Spherical conjugacy classes and involutions in the Weyl group

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Abstract

Let G be a simple algebraic group over an algebraically closed field of characteristic zero or positive odd, good characteristic. Let B be a Borel subgroup of G. We show that the spherical conjugacy classes of G intersect only the double cosets of B in G corresponding to involutions in the Weyl group of G. This result is used in order to prove that for a spherical conjugacy class $\mathcal{O}$ with dense B-orbit $v_0 \subset BwB$ there holds $\ell(w) + \mathrm{rk}(1-w) = \dim \mathcal{O}$ extending to the case of groups over fields of odd, good characteristic a characterization of spherical conjugacy classes obtained by Cantarini, Costantini and the author. It is also shown that the weights occurring in the G-module decomposition of the ring of regular functions on $\mathcal{O}$ are self-adjoint and they lie in the -1-eigenspace of the element w.

Introduction

If an algebraic group acts with finitely many orbits, a natural way to understand the action is given by the combinatorics of the Zariski closures of such orbits. In [25], [29], a detailed description of the combinatorics of the closures of orbits for a Borel subgroup B in a symmetric space G/K is given. The description is provided in terms of an action, on the set of these orbits, of a monoid $M(W)$ related to the Weyl group W of G. This action is best understood considering the decomposition into B-orbits of an orbit of a minimal parabolic subgroup. Through this approach several invariants of the B-orbits can be determined, including their dimension. To each B-orbit it is possible to associate a Weyl group element and the Weyl group element corresponding to the (unique) dense B-orbit in the symmetric space can be described in combinatorial terms. A formula for the dimension of each B-orbit v is provided in terms of its associated Weyl group element and the sequence of elements in the monoid that are necessary to reach v from a closed B-orbit. When the symmetric space corresponds to an inner involution, that is, if it corresponds to a conjugacy class in G, the attached Weyl group element is just the element corresponding to the Bruhat cell containing the B-orbit.

The monoid action can be carried over to homogeneous spaces of algebraic groups for which the action of the Borel subgroup has finitely many orbits, i.e., the spherical homogeneous spaces

([28]) and it can be used to define representations of the Hecke algebra ([18]). When the homogeneous space is a conjugacy class the natural map from the set of B-orbits to the Weyl group given in terms of the Bruhat decomposition is still defined. A more geometric approach to a Bruhat order on spherical varieties has been addressed in [7]. Besides, a genuine Weyl group action on the set of B-orbits on a spherical homogeneous space was defined in [17].

The action of $M(W)$ on a spherical homogeneous space does not afford all nice properties that it had in the symmetric case (see [8] for some key counterexamples) and it is natural to ask which properties still hold for spherical conjugacy classes. One of the main differences between the general spherical case and the symmetric case is that there are B-orbits that do not lie in the $M(W)$-orbit of a closed one. However, every B-orbit can be reached from a closed B-orbit through a sequence of moves involving either the $M(W)$-action or the W-action ([28]).

A natural question is whether we can provide formulas for the dimension of each B-orbit in a spherical conjugacy class in terms of the actions of $M(W)$ and W. Although not all results in [25] hold at this level of generality, there are properties that hold true in general. For instance, the dimension of the dense B-orbit in a spherical conjugacy class is governed by a formula analogous to the formula for the dimension of the dense B-orbit in a symmetric conjugacy class. This result, when the base field is $\mathbb{C}$, was achieved in [10], leading to a characterization of spherical conjugacy classes in complex simple algebraic groups. The interest in this formula lied in the verification of De Concini-Kac-Procesi conjecture on the dimension of irreducible representations of quantum groups at the roots of unity ([11]) in the case of spherical conjugacy classes. For this reason, the analysis was restriced to the case of an algebraic group over an algebraically closed field of characteristic zero. In order to obtain the characterization, a classification of all spherical conjugacy classes in a simple algebraic group was needed, and part of the results were obtained through a case-by-case analysis involving this classification.

In the present paper we apply the combinatorics of $M(W)$-action and W-action on the set of B-orbits of a spherical conjugacy class in a simple algebraic group to retrieve the formula in [10]. This will show that the characterization of spherical conjugacy classes can be achieved without using their classification and without drastic restrictions on the characteristic of the base field.

A first question to be answered concerns which Bruhat cell may contain a B-orbit of a spherical conjugacy class. In the case of a symmetric conjugacy class it is immediate to see that the corresponding Weyl group elements are involutions. An analysis of the actions of $M(W)$ and W allows us to generalize this result to all spherical conjugacy classes.

Theorem 1 *All B-orbits in a spherical conjugacy class lie in Bruhat cells corresponding to involutions in the Weyl group.*

In order to understand the Weyl group elements associated with the dense B-orbit we analyze the variation of the Weyl group element with respect to the action of the monoid $M(W)$. This analysis leads to a description of the stationary points, i.e., of those B-orbits for which the associated Weyl group element does not change under the action of all standard generators of $M(W)$. Stationary points other than the dense B-orbit do not exist in symmetric conjugacy classes but they exist, for instance, in spherical unipotent conjugacy classes.

The results in [29] allow us to describe the Weyl group element corresponding to a stationary point, and more precisely, the one associated with the dense B-orbit.

Combining the analysis of representatives of the dense B-orbit with Theorem 1 yields a new proof of the formula in [10], that holds now in almost all characteristcs and does not require the classification of spherical conjugacy classes:

Theorem 2 *Let $\mathcal{O}$ be a spherical conjugacy class in a simple algebraic group G, let v_0 be its dense B-orbit and let $BwB \supset v_0$. Then $\dim \mathcal{O} = \ell(w) + \mathrm{rk}(1-w)$.*

It is proved in [10] with a characteristic-free argument that if a conjugacy class $\mathcal{O}$ intersects some BwB with $\ell(w) + \mathrm{rk}(1-w) = \dim \mathcal{O}$ then $\mathcal{O}$ is spherical, hence the results in the present paper provide a characteristic-free proof of the characterization of spherical conjugacy class given in [10].

