$(3h)$	$D_{4,4,-1}$	$G(16,4)$	(4^3)	$\langle x,y \mid x^4=y^4=1,\ xyx^{-1}=y^{-1}\rangle$
$(3i)$	$\mathbb{Z}_2\times D_4$	$G(16,11)$	$(2^3,4)$	$\langle z \mid z^2=1\rangle\times\langle x,y \mid x^2=y^4=1,\ xyx^{-1}=y^{-1}\rangle$
$(3j)$	$\mathbb{Z}_2\ltimes(\mathbb{Z}_2\times\mathbb{Z}_4)$	$G(16,13)$	$(2^3,4)$	$\langle x,y,z \mid x^2=y^2=z^4=1,$ $[x,z]=[y,z]=1,\ xyx^{-1}=yz^2\rangle$
$(3k)$	$D_{3,7,2}$	$G(21,1)$	$(3^2,7)$	$\langle x,y \mid x^3=y^7=1,\ xyx^{-1}=y^2\rangle$
$(3l)$	$D_{2,12,5}$	$G(24,5)$	$(2,4,12)$	$\langle x,y \mid x^2=y^{12}=1,\ xyx^{-1}=y^5\rangle$
$(3m)$	$\mathbb{Z}_2\times A_4$	$G(24,13)$	$(2,6^2)$	$\langle z \mid z^2=1\rangle\times\langle x,y \mid x=(12)(34),\ y=(123)\rangle$
$(3n)$	$\mathrm{SL}_2(\mathbb{F}_3)$	$G(24,3)$	$(3^2,6)$	$\langle x,y \mid x=\begin{pmatrix}1&1\\0&1\end{pmatrix},\ y=\begin{pmatrix}0&1\\-1&-1\end{pmatrix}\rangle$
$(3o)$	S_4	$G(24,12)$	$(3,4^2)$	$\langle x,y \mid x=(12),\ y=(1234)\rangle$
$(3p)$	S_4	$G(24,12)$	$(2^3,3)$	$\langle x,y \mid x=(12),\ y=(1234)\rangle$
$(3q)$	$\mathbb{Z}_2\ltimes(\mathbb{Z}_2\times\mathbb{Z}_8)$	$G(32,9)$	$(2,4,8)$	$\langle x,y,z \mid x^2=y^2=z^8=1,$ $[x,y]=[y,z]=1,\ xzx^{-1}=yz^3\rangle$
$(3r)$	$\mathbb{Z}_2\ltimes D_{2,8,5}$	$G(32,11)$	$(2,4,8)$	$\langle x,y,z \mid x^2=y^2=z^8=1,$ $yzy^{-1}=z^5,\ xyx^{-1}=yz^4,\ xzx^{-1}=yz^3\rangle$
$(3s)$	$\mathbb{Z}_2\times S_4$	$G(48,48)$	$(2,4,6)$	$\langle z \mid z^2=1\rangle\times\langle x,y \mid x=(12),\ y=(1234)\rangle$
$(3t)$	$G(48,33)$	$G(48,33)$	$(2,3,12)$	$\langle x,y,z,w,t \mid x^2=z^2=w^2=t,\ y^3=t^2=1,$ $yzy^{-1}=w,\ ywy^{-1}=zw,\ zwz^{-1}=wt,$ $[x,y]=[x,z]=1\rangle$
$(3u)$	$\mathbb{Z}_3\ltimes(\mathbb{Z}_4)^2$	$G(48,3)$	$(3^2,4)$	$\langle x,y,z \mid x^3=y^4=z^4=1,$ $[y,z]=1,\ xyx^{-1}=z,\ xzx^{-1}=(yz)^{-1}\rangle$
$(3v)$	$S_3\ltimes(\mathbb{Z}_4)^2$	$G(96,64)$	$(2,3,8)$	$\langle x,y,z,w \mid x^2=y^3=z^4=w^4=1,$ $[z,w]=1,\ xyx^{-1}=y^{-1},\ xzx^{-1}=w,$ $xwx^{-1}=z,\ yzy^{-1}=w,\ ywy^{-1}=(zw)^{-1}\rangle$
$(3w)$	$\mathrm{PSL}_2(\mathbb{F}_7)$	$G(168,42)$	$(2,3,7)$	$\langle x,y \mid x=(375)(486),\ y=(126)(348)\rangle$

Table 1. Nonabelian automorphism groups with rational quotient on Riemann surfaces of genus 3.

