Take Home Exam
Representation Theory of Groups
Deadline for solutions: 19th of august 2008

1. Let $G = GL_2(\mathbb{F}_{13})$, the group of 2×2 invertible matrices with coefficients in the field with 13 elements and let $H = D_{26}$ be the dihedral group with 52 elements.

   (a) Let B be the subgroup of G of (invertible) upper triangular matrices and let $s = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Show that, as sets, $G = B \cup BsB$ so G is generated by B and s. In particular you might need for the following that G is generated by B, s and sBs^{-1}.

   (b) Let $SL_2(\mathbb{F}_{13})$ be the subgroup of G of matrices of determinant 1 and let $B_1 = B \cap SL_2(\mathbb{F}_{13})$. Show that, as sets, $SL_2(\mathbb{F}_{13}) = B_1 \cup B_1 s B_1$ so it is generated by B_1 and s.

   (c) Show that $[G, G] = SL_2(\mathbb{F}_{13})$.

   (d) List all degree 1 representations of G over the following fields: $k = \mathbb{C}$; $k = \mathbb{R}$; $k = \mathbb{Q}(i, \zeta)$ for ζ a primitive 3-rd root of 1; $\mathbb{F}_2$.

   (e) List all 1-dimensional representations of H over $k = \mathbb{C}$; $k = \mathbb{R}$; $k = \mathbb{Q}(\zeta)$ for ζ a primitive 3-rd root of 1; $\mathbb{F}_2$.

   (f) Determine which of the fields: $k = \mathbb{C}$; $k = \mathbb{R}$; $k = \mathbb{Q}(\nu)$ for ν a primitive 13-th root of 1; $\mathbb{F}_2(\xi)$, where ξ is a primitive 13-th root of 1 is a splitting field for H, motivating your answer.

2. Let G be a group, let $\gamma = \frac{1}{2}(-1 + i\sqrt{7})$ and let the table below be part of its (complex) character table, where one or more irreducible characters are missing.

	1	a	b	c	d	e
χ_1	1	1	1	1	1	1
χ_2	3	-1	γ	$\overline{\gamma}$	1	0
χ_3	3	-1	$\overline{\gamma}$	γ	1	0
χ_4	6	2	-1	-1	0	0
χ_5	8	0	1	1	0	-1

   with $|Cl(a)| = 21$; $|Cl(b)| = |Cl(c)| = 24$, $|Cl(d)| = 42$ and $|Cl(e)| = 56$ for fixed a, b, c, d, $e \in G$. Here $Cl(g)$ denotes the conjugacy class of $g \in G$.

   (a) Determine the order of G, the number and the dimension of each irreducible character of G.

   (b) Complete the character table of G.

   (c) Determine all normal subgroups of G and for each of them determine the character table of the quotient group.

   (d) Determine $[G, G]$ and $Z(G)$.

   (e) Determine whether G is solvable or not;

(f) Determine whether G is nilpotent or not;

(g) Show that a is conjugate to its inverse and that b^{-1} is conjugate to c.

(h) Consider the product of the characters χ_4 and χ_3. Decompose it as a sum of irreducible characters.

3. Let $G = C_6 \ltimes C_{13} = \langle g, h \mid g^6 = h^{13} = 1, \quad gh = h^4 g\rangle$ be a semi-direct product of the cyclic group of order 6 with the cyclic group of order 13, with the above presentation.

   (a) Provide the (complex) character table of G in terms of characters induced by subgroups containing $\langle h\rangle \cong C_{13}$.

   (b) Let $C = \langle g\rangle$ be the subgroup of G generated by g. For every irreducible character χ of G verify whether its restriction to C is irreducible or not.

4. Let $G = S_n$, the symmetric group on n letters. Let λ be a partition of n with associated standard tableau T_λ. Let P_λ and a_λ be as in Fulton-Harris.

   (a) Show that the natural S_n-representation on $\mathbb{C}[S_n]a_\lambda$ given by left multiplication is isomorphic to $\mathrm{Ind}_{P_\lambda}^{S_n}(\rho_0)$ where ρ_0 is the trivial representation of P_λ.

   (b) Determine the degree of the representations in the previous question, for $n = 3, 4$ and every partition of n.

5. Let $G = GL_n(k)$ for a finite field k of characteristic p.

   (a) Show that $g \in G$ is a p-element if and only if all its eigenvalues are equal to 1.

   (b) Show that $g \in G$ is a p-regular element if and only if it is *semisimple*, that is, diagonalizable over a finite extension of k.

   (c) Provide an explicit example of a 3-element and of a 3-regular element in $GL_3(\mathbb{F}_9)$.

   (d) Provide an example of an element $g \in GL_3(\mathbb{F}_9)$ which is neither 3-regular nor a 3-element and decompose it as $g = g_1 g_2$ with g_1 a 3-regular element and g_2 a 3-element commuting with g_1.

6. Let G and H be finite groups and let ρ be a complex representation of G and σ be a complex representation of H, of character χ_ρ and χ_σ, respectively. Consider the representation $\rho \otimes \sigma$ of $G \times H$, with character χ.

   (a) Show that $Z(\chi) = Z(\chi_\rho) \times Z(\chi_\sigma)$.

   (b) Let ρ and σ be irreducible and faithful. Show that $\rho \otimes \sigma$ is irreducible and faithful if and only if $\gcd(|Z(G)|, |Z(H)|) = 1$.

7. Let $G = C_{11}$, the cyclic group of order 11.

(a) Let W be a non-trivial irreducible $\mathbb{C}$-representation of G and let χ be its character. Find an irreducible $\mathbb{Q}$-representation V of G such that W is a composition factor of $V^{\mathbb{C}}$. (Hints: all irreducible representations are components of the regular representation. Splitting a representation of a finite cyclic group is equivalent to putting the image of a generator in block diagonal form. Over the complex field this is equivalent to putting the image of a generator in diagonal form).

(b) Compute $m_{\mathbb{Q}}(\chi)$, the Schur index of χ over $\mathbb{Q}$ using the definition in Isaacs.

(c) Determine $\mathrm{Hom}_{\mathbb{Q}[G]}(V, V)$, $\mathrm{Hom}_{\mathbb{C}[G]}(W, W)$, $\mathrm{Hom}_{\mathbb{C}[G]}(V^{\mathbb{C}}, V^{\mathbb{C}})$.
Compute the characters of V, W and $V^{\mathbb{C}}$.

(d) Compute the Schur index m_V of V using the definition in Serre.