The element w corresponding to the dense B-orbit plays a role in the G-module decomposition of the ring $k[\mathcal{O}]$ of regular functions on $\mathcal{O}$, which is multiplicity-free ([9],[15]). Indeed, $w = w_0 w_\Pi$, the product of the longest element in W and the longest element of a suitable parabolic subgroup W_Π of W. All weights of eigenvectors of the B-action on the function field $k(\mathcal{O})$ are orthogonal to the root subsystem $\Phi(\Pi)$ and we have:

Theorem 3 *Let $\mathcal{O}$ be a spherical conjugacy class in a simple algebraic group G, let v_0 be its dense B-orbit and let $BwB \supset v_0$. The weights occurring in the G-module decomposition of $k[\mathcal{O}]$ are self-adjoint and lie in $P^+ \cap Q \cap \mathrm{Ker}(1+w)$.*

Explicit examples of the G-module decomposition of $k[\mathcal{O}]$ can be found in [1], [20], [23], [32].

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1 Preliminaries

Let G be a simple algebraic group over an algebraically closed field k of characteristic 0 or odd and good ([30, §4.3]). Let B be a Borel subgroup of G, let T be a maximal torus contained in B and B^- the Borel subgroup opposite to B. Let U (respectively U^-) be the unipotent radical of B (respectively B^-). For an algebraic group K we shall denote by K° its identity component.

We shall denote by Φ the set of roots relative to (B,T); by Φ^+ the corresponding positive roots; by $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ the corresponding set of simple roots. We shall use the numbering of the simple roots in [4, Planches I-IX]. The height of a root α will be indicated by $\mathrm{ht}(\alpha)$. We shall indicate by P^+ and Q, respectively, the set of dominant weights and the root lattice associated with Φ and (B,T). For a co-character $\alpha^\vee \colon k^\cdot \to G$ and a nonzero scalar $h \in k$ we shall denote by $\alpha^\vee(h) \in T$ the image of h through $\alpha^\vee$.

We shall denote by W the Weyl group associated with G and by s_α the reflection corresponding to the root α. By $\ell(w)$ we shall denote the length of the element $w \in W$ and by $\mathrm{rk}(1-w)$ we shall mean the rank of $1-w$ in the standard representation of the Weyl group. By w_0 we shall denote the longest element in W and by ϑ we shall denote the automorphism of Φ given by $-w_0$. By Π we shall always denote a subset of Δ and $\Phi(\Pi)$ will indicate the corresponding root subsystem. We shall denote by W_Π the parabolic subgroup of W generated by the simple reflections in Π. Given an element $w \in W$ we shall denote by $\dot{w}$ a representative of w in the normalizer $N(T)$ of T. For any root α in Φ we shall denote by $x_\alpha(t)$ the elements of the corresponding root subgroup X_α of G.

We assume that we have fixed an ordering of the positive roots so that every $u \in U$ is written uniquely as an ordered product of elements of the form $x_\alpha(l)$, for $l \in k$ and $\alpha \in \Phi^+$. Given an element $u \in U$ by abuse of language we will say that a root $\gamma \in \Phi^+$ *occurs in* u if for the expression of u as an ordered product of $x_\alpha(l_\alpha)$'s we have $l_\gamma \neq 0$. If $\alpha \in \Delta$ we shall indicate by P_α the minimal non solvable parabolic subgroup containing $X_{-\alpha}$ and by P_α^u its unipotent radical.

For $w \in W$, we shall denote by U^w (respectively, U_w) the subgroup generated by the root subgroups X_α corresponding to those $\alpha \in \Phi^+$ for which $w^{-1}(\alpha) \in -\Phi^+$ (respectively, Φ^+). We shall denote by T^w the subgroup of the torus that is centralized by any representative $\dot{w}$ of w.

Given an element $x \in G$ we shall denote by $\mathcal{O}_x$ the conjugacy class of x in G and by G_x (resp. B_x, resp. T_x) the centralizer of x in G (resp. B, resp. T). The center of a group H will be indicated by $Z(H)$. For a conjugacy class $\mathcal{O} = \mathcal{O}_x$ we shall denote by $\mathcal{V}$ the set of B-orbits into which $\mathcal{O}$ can be decomposed.

Definition 1.1 *Let K be a connected algebraic group over k. A homogeneous K-space is called spherical if it has a dense orbit for some Borel subgroup of K.*

It is well-known ([6], [31] in characteristic 0, [12], [17] in positive characteristic) that X is a spherical homogeneous G-space if and only if the set of B-orbits in X is finite.

2 B-orbits and Bruhat decomposition

Let $\mathcal{O}$ be a conjugacy class of G and let $\mathcal{V}$ be the set of B-orbits in $\mathcal{O}$. There is a natural map $\phi\colon \mathcal{V} \to W$ associating to $v \in \mathcal{V}$ the element w in the Weyl group of G for which $v \subset BwB$. The set $\mathcal{V}$ carries a partial order given by: $v \leq v'$ if $\overline{v} \subset \overline{v'}$. If $\mathcal{O}$ is spherical the minimal B-orbits are the closed ones and there is a unique maximal orbit, namely the dense B-orbit v_0 in $\mathcal{O}$.

Lemma 2.1 *Let $\mathcal{O}$ be a conjugacy class and let $v, v' \in \mathcal{V}$. If $v \leq v'$ then $\phi(v) \leq \phi(v')$ in the Bruhat order in W.*

Proof. We have: $v \subset \overline{v} \subset \overline{v'} \subset \overline{B\phi(v')B} = \dot{\cup}_{\sigma \leq \phi(v')} B\sigma B$ so $\phi(v) \leq \phi(v')$. $\square$

Lemma 2.2 *Let $x \in G$ be either semisimple or unipotent and let $\mathcal{O}_x$ be a spherical conjugacy class. The image through ϕ of a closed B-orbit in $\mathcal{O}$ is 1.*

Proof. If $\mathcal{O} = \mathcal{O}_x$ and $H = G_x$, the B-orbit of gxg^{-1} corresponds to the double coset BgH through the natural morphism from G to $\mathcal{O}_x$ mapping g to gxg^{-1}. Borrowing an argument from [28, §3.4 (b)] we see that the closed B-orbits correspond to closed double cosets BgH so that $B_H = (H \cap g^{-1}Bg)$ is a Borel subgroup of H and of H°. Let x be semisimple. Since it is not restrictive to assume that G is simply connected, we have $H = H^\circ$ and $x \in Z(H) = Z(B_H)$ by [27, Corollary 6.2.9]. Hence, the representative gxg^{-1} of the closed B-orbit lies in B.