Case	G	IdSmallGroup(G)	$\mathbf{m}$	Presentation
$(4a)$	S_3	$G(6,1)$	(2^6)	$\langle x,y \mid x=(12),\ y=(123)\rangle$
$(4b)$	S_3	$G(6,1)$	$(2^2,3^3)$	$\langle x,y \mid x=(12),\ y=(123)\rangle$
$(4c)$	D_4	$G(8,3)$	$(2^4,4)$	$\langle x,y \mid x^2=y^4=1,\ xyx^{-1}=y^{-1}\rangle$
$(4d)$	Q_8	$G(8,4)$	$(2,4^3)$	$\langle i,j,k,-1 \mid i^2=j^2=k^2=-1,$ $ij=k,\ jk=i,\ ki=j\rangle$
$(4e)$	D_5	$G(10,1)$	$(2^2,5^2)$	$\langle x,y \mid x^2=y^5=1,\ xyx^{-1}=y^{-1}\rangle$
$(4f)$	A_4	$G(12,3)$	$(2,3^3)$	$\langle x,y \mid x=(12)(34),\ y=(123)\rangle$
$(4g)$	D_6	$G(12,4)$	(2^5)	$\langle x,y \mid x^2=y^6=1,\ xyx^{-1}=y^{-1}\rangle$
$(4h)$	D_6	$G(12,4)$	$(2^2,3,6)$	$\langle x,y \mid x^2=y^6=1,\ xyx^{-1}=y^{-1}\rangle$
$(4i)$	D_8	$G(16,7)$	$(2^3,8)$	$\langle x,y \mid x^2=y^8=1, xyx^{-1}=y^{-1}\rangle$
$(4j)$	$G(16,9)$	$G(16,9)$	$(4^2,8)$	$\langle x,y,z,w \mid x^2=y^2=z^2=w,$ $w^2=1,\ xzx^{-1}=z^{-1},$ $yzy^{-1}=z^{-1},\ yxy^{-1}=(xz)^{-1}\rangle$
$(4k)$	$\mathbb{Z}_3\times S_3$	$G(18,3)$	$(2^2,3^2)$	$\langle z \mid z^3=1\rangle\times\langle x,y \mid x=(12),\ y=(123)\rangle$
$(4l)$	$\mathbb{Z}_3\times S_3$	$G(18,3)$	$(3,6^2)$	$\langle z \mid z^3=1\rangle\times\langle x,y \mid x=(12),\ y=(123)\rangle$
$(4m)$	$\mathbb{Z}_2\ltimes(\mathbb{Z}_3)^2$	$G(18,4)$	$(2^2,3^2)$	$\langle x,y,z \mid x^2=y^3=z^3=1,$ $xyx^{-1}=y^{-1},\ xzx^{-1}=z^{-1},\ [y,z]=1\rangle$
$(4n)$	$\mathbb{Z}_2\times D_5$	$G(20,4)$	$(2^3,5)$	$\langle z \mid z^2=1\rangle\times\langle x,y \mid x^2=y^5=1,\ xyx^{-1}=y^{-1}\rangle$
$(4o)$	$D_{4,5,-1}$	$G(20,1)$	$(4^2,5)$	$\langle x,y \mid x^4=y^5=1,\ xyx^{-1}=y^{-1}\rangle$
$(4p)$	$D_{4,5,2}$	$G(20,3)$	$(4^2,5)$	$\langle x,y \mid x^4=y^5=1,\ xyx^{-1}=y^2\rangle$
$(4q)$	S_4	$G(24,12)$	$(2^3,4)$	$\langle x,y \mid x=(12),\ y=(1234)\rangle$
$(4r)$	$D_{2,12,7}$	$G(24,10)$	$(2,6,12)$	$\langle x,y \mid x^2=y^{12}=1,\ xyx^{-1}=y^7\rangle$
$(4s)$	$\mathrm{SL}_2(\mathbb{F}_3)$	$G(24,3)$	$(3,4,6)$	$\langle x,y \mid x=\begin{pmatrix}1&1\\0&1\end{pmatrix},\ y=\begin{pmatrix}0&1\\-1&-1\end{pmatrix}\rangle$
$(4t)$	$D_{2,16,7}$	$G(32,19)$	$(2,4,16)$	$\langle x,y \mid x^2=y^{16}=1,\ xyx^{-1}=y^7\rangle$