Let x be unipotent. By [30, §3.15] with $S = \{x\}$ we have $x \in H^\circ$, hence $x \in Z(H^\circ) \subset B_H$ and the statement follows. $\square$

Remark 2.3 All closed B-orbits in a spherical conjugacy class $\mathcal{O}_x$ have the same dimension ([28, §3.4 (b)]) namely $\dim B - \dim B^\circ_{G_x}$ where B_{G_x} denotes a Borel subgroup in the centralizer of x.

Remark 2.4 The converse of Lemma 2.2 does not hold for spherical unipotent elements. For instance, if $\mathcal{O}$ is a spherical unipotent conjugacy class in $G = SL_n(\mathbb{C})$ the combinatorics of the closures of the B-orbits that are contained in B is described in [19]: if $\mathcal{O}$ is the minimal unipotent orbit in G the B-orbits that are contained in B are in bijection with the transpositions in S_n, and only $B.x_\beta(1) = X_\beta \setminus \{1\}$, for β the highest root in Φ, is closed.

Remark 2.5 In a spherical semisimple conjugacy class, v being closed is equivalent to $\phi(B.x) = 1$ and to $v \cap T \neq \emptyset$. Indeed if v is closed then v is contained in B so a representative $x \in v$ is conjugate in B to some element in T. Viceversa, if $v = B.t$ for some $t \in T$ then t normalizes B and by [3, Theorem 9.2] the conjugacy class $B.t$ is closed.

Let $M = M(W)$ be the monoid with elements $m(w)$ indexed by the elements $w \in W$ with relations

$$m(s)m(w) = m(sw), \text{ if } \ell(sw) > \ell(w), \; m(s)m(w) = m(w), \text{ if } \ell(sw) < \ell(w).$$

The monoid $M(W)$ is generated by the elements $m(s)$ corresponding to simple reflections, subject to the braid relations and to the relation $m(s)^2 = m(s)$. In [25] an action of the monoid $M(W)$ on the set of B-orbits of a symmetric space is defined. This action can be generalized to an action of $M(W)$ on the set $\mathcal{V}$ of B-orbits of a spherical homogeneous space (see, for instance, [28, §3.6]). The action of $m(s)$, for a simple reflection $s = s_\alpha$ is given as follows. If P_α is the minimal parabolic subgroup corresponding to α and $v \in \mathcal{V}$ then $m(s).v$ is the dense B-orbit in $P_\alpha v$. This action is analyzed in [8], [17],[18, §4.1], [25]. We provide an account of the information we will need.

Given $v \in \mathcal{V}$, choose $y \in v$ with stabilizer $(P_\alpha)_y$ in P_α. Then $(P_\alpha)_y$ acts on $P_\alpha/B \cong \mathbb{P}^1$ with finitely many orbits. Let $\psi\colon (P_\alpha)_y \to PGL_2(k)$ be the corresponding group morphism. The kernel of ψ is $\mathrm{Ker}(\alpha)P^u_\alpha$. The image H of $(P_\alpha)_y$ in $PGL_2(k)$ is either: $PGL_2(k)$; or solvable and contains a nontrivial unipotent subgroup; or a torus; or the normalizer of a torus. Here is a list of the possibilities that may occur.

- I $P_\alpha v = v$ so $H = PGL_2(k)$;
- IIa $P_\alpha v = v \cup m(s)v$, with $\dim v = \dim P_\alpha v - 1$. We may choose $y \in v$ such that $\psi(X_\alpha) \subset H \subset \psi(B)$.
- IIb $P_\alpha v = v \cup v'$, with $\dim v' = \dim v - 1$ and v is open in $P_\alpha v$ so $m(s)v = v$. We may choose $y \in v$ such that $\psi(X_{-\alpha}) \subset H \subset \psi(B^-)$.
- IIIa $P_\alpha v = v \cup v' \cup m(s)v$, with $\dim v = \dim v' = \dim P_\alpha v - 1$ and $v \neq v'$. We may choose $y \in v$ such that $H = \psi(T)$.
- IIIb $Pv = v \cup v' \cup v''$, with $\dim v - 1 = \dim v' = \dim v''$ and v is open in $P_\alpha v$ so $m(s)v = v$. We may choose $y \in v$ such that $H = \psi(\dot{s}_\alpha x_\alpha(-1) T x_\alpha(1) \dot{s}_\alpha^{-1})$.

IVa $Pv = v \cup m(s)v$, with $\dim v = \dim P_\alpha v - 1$. We may choose $y \in v$ such that $H = \psi(N(T))$.

IVb $Pv = v \cup v'$, with $\dim v = \dim v' + 1$, and v is open in $P_\alpha v$ so $m(s)v = v$. We may choose $y \in v$ such that $H = \psi(N(\dot{s}_\alpha T \dot{s}_\alpha^{-1}))$.

Based on the structure of H, cases II, III, and IV are also called type U, type T and type N, respectively.

A W-action on $\mathcal{V}$ can be defined ([17], [18, §4.2.5, Remark]) as follows: in case II the two B-orbits are interchanged; in case III the two non-dense orbits are interchanged, in all other cases the B-orbits are fixed. The image of $v \in \mathcal{V}$ through the action of a simple reflection $s \in W$ will be denoted by $s.v$.

We recall ([28, §3.6]) that a *reduced decomposition* of $v \in \mathcal{V}$ is a pair $(\mathbf{v}, \mathbf{s})$ with $\mathbf{v} = (v(0), v(1), \ldots, v(r))$ a sequence of distinct elements in $\mathcal{V}$ and $\mathbf{s} = (s_{i_1}, \ldots, s_{i_r})$ a sequence of simple reflections such that: $v(0)$ is closed; $v(j) = m(s_{i_j}).(v(j-1))$ for $1 \leq j \leq r-1$; $\dim(v(j)) = \dim(v(j-1)) + 1$ and $v(r) = v$.

All B-orbits in a symmetric homogeneous space admit a reduced decomposition ([25, §7]). This is still the case for the dense B-orbit in spherical homogeneous spaces but it is not always the case for general B-orbits. The reader can refer to [8] for a series of counterexamples. We will use a weaker notion of decomposition that exists for every $v \in \mathcal{V}$.

Given a reduced decomposition $(\mathbf{v}, \mathbf{s}) = ((v(0), \ldots, v(r)), (s_{i_1}, \ldots, s_{i_r}))$ of $v \in \mathcal{V}$ a *subexpression* of $(\mathbf{v}, \mathbf{s})$ ([28, §3.6]) is a sequence $\mathbf{x} = (v'(0), v'(1), \ldots, v'(r))$ of elements in $\mathcal{V}$ with $v'(0) = v(0)$ and such that for $1 \leq i \leq r$ only one of the following alternatives occurs:

(a) $v'(j-1) = v'(j)$;

(b) $v'(j-1) \neq v'(j)$, $\dim v'(j-1) = \dim v'(j)$ and $v'(j) = s_{i_j}.v'(j-1)$;

(c) $\dim v'(j-1) = \dim v'(j) - 1$ and $v'(j) = m(s_{i_j}).(v'(j-1))$.

The element $v'(r)$ is called the *final term* of the subexpression. Even though some B-orbits in a spherical homogeneous space might not have a reduced decomposition, every $v \in \mathcal{V}$ is the final term of a subexpression of a reduced decomposition of the dense B-orbit v_0. This is to be found in [28, §3.6 Proposition 2] and it holds also in positive odd characteristic.

Lemma 2.6 *Let $\mathcal{O}$ be a spherical conjugacy class, let $v \in \mathcal{V}$ and let $s = s_\alpha$ be a simple reflection. If $w = \phi(v)$ is an involution then $w' = \phi(m(s).v)$ is an involution.*

Proof. We consider $P_\alpha v = v \cup (BsB).v$. If $m(s)v = v$ there is nothing to prove. Let us assume that $m(s)v \subset (Bs).v$. Then $m(s).v \subset BsBwBsB$. The following four possibilities may occur.

1. $\ell(sws) = \ell(w) + 2$. By [27, Lemma 8.3.7] we have $w' = sws$ and the statement holds.

2. $\ell(sw) > \ell(w)$ and $\ell(sws) = \ell(w)$. Since w is an involution, by [29, Lemma 3.2 (ii)] with $\theta = \mathrm{id}$ we have $sw = ws$. Besides, by [27, Lemma 8.3.7] we have

$$m(s).v \subset BswBsB = BswB \cup BswsB$$

hence $w' \in \{sw, w\}$ is an involution.

3. $\ell(sw) < \ell(w)$ and $\ell(sws) = \ell(w)$. Again by [29, Lemma 3.2 (i)] with $\theta = \mathrm{id}$ we have $sw = ws$. Then
$$m(s).v \subset BsBwBsB \subset BswsB \cup BswB$$
so $w' \in \{w, ws\}$ is an involution.

4. $\ell(sws) = \ell(w) - 2$. We have
$$m(s).v \subset BsBwBsB \subset BswB \cup BswsB \cup BwsB \cup BwB.$$
By [28, §3.6 Proposition 1 (a)] we have $v \leq m(s).v$ so by Lemma 2.1 there holds $w \leq w'$ hence $w' \neq sw, ws, sws$ and $w' = w$ is an involution. □

Theorem 2.7 *Let $\mathcal{O}$ be a spherical conjugacy class, and let $\phi\colon \mathcal{V} \to W$ be the natural map. Then the image of ϕ consists of involutions.*

Proof. We first consider a spherical semisimple conjugacy class. Let $v \in \mathcal{V}$ and let $\mathbf{x}$ be a subexpression of a reduced decomposition of the dense B-orbit v_0 with initial term a closed B-orbit $v(0)$ and final term v. We proceed by induction on $\dim v$. If v has minimal dimension then it is closed, otherwise it would contain in its closure a B-orbit of strictly smaller dimension. It follows from Lemma 2.2 that $\phi(v) = 1$.

Let us assume then that $\dim v(r) > \dim v(0)$. If $v(r) = v(r-1)$ we may shorten the sequence replacing r by $r-1$. Hence we may assume that $v(r) = m(s)v(r-1)$ or $v(r) = s.v(r-1)$ for some simple reflection $s = s_\alpha$. If $v(r) = m(s)v(r-1)$ then $\dim v(r)$ is strictly larger than $\dim v(r-1)$ so we may use the induction hypothesis and Lemma 2.6. Let us assume that $v(r) = s.v(r-1)$. Then $\dim(v(r-1)) = \dim v(r)$. If we proceed downwards along the terms of the subexpression we might have a sequence of steps in which either the B-orbit does not change or it changes through the W-action, but we will eventually reach a step at which $v(j) = m(s')(v(j-1))$ with $\dim(v(j)) > \dim v(j-1)$, where we can apply Lemma 2.6. Hence there is a B-orbit $v' \neq v$ in the sequence with $\dim v' = \dim v$ and $\phi(v') = w$ an involution. Therefore we may reduce to the case in which $v' = v(r-1)$ and $v = v(r) = s.v(r-1)$ with $v(r-1) \neq v(r)$. The analysis of the decomposition into B-orbits of $P_\alpha v$ shows that we are in case IIIa. Then $v \subset B\dot{s}v'\dot{s}^{-1}B \subset BsBwBsB$.

If $\ell(sws) = \ell(w) + 2$ then $v \subset BswsB$ and we have the statement.

If $sw > w$ and $\ell(sws) = \ell(w)$ then $sw = ws$ and $v \subset BswB \cup BswsB$ so $\phi(v)$ is an involution.

If $sw < w$ and $\ell(w) = \ell(sws)$ then $sw = ws$ and $v \subset BwB \cup BswB$ so $\phi(v)$ is an involution.

Let us assume that $\ell(sws) = \ell(w) - 2$. It follows from the proof of Lemma 2.1 in this case that $\phi(m(s)v) = \phi(m(s)v') = w = \phi(v')$. By [14, Lemma 1.6] we have $w^{-1}\alpha = w\alpha \in -\Phi^+$. Thus X_α lies in U^w and we may choose representatives $x = u_x\dot{w}\mathrm{v}_x$ and $y = u_y\dot{w}\mathrm{v}_y$ of the same B-orbit v' with $u_x, u_y \in U^w$ and $\mathrm{v}_x, \mathrm{v}_y \in U$ for which $u_x,\ \mathrm{v}_y \in P_\alpha^u$.