$(4u)$	$S_3\times S_3$	$G(36,10)$	$(2^3,3)$	$\langle x,\ y,\ z,\ w \mid x=(12), y=(123), z=(45), w=(456)\rangle$
$(4v)$	$S_3\times S_3$	$G(36,10)$	$(2,6^2)$	$\langle x,\ y,\ z,\ w \mid x=(12), y=(123), z=(45), w=(456)\rangle$
$(4w)$	$\mathbb{Z}_6\times S_3$	$G(36,12)$	$(2,6^2)$	$\langle z \mid z^6=1\rangle\times\langle x,y \mid x=(12),\ y=(123)\rangle$
$(4x)$	$\mathbb{Z}_3\times A_4$	$G(36,11)$	$(3^2,6)$	$\langle z \mid z^3=1\rangle\times\langle x,\ y \mid x=(12)(34),\ y=(123)\rangle$
$(4y)$	$\mathbb{Z}_4\ltimes(\mathbb{Z}_3)^2$	$G(36,9)$	$(3,4^2)$	$\langle x,y,z \mid x^4=y^3=z^3=1,$ $xyx^{-1}=yz^2,\ xzx^{-1}=y^2z^2,\ [y,z]=1\rangle$
$(4z)$	$D_4\ltimes\mathbb{Z}_5$	$G(40,8)$	$(2,4,10)$	$\langle x,y,z \mid x^2=y^4=z^5=1,$ $xyx^{-1}=y^{-1},\ xzx^{-1}=z,\ yzy^{-1}=z^{-1}\rangle$
$(4aa)$	A_5	$G(60,5)$	$(2,5^2)$	$\langle x,y \mid x=(12)(34),\ y=(12345)\rangle$
$(4ab)$	$\mathbb{Z}_3\times S_4$	$G(72,42)$	$(2,3,12)$	$\langle z \mid z^3=1\rangle\times\langle x,\ y \mid x=(12),\ y=(1234)\rangle$
$(4ac)$	$D_4\ltimes(\mathbb{Z}_3)^2$	$G(72,40)$	$(2,4,6)$	$\langle x,y,z,w \mid x^2=y^4=z^3=w^3=1,$ $xyx^{-1}=y^{-1},\ xzx^{-1}=w,\ yzy^{-1}=w,$ $ywy^{-1}=z^2,\ [z,w]=1\rangle$
$(4ad)$	S_5	$G(120,34)$	$(2,4,5)$	$\langle x,y \mid x=(12),\ y=(12345)\rangle$

TABLE 2. Nonabelian automorphism groups with rational quotient on Riemann surfaces of genus 4.

Case	G	IdSmall Group(G)	$\mathbf{m}$	Presentation
$(5a)$	S_3	$G(6,1)$	$(2^4,3^2)$	$\langle x,y \mid x=(12),\ y=(123)\rangle$
$(5b)$	D_4	$G(8,3)$	(2^6)	$\langle x,y \mid x^2=y^4=1,\ xyx^{-1}=y^{-1}\rangle$
$(5c)$	D_4	$G(8,3)$	$(2^3,4^2)$	$\langle x,y \mid x^2=y^4=1,\ xyx^{-1}=y^{-1}\rangle$
$(5d)$	Q_8	$G(8,4)$	(4^4)	$\langle i,j,k,-1 \mid i^2=j^2=k^2=-1,$ $ij=k,\ jk=i,\ ki=j\rangle$
$(5e)$	D_5	$G(10,1)$	$(2^4,5)$	$\langle x,y \mid x^2=y^5=1,\ xyx^{-1}=y^{-1}\rangle$
$(5f)$	D_6	$G(12,4)$	$(2^4,3)$	$\langle x,y \mid x^2=y^6=1,\ xyx^{-1}=y^{-1}\rangle$
$(5g)$	D_6	$G(12,4)$	$(2^2,6^2)$	$\langle x,y \mid x^2=y^6=1,\ xyx^{-1}=y^{-1}\rangle$
$(5h)$	$D_{4,3,-1}$	$G(12,1)$	$(2,3,4^2)$	$\langle x,y \mid x^4=y^3=1,\ xyx^{-1}=y^{-1}\rangle$