Conjugation by $\dot{s}$ maps x in $BswB \cup BswsB$, hence $\dot{s}x\dot{s}^{-1} \in v$ and $\phi(v) \in \{sw, sws\}$. On the other hand, conjugation by $\dot{s}$ maps y in $BswsB \cup BwsB$, hence $\dot{s}y\dot{s}^{-1} \in v$ and $\phi(v) \in \{sw, sws\} \cap \{ws, sws\}$ is an involution. Thus we have the statement for spherical semisimple conjugacy classes.

Let us consider the spherical conjugacy class of an element $x \in G$ with Jordan decomposition su. The proof will follow by induction as in the previous case once we show that the image through ϕ of a closed B-orbit is an involution. As in the proof of Lemma 2.2, if $y = gxg^{-1}$ is a representative of a closed B-orbit then $(G_x \cap g^{-1}Bg)^\circ$ is a Borel subgroup of G_x° and $u \in Z(G_x^\circ)$ by [30, §3.15]. Thus the unipotent part gug^{-1} of y lies in B and $\phi(B.y) = \phi(B.gsg^{-1})$. The conjugacy class $\mathcal{O}_s$ is spherical because if $BgG_x = Bg(G_s \cap G_u)$ is dense in G then BgG_s is dense in G. By the first part of the proof $\phi(B.y)$ is an involution. $\square$

Remark 2.8 The reader is referred to [10, §1.4, Remark 4] for a different proof, in characteristic zero, that the image through ϕ of the dense B-orbit v_0, denoted by $z(\mathcal{O})$, is an involution. In the same paper $\phi(v_0)$ for all spherical conjugacy classes of a simple algebraic group over $\mathbb{C}$ is explicitely computed.

3 Stationary points

In this Section we shall analyze those elements in $v \in \mathcal{V}$ for which $\phi(m(s)v) = \phi(v)$ for all simple reflections $s \in W$.

Definition 3.1 *Let $v \in \mathcal{V}$, let $w = \phi(v)$ and let α be a simple root. We say that v is a* stationary point with respect to α *if $\phi(v) = \phi(m(s_\alpha)v)$. We say that v is a* stationary point *if it is a stationary point with respect to all simple roots.*

It follows from the results in [25] that stationary points different from the dense B-orbit do not exist in symmetric conjugacy classes. They do exist in unipotent spherical conjugacy classes.

Example 3.2 Let $\mathcal{O}_{\min}$ be the minimal nontrivial unipotent conjugacy class in a group G of semisimple rank at least 2. It is well-known that $\mathcal{O}_{\min}$ is spherical. If β denotes the highest root in Φ then $B.x_\beta(1) = X_\beta \setminus \{1\}$ is a stationary point. Indeed, if α is a simple root, $P_\alpha = Bs_\alpha X_\alpha \cup B$ and $P_\alpha.x_\beta(t) \subset B.X_{s_\alpha(\beta)} \cup B.x_\beta(1) \subset B$ so $\phi(B.x_\beta(1)) = \phi(m(s_\alpha)(B.x_\beta(1)) = 1$.

The following lemmas describe stationary points with respect to a simple root.

Lemma 3.3 *Let $v \in \mathcal{V}$ with $w = \phi(v)$. Let α be a simple root such that $s_\alpha w < w$ in the Bruhat order. Then v is a stationary point with respect to α.*

Proof. Let us put $s = s_\alpha$. If $sw < w$ then $\phi(m(s)v) \in \{sws, sw, ws, w\}$ and $\phi(m(s)v) \geq \phi(v)$. If $sw = ws$ the statement follows because $ws < w$ and $sws = w$. Otherwise it follows because $sw, ws, sws < w$. $\square$

Lemma 3.4 *Let $v \in \mathcal{V}$ with $w = \phi(v)$. Let α be a simple root such that $s_\alpha w > w$ in the Bruhat order. Let $x = u\dot{w}\mathrm{v} \in v$ with $u \in U^w$, $\dot{w} \in N(T)$ and $\mathrm{v} \in U$. Then v is a stationary point with respect to α if and only if the following conditions hold:*

1. $s_\alpha w = ws_\alpha$;
2. $\mathrm{v} \in P_\alpha^u$, *the unipotent radical of* P_α;
3. $X_{\pm\alpha}$ *commutes with* $\dot{w}$.

Proof. Let v be a stationary point with respect to α. Theorem 2.7 ensures that w is an involution. We have either $s_\alpha w s_\alpha > s_\alpha w$ or $s_\alpha w s_\alpha = w$. If the first case were possible, we would have $\dot{s}_\alpha x \dot{s}_\alpha^{-1} \in B s_\alpha w s_\alpha B$ and v would not be a stationary point. Hence 1 holds, $w\alpha = \alpha$ and α does not occur in $u \in U^w$.

Let us consider $y = \dot{s}_\alpha x \dot{s}_\alpha^{-1}$. The element

$$y = (\dot{s}_\alpha u \dot{s}_\alpha^{-1})(\dot{s}_\alpha \dot{w} \dot{s}_\alpha^{-1})(\dot{s}_\alpha \mathrm{v} \dot{s}_\alpha^{-1}) \in B\dot{w}(\dot{s}_\alpha \mathrm{v} \dot{s}_\alpha^{-1}).$$

If α would occur in v then by [27, Lemmas 8.1.4, 8.3.7] we would have $y \in Bws_\alpha B \cap P_\alpha v$ with $ws_\alpha = s_\alpha w > \phi(m(s_\alpha)v)$, a contradiction. Hence 2 holds for any representative x.

Let then $x \in v$, let $l \in k^{\cdot}$ and let $x_1 = x_\alpha(l) x x_\alpha(-l) = u_1 \dot{w}_1 \mathrm{v}_1$. Since $\alpha \in \Delta$ and $u, \mathrm{v} \in P_\alpha^u$ we have

$$\dot{w}_1 = \dot{w} \quad \text{and} \quad \mathrm{v}_1 = (\dot{w}^{-1} x_\alpha(l) \dot{w}) \mathrm{v} x_\alpha(-l) \in P_\alpha^u.$$

By Chevalley's commutator formula $\mathrm{v}_1 \in P_\alpha^u$ only if $\dot{w}^{-1} x_\alpha(l) \dot{w} = x_\alpha(l)$, that is, only if 3 holds for X_α.

Let $l \in k^{\cdot}$ and let $x_2 = x_{-\alpha}(l) x x_{-\alpha}(-l)$. Since $\alpha \in \Delta$ and $u, \mathrm{v} \in P_\alpha^u$ the element $x_{-\alpha}(l) u x_{-\alpha}(-l)$ lies in U so

$$x_2 \in U\dot{w}(\dot{w}^{-1} x_{-\alpha}(l) \dot{w}) \mathrm{v} x_{-\alpha}(-l).$$

If $(\dot{w}^{-1} x_{-\alpha}(l) \dot{w}) \mathrm{v} x_{-\alpha}(-l)$ would not lie in U we would have $\phi(B.x_2) = ws_\alpha > w$, a contradiction. Hence $(\dot{w}^{-1} x_{-\alpha}(l) \dot{w}) \mathrm{v} x_{-\alpha}(-l) \in U$. By Chevalley's commutator formula this is possible only if $\dot{w}^{-1} x_{-\alpha}(l) \dot{w} = x_{-\alpha}(l)$, that is, only if 3 holds for $X_{-\alpha}$.

Let x satisfy 2, and 3. Then $P_\alpha v = P_\alpha x \subset B\dot{s}_\alpha X_\alpha.x \cup v$. Properties 2 and 3 imply that $B\dot{s}_\alpha X_\alpha.x \subset BwB$ so v is stationary. $\square$

Lemma 3.5 *Let $\mathcal{O}$ be a spherical conjugacy class, let $v \in \mathcal{V}$ be a stationary point and let $w = \phi(v)$. Let $\Pi = \{\alpha \in \Delta \mid w(\alpha) = \alpha\}$ and w_Π be the longest element in W_Π. Then $w = w_\Pi w_0$.*

Proof. By Theorem 2.7 the element $w \in W$ is an involution. By Lemma 3.4 if $\alpha \in \Delta$ and $w\alpha \in \Phi^+$ then $w\alpha = \alpha$. The statement follows from [29, Proposition 3.5]. $\square$

Example 3.6 Let G be of type A_n and let $\mathcal{O}$ be a spherical conjugacy class. Then the image through ϕ of the dense B-orbit is $w = w_0 w_\Pi$ for some $\Pi \subset \Delta$. The set Π must be stabilized by $\vartheta = -w_0$ because for $\alpha \in \Pi$ we have

$$\alpha = w\alpha = w_0 w_\Pi \alpha \in -w_0 \Pi.$$

Besides, if α_j lies in Π then $\alpha_j = w(\alpha_j) = -w_\Pi(\alpha_{n-j+1})$ so α_j and α_{n-j+1} must lie in the same connected component of Π. Hence, $\Pi = \{\alpha_t, \alpha_{t+1}, \cdots, \alpha_{n-t+1}\}$ for some t and

$$w = (s_{\beta_t} \cdots s_{\beta_{[\frac{n}{2}]}}) w_0 = s_{\beta_1} \cdots s_{\beta_{t-1}}$$

where $\beta_1, \cdots, \beta_{[\frac{n}{2}]}$ is the sequence given by the highest root, the highest root of the root system orthogonal to β_1, and so further.

The argument in the example above shows that, for any stationary point, Π has to be invariant with respect to $\vartheta = -w_0$. Besides, the restriction of w_0 to $\Phi(\Pi)$ always coincides with w_Π.

If $w = \phi(v)$ for some stationary point $v \in \mathcal{V}$, the involution w may be written as a product of reflections with respect to $\mathrm{rk}(1-w)$ mutually orthogonal roots $\gamma_1, \ldots, \gamma_m$ ([24, Page 910]). Thus, U_w (notation as in Section 1) is the subgroup generated by the root subgroups X_γ with $(\gamma, \gamma_j) = 0$ for every j. In other words, U_w is the subgroup generated by X_γ for $\gamma \in \Phi(\Pi)$ and U^w is normalized by U_w.

4 The dense B-orbit

We shall turn our attention to the special stationary point given by the dense B-orbit v_0. We will first analyze the possible Π for which $\phi(v_0) = w_0 w_\Pi$. These are subsets of Δ for which the restriction of w_0 to $\Phi(\Pi)$ coincides with the longest element w_Π of the parabolic subgroup W_Π of W. Next step will be to show which connected components of Π may not consist of isolated roots.

Lemma 4.1 *Let $\mathcal{O}$ be a spherical conjugacy class, let v_0 be its dense B-orbit and let $w = w_0 w_\Pi = \phi(v_0)$. Let α and β be simple roots with the following properties:*

- $(\beta,\, \beta) = (\alpha,\, \alpha)$*;*
- $w_0(\beta) = -\beta$*;*
- $\beta \not\perp \alpha$*;*
- $\beta \perp \alpha'$ *for every* $\alpha' \in \Pi \setminus \{\alpha\}$.

Then $\{\alpha\}$ cannot be a connected component of Π.

Proof. Let us assume that, in the hypothesis of the Lemma, $\{\alpha\}$ is a connected component of Π so that w is the product of $w_0 s_\alpha$ with the longest element $w_{\Pi'}$ of the parabolic subgroup $W_{\Pi'}$ of W associated with the complement Π' of α in Π. Let us choose a representative of v_0 of the form $x = \dot{w}\mathrm{v}$. We claim that $\mathrm{v} \in P^u_\beta$. Otherwise, given a representative $\dot{s}_\beta$ of s_β in $N(T)$, we consider $y = \dot{s}_\beta \dot{w} \mathrm{v} \dot{s}_\beta^{-1}$. Then, as s_β commutes with w_0 and $w_{\Pi'}$ by assumption, we would have, for some $l \in k^\cdot$:

$$y = \dot{w}_0 \dot{s}_{\alpha+\beta} \dot{w}_{\Pi'} \mathrm{v}_1 x_{-\beta}(l) \mathrm{v}_2 \in B w_0 s_{\alpha+\beta} w_{\Pi'} B s_\beta B; \quad \text{ with } \mathrm{v}_1,\, \mathrm{v}_2 \in P^u_\beta.$$

Besides $w_0 s_{\alpha+\beta} w_{\Pi'}(\beta)$ is positive and different from β. Thus, $w_0 s_{\alpha+\beta} w_{\Pi'} s_\beta > w_0 s_{\alpha+\beta} w_{\Pi'}$ and in this case $\phi(B.y) = w_0 s_{\alpha+\beta} w_{\Pi'} s_\beta$ would not be an involution contradicting Lemma 2.7. Hence, $\mathrm{v} \in P^u_\beta$ and $\mathrm{v}' = \dot{s}_\beta \mathrm{v} \dot{s}_\beta^{-1} \in U$.