$(5i)$	A_4	$G(12,3)$	(3^4)	$\langle x,y \mid x=(12)(34),\ y=(123)\rangle$
$(5j)$	D_8	$G(16,7)$	(2^5)	$\langle x,y \mid x^2=y^8=1,\ xyx^{-1}=y^{-1}\rangle$
$(5k)$	$\mathbb{Z}_2\times D_4$	$G(16,11)$	(2^5)	$\langle z \mid z^2=1\rangle\times\langle x,\ y \mid x^2=y^4=1,\ xyx^{-1}=y^{-1}\rangle$
$(5l)$	$\mathbb{Z}_4\ltimes(\mathbb{Z}_2)^2$	$G(16,3)$	$(2^2,4^2)$	$\langle x,y,z \mid x^4=y^2=z^2=1,$ $xyx^{-1}=yz,\ [x,z]=[y,z]=1\rangle$
$(5m)$	$D_{2,8,3}$	$G(16,8)$	$(2^2,4^2)$	$\langle x,y \mid x^2=y^8=1,\ xyx^{-1}=y^3\rangle$
$(5n)$	$\mathbb{Z}_2\times D_4$	$G(16,11)$	$(2^2,4^2)$	$\langle z \mid z^2=1\rangle\times\langle x,\ y \mid x^2=y^4=1,\ xyx^{-1}=y^{-1}\rangle$
$(5o)$	$\mathbb{Z}_2\ltimes(\mathbb{Z}_2\times\mathbb{Z}_4)$	$G(16,13)$	$(2^2,4^2)$	$\langle x,y,z \mid x^2=y^2=z^4=1,$ $[x,z]=[y,z]=1,\ xyx^{-1}=yz^2\rangle$
$(5p)$	$D_{2,8,5}$	$G(16,6)$	$(4,8^2)$	$\langle x,y \mid x^2=y^8=1,\ xyx^{-1}=y^5\rangle$
$(5q)$	D_{10}	$G(20,4)$	$(2^3,10)$	$\langle x,y \mid x^2=y^{10}=1,\ xyx^{-1}=y^{-1}\rangle$
$(5r)$	$D_{4,5,-1}$	$G(20,1)$	$(4^2,10)$	$\langle x,y \mid x^4=y^5=1,\ xyx^{-1}=y^{-1}\rangle$
$(5s)$	$\mathbb{Z}_2\ltimes((\mathbb{Z}_2)^2\times\mathbb{Z}_3)$	$G(24,8)$	$(2^3,6)$	$\langle x,y,z,w \mid x^2=y^2=z^2=w^3=1,$ $[x,\ y]=[y,\ z]=[y,\ w]=[z,\ w]=1,$ $xzx^{-1}=zy,\ xwx^{-1}=w^2\rangle$
$(5t)$	$(\mathbb{Z}_2)^2\times S_3$	$G(24,14)$	$(2^3,6)$	$\langle z,w \mid z^2=w^2=[z,w]=1\rangle\times S_3$
$(5u)$	S_4	$G(24,12)$	$(2^2,3^2)$	$\langle x,y \mid x=(12),\ y=(1234)\rangle$
$(5v)$	$\mathbb{Z}_2\times A_4$	$G(24,13)$	$(2^2,3^2)$	$\langle z \mid z^2=1\rangle\times\langle x,y \mid x=(12)(34),\ y=(123)\rangle$
$(5w)$	$\mathbb{Z}_2\times A_4$	$G(24,13)$	$(3,6^2)$	$\langle z \mid z^2=1\rangle\times\langle x,y \mid x=(12)(34),\ y=(123)\rangle$
$(5x)$	$D_{4,6,-1}$	$G(24,7)$	$(4^2,6)$	$\langle x,y \mid x^4=y^6=1,\ xyx^{-1}=y^{-1}\rangle$
$(5y)$	$D_{6,5,-1}$	$G(30,2)$	$(2,6,15)$	$\langle x,y \mid x^6=y^5=1,\ xyx^{-1}=y^{-1}\rangle$