Let us consider a representative $\dot{s}_\alpha$ of s_α in $N(T)$ and the element:

$$z = \dot{s}_\alpha \dot{s}_\beta \dot{w} \mathrm{v} \dot{s}_\beta^{-1} \dot{s}_\alpha^{-1} = (\dot{w}_0 \dot{s}_\beta \dot{w}_{\Pi'})(\dot{s}_\alpha \dot{s}_\beta \mathrm{v} \dot{s}_\beta^{-1} \dot{s}_\alpha^{-1}) \in B w_0 s_\beta w_{\Pi'} B (\dot{s}_\alpha \mathrm{v}' \dot{s}_\alpha^{-1}) B.$$

Here we have used that $w_0(\alpha) = -\alpha$ because α is an isolated root in Π. If v' lies in P^u_α then $\phi(B.z) = w_0 s_\beta w_{\Pi'}$. If v' does not lie in P^u_α then

$$z \in B w_0 s_\beta w_{\Pi'} B (\dot{s}_\alpha \mathrm{v}' \dot{s}_\alpha^{-1}) B \subset B w_0 s_\beta w_{\Pi'} B \cup B w_0 s_\beta s_\alpha w_{\Pi'} B.$$

It follows from Theorem 2.7 that also in this case $\phi(B.z) = w_0 s_\beta w_{\Pi'}$ because $(w_0 s_\beta s_\alpha w_{\Pi'})^2 = s_\alpha s_\beta \neq 1$. On the other hand, $\ell(\phi(B.z)) = \ell(w_0 s_\alpha w_{\Pi'}) = \ell(\phi(v_0))$ with $\phi(B.z) \neq \phi(v_0)$, contradicting $\phi(B.z) \leq \phi(v_0)$. $\square$

Corollary 4.2 *Let $\mathcal{O}$ be a noncentral spherical conjugacy class, let v_0 be its dense B-orbit and let $w = w_0 w_\Pi = \phi(v_0)$. Then Π is either empty or it is one of the following subsets of Δ:*
Type A_n

$$\circ \cdots \circ - - \bullet \cdots \bullet - - \circ \cdots \circ \qquad \Pi = \{\alpha_l, \cdots, \alpha_{n-l+1}\}, \quad 2 \leq l \leq \left[\frac{n}{2}\right]$$

Type B_n

$$\circ \cdots \circ - - \bullet \cdots \bullet =>= \bullet \quad \Pi_1 = \{\alpha_l, \cdots, \alpha_n\}, \quad 2 \leq l \leq n$$

$$\bullet - - \circ \cdots \bullet - - \circ - - \bullet \cdots \bullet =>= \bullet$$

$$\Pi_2 = \{\alpha_1, \alpha_3, \cdots, \alpha_{2l-1}, \alpha_{2l+1}, \alpha_{2l+2}, \cdots, \alpha_n\}, \quad 1 \leq l \leq \frac{n}{2}$$

Type C_n

$$\circ \cdots \circ - - \bullet \cdots \bullet =<= \bullet \quad \Pi_1 = \{\alpha_l, \cdots, \alpha_n\}, \quad 2 \leq l \leq n$$

$$\bullet - - \circ \cdots \bullet - - \circ - - \bullet \cdots \bullet =<= \bullet$$

$$\Pi_2 = \{\alpha_1, \alpha_3, \cdots, \alpha_{2l-1}, \alpha_{2l+1}, \alpha_{2l+2}, \cdots, \alpha_n\}, \quad 1 \leq l \leq \frac{n}{2}$$

Type D_n

$$\begin{array}{ccc} \circ \cdots \circ - - \bullet \cdots & \bullet & - - \bullet \\ & | & \\ & \bullet & \end{array} \qquad \Pi_1 = \{\alpha_{2l+1}, \cdots, \alpha_n\}, \quad 2 \leq l \leq \frac{n}{2} - 1$$

$$\begin{array}{ccc} \bullet - - \circ - - \bullet - - \circ \cdots \circ - - \bullet \cdots & \bullet & - - \bullet \\ & | & \\ & \bullet & \end{array}$$

$$\Pi_2 = \{\alpha_1, \alpha_3, \cdots, \alpha_{2l-1}, \alpha_{2l+1}, \alpha_{2l+2}, \cdots, \alpha_n\}, \quad 1 \leq l < \frac{n}{2}$$

D_{2m+1}

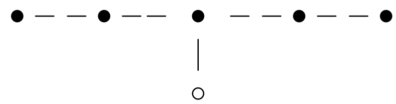
$$\begin{array}{ccc} \bullet - - \circ - - \cdots \bullet - - \circ - - & \bullet & - - \circ \\ & | & \\ & \circ & \end{array} \qquad \Pi_3 = \{\alpha_1, \alpha_3, \cdots, \alpha_{2m-1}\}$$

D_{2m}

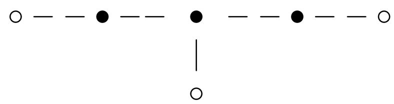
$$\begin{array}{ccc} \bullet - - \circ - - \cdots - - \bullet - - & \circ & - - \bullet \\ & | & \\ & \circ & \end{array} \qquad \Pi_4 = \{\alpha_1, \alpha_3, \cdots, \alpha_{2m-3}, \alpha_{2m-1}\}$$

$$\begin{array}{ccc} \bullet - - \circ - - \cdots - - \bullet - - & \circ & - - \circ \\ & | & \\ & \bullet & \end{array} \qquad \Pi_5 = \{\alpha_1, \alpha_3, \cdots, \alpha_{2m-3}, \alpha_{2m}\}$$

Type E_6

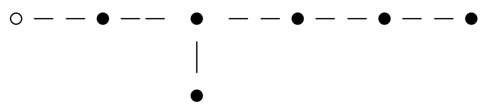


$\Pi_1 = \{\alpha_1, \alpha_3, \alpha_4, \alpha_5, \alpha_6\}$

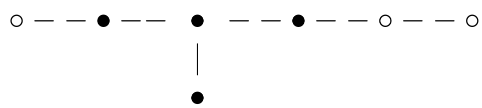


$\Pi_2 = \{\alpha_3, \alpha_4, \alpha_5\}$

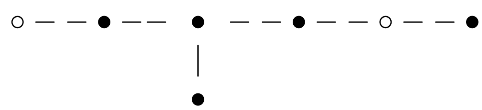
Type E_7



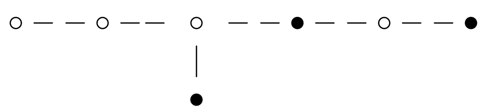
$\Pi_1 = \{\alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7\}$



$\Pi_2 = \{\alpha_2, \alpha_3, \alpha_4, \alpha_5\}$

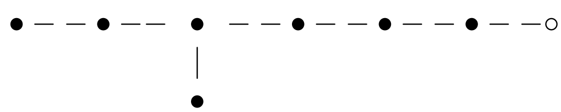


$\Pi_2 \cup \{\alpha_7\}$

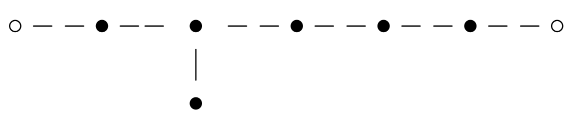


$\Pi_3 = \{\alpha_2, \alpha_5, \alpha_7\}$

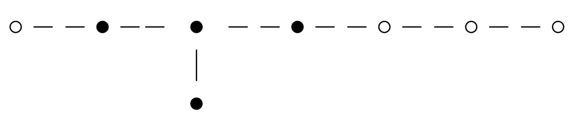
Type E_8



$\Pi_0 = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7\}$

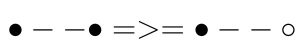


$\Pi_1 = \{\alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7\}$

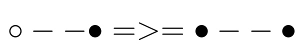


$\Pi_2 = \{\alpha_2, \alpha_3, \alpha_4, \alpha_5\}$

Type F_4



$\Pi_1 = \{\alpha_1, \alpha_2, \alpha_3\}$



$\Pi_2 = \{\alpha_2, \alpha_3, \alpha_4\}$

$$\circ - - \bullet => = \bullet - - \circ \qquad \Pi_3 = \{\alpha_2, \alpha_3\}$$

Type G_2

$$\bullet \equiv<\equiv \circ \qquad \Pi_1 = \{\alpha_1\}$$

$$\circ \equiv<\equiv \bullet \qquad \Pi_2 = \{\alpha_2\}.$$

Proof. Most of the restrictions are due to the fact that $w_\Pi = w_0|_{\Phi(\Pi)}$. The isolated roots occurring as connected components of Π are necessarily alternating, or differ by a node, or their length is different from the length of all adjacent roots, as it happens to $\{\alpha_n\}$ in type C_n. □

Remark 4.3 All the above diagrams actually occur when $k = \mathbb{C}$ (cfr. [10]). They are strictly more than the Araki-Satake diagrams for symmetric conjugacy classes (see [2], [13], [26, Table 1]).

Theorem 4.4 *Let $\mathcal{O}$ be a spherical conjugacy class, let v_0 be its dense B-orbit and let $w = \phi(v_0) = w_0 w_\Pi$. Then*

$$\dim(\mathcal{O}) = \ell(w) + \mathrm{rk}(1-w).$$

Proof. Let π_0 be the restriction to v_0 of the natural map $\pi\colon G \to G/B = \cup_{\sigma\in W} B\sigma B/B$. Its image is precisely the Bruhat cell $C_w = BwB/B$ and the image of $\mathcal{O}$ through π lies in the corresponding Schubert variety $\overline{C_w} = \cup_{\sigma\leq w\in W} B\sigma B/B$. By [27, Theorem 5.1.6 (ii)] for a generic point $gB \in C_w$ and for every irreducible component C of the fiber $\pi_0^{-1}(gB)$ we have:

$$\dim(\mathcal{O}) = \dim(v_0) = \dim(C_w) + \dim(C) = \ell(w) + \dim(C).$$

Let $g = u\dot{w}\mathrm{v} \in v_0$ with $u \in U^w$, $\dot{w} \in N(T)$ and $\mathrm{v} \in U$. Then

$$\pi_0^{-1}(gB) = \{x \in v_0 \mid x = u\dot{w}b, \text{ for some } b \in B\}.$$

Let us consider $g' = u^{-1}gu = \dot{w}\mathrm{v}u \in v_0$. Then x lies in $\pi_0^{-1}(gB)$ if and only if $x = ag'a^{-1} = u\dot{w}b$ for some $a,\, b \in B$. Moreover, if $a = a^w a_w t \in U^w U_w T$ we necessarily have $a^w = u$ so $\pi_0^{-1}(gB) = u\pi_0^{-1}(g'B)u^{-1}$. Besides, $\pi_0^{-1}(g'B)$ is the TU_w-orbit of g' so it is irreducible and $\dim C = \dim(TU_w.g')$.

Let $\phi_w\colon T \to T$ be the group morphism $t \mapsto \dot{w}^{-1}t\dot{w}t^{-1}$ so that $T^w = \mathrm{Ker}(\phi_w)$. For $tu_w \in TU_w$ we have $tu_w g' u_w^{-1} t^{-1} = \dot{w}\phi_w(t)tu_w\mathrm{v}uu_w^{-1}t^{-1}$ by Lemma 3.4 and $v_0 \cap \dot{w}U$ is precisely the $T^w U_w$-orbit of g'. Then, $\pi_0^{-1}(g'B)$ is parametrized by pairs $(tT^w,\, \dot{w}\mathrm{v}')$ in $T/T^w \times (v_0 \cap \dot{w}U)$ so $\dim(C) = \mathrm{rk}(1-w) + \dim(v_0 \cap \dot{w}U)$.

The theorem follows if we show that $\dim(v_0 \cap \dot{w}U) = 0$, or, equivalently, that the identity component $(T^w)^\circ$ of T^w and U_w centralize an element in $v_0 \cap \dot{w}U$. We shall provide a description of the elements in $\mathcal{O} \cap \dot{w}B$ that will lead to the knowledge of $v_0 \cap \dot{w}U$.

For Π as above we shall denote by P_Π the standard parabolic subgroup corresponding to Π and by L_Π its standard Levi factor generated by T and the one-parameter subgroups $X_{\pm\beta}$ with $\beta \in \Pi$. Since $U_w = \langle X_\alpha,\, \alpha