$(5z)$	$\mathbb{Z}_2\ltimes(\mathbb{Z}_2)^4$	$G(32,27)$	$(2^3,4)$	$\langle x,y,z,w,t \mid x^2=y^2=z^2=w^2=t^2=1,$ $[y,\ z]=[x,\ w]=[y,\ w]=[z,\ w]=1$ $[x,\ t]=[y,\ t]=[z,\ t]=[w,\ t]=1$ $[x,y]=w,\ [x,z]=t\rangle$
$(5aa)$	$\mathbb{Z}_2\ltimes(\mathbb{Z}_4\times(\mathbb{Z}_2)^2)$	$G(32,28)$	$(2^3,4)$	$\langle x,y,z,w \mid x^2=y^4=z^2=w^2=1,\ xyx^{-1}=y^{-1},$ $xzx^{-1}=wz,\ [y,\ z]=[y,\ w]=[z,\ w]=[x,\ w]=1\rangle$
$(5ab)$	$\mathbb{Z}_2\ltimes(D_4\times\mathbb{Z}_2)$	$G(32,43)$	$(2^3,4)$	$\langle x,y,z,w \mid x^2=y^2=z^4=w^2=1,$ $xyx^{-1}=yz,\ xzx^{-1}=z^{-1},\ yzy^{-1}=z^{-1},$ $xwx^{-1}=z^2w,\ [y,\ w]=[z,\ w]=1\rangle$
$(5ac)$	$\mathbb{Z}_8\ltimes(\mathbb{Z}_2)^2$	$G(32,5)$	$(2,8^2)$	$\langle x,y,z \mid x^8=y^2=z^2=1,$ $[y,z]=[x,\ z]=1,\ [x,\ y]=z\rangle$
$(5ad)$	$\mathbb{Z}_2\ltimes D_{2,8,5}$	$G(32,7)$	$(2,8^2)$	$\langle x,y,z \mid x^2=y^2=z^8=1,\ yzy^{-1}=z^5,$ $xzx^{-1}=yz,\ [x,\ y]=1\rangle$
$(5ae)$	$\mathbb{Z}_4\ltimes(\mathbb{Z}_4\times\mathbb{Z}_2)$	$G(32,2)$	(4^3)	$\langle x,y,z \mid x^4=y^4=z^2=1,$ $xyx^{-1}=yz,\ [y,\ z]=[x,\ z]=1\rangle$

Table 3 (continued on the next page).

Case	G	`IdSmall Group`(G)	$\mathbf{m}$	Presentation
$(5af)$	$\mathbb{Z}_4 \ltimes (\mathbb{Z}_2)^3$	$G(32,6)$	(4^3)	$\langle x,y,z,w \mid x^4=y^2=z^2=w^2=1,$ $xyx^{-1}=yz,\ xzx^{-1}=zw,$ $[y,\,w]=[z,\,w]=[y,\,z]=[x,\,w]=1\rangle$
$(5ag)$	$D_{2,20,9}$	$G(40,5)$	$(2,4,20)$	$\langle x,y \mid x^2=y^{20}=1,\ xyx^{-1}=y^9\ \rangle$
$(5ah)$	$\mathbb{Z}_2 \times S_4$	$G(48,48)$	$(2^3,3)$	$\langle z \mid z^2=1\rangle \times \langle x,\,y,\ \mid x=(12),\,y=(1234)\rangle$
$(5ai)$	$\mathbb{Z}_2 \ltimes (\mathbb{Z}_{12}\times\mathbb{Z}_2)$	$G(48,14)$	$(2,4,12)$	$\langle x,y,z \mid x^2=y^{12}=z^2=1,$ $xyx^{-1}=y^5z,\ [x,\,z]=[y,\,z]=1\rangle$
$(5aj)$	$\mathbb{Z}_4 \ltimes A_4$	$G(48,30)$	$(3,4^2)$	$\langle x,y,z \mid x^4=1,\ y=(12)(34),\ z=(123),$ $x(123)x^{-1}=(132),\ x(12)(34)x^{-1}=(13)(24)\rangle$
$(5ak)$	$(\mathbb{Z}_2)^2\times A_4$	$G(48,49)$	$(2,6^2)$	$\langle z,\,w \mid z^2=w^2=[z,\,w]=1\rangle\times$ $\langle x,\,y \mid x=(12)(34),\,y=(123)\rangle$
$(5al)$	A_5	$G(60,5)$	$(3^2,5)$	$\langle x,y \mid x=(12)(34),\ y=(12345)\rangle$
$(5am)$	$\mathbb{Z}_4 \ltimes (D_4\times\mathbb{Z}_2)$	$G(64,8)$	$(2,4,8)$	$\langle x,y,z,w \mid x^4=y^2=z^4=w^2=1,$ $xyx^{-1}=yz,\ yzy^{-1}=z^3,\ xzx^{-1}=zw,$ $[x,\,w]=[y,\,w]=[z,\,w]=1\rangle$
$(5an)$	$\mathbb{Z}_4 \ltimes (\mathbb{Z}_2)^4$	$G(64,32)$	$(2,4,8)$	$\langle x,y,z,w,v \mid x^4=y^2=z^2=w^2=1,$ $xyx^{-1}=yz,\ xzx^{-1}=zw,\ xwx^{-1}=wv,$ $[x,\,v]=[y,\,z]=[y,\,w]=[y,\,v]=1,$ $[z,\,w]=[z,\,v]=[w,\,v]=1\rangle$
$(5ao)$	$\mathbb{Z}_5 \ltimes (\mathbb{Z}_2)^4$	$G(80,49)$	$(2,5^2)$	$\langle x,y,z,w,v \mid x^5=y^2=z^2=w^2=1,$ $xyx^{-1}=v,\ xzx^{-1}=yv,\ xwx^{-1}=yzv,$ $xvx^{-1}=yzwv,\ [y,\,z]=[y,\,w]=[y,\,v]=1,$ $[z,\,w]=[z,\,v]=[w,\,v]=1\rangle$
$(5ap)$	$G(96,3)$	$G(96,3)$	$(3^2,4)$	$\langle x,y,z,w,v,u \mid x^3=w^2=v^2=u^2=1,$ $y^2=wv,\ z^2=w,\ xyx^{-1}=z,\ xzx^{-1}=yzv,$ $xwx^{-1}=vu,\ xvx^{-1}=wvu,\ yzy^{-1}=zu,$ $[x,\,u]=[y,\,w]=[y,\,v]=[y,\,u]=1,$ $[z,\,u]=[z,\,w]=[z,\,v]=[w,\,v]=1,\rangle$
$(5aq)$	$S_3 \ltimes (\mathbb{Z}_2)^4$	$G(96,195)$	$(2,4,6)$	$\langle x,y,z,w,v,u \mid x^2=y^3=z^2=w^2=1,$ $v^2=u^2=1,\ xyx^{-1}=y^2,\ xzx^{-1}=zw,\ xvx^{-1}=u,$ $xux^{-1}=v,\ yvy^{-1}=u,\ yuy^{-1}=vu,\ [x,\,w]=1,$ $[y,\,z]=[y,\,w]=[z,\,w]=[z,\,v]=1,$ $[w,\,u]=[w,\,v]=[u,\,v]=1\rangle$
$(5ar)$	$\mathbb{Z}_2 \times A_5$	$G(120,35)$	$(2,3,10)$	$\langle z \mid z^2=1\rangle \times \langle x,y \mid x=(12)(34),\ y=(12345)\rangle$
$(5as)$	$D_5 \ltimes (\mathbb{Z}_2)^4$	$G(160,234)$	$(2,4,5)$	$\langle x,y,z,w,v,u \mid x^2=y^5=z^2=w^2=1,$ $v^2=u^2=1,\ xyx^{-1}=y^{-1},\ xzx^{-1}=zw,$ $yzy^{-1}=zw,\ ywy^{-1}=wv,\ xvx^{-1}=zwu,$ $yvy^{-1}=vu,\ xux^{-1}=zv,\ yuy^{-1}=z,\ [x,\,w]=1,$ $[z,\,w]=[z,\,u]=[z,\,v]=[w,\,u]=[w,\,v]=[v,\,u]=1\rangle$
$(5at)$	$G(192,181)$	$G(192,181)$	$(2,3,8)$	$\langle x,y,z,w,v,u,t \mid x^2=y^3=v^2=u^2=t^2=1,$ $z^2=vu,\ w^2=v,\ xyx^{-1}=y^2,$ $xzx^{-1}=w,\ yzy^{-1}=wt,\ xwx^{-1}=z,$ $ywy^{-1}=zwu,\ xvx^{-1}=vu,$ $yvy^{-1}=ut,\ yuy^{-1}=vut,\ [x,\,u]=[x,\,t]=1,$ $[y,\,t]=[z,\,v]=[z,\,u]=[z,\,t]=1,$ $[v,\,u]=[v,\,t]=[u,\,t]=1\rangle$

Table 3. Automorphism groups with rational quotient on Riemann surfaces of genus 5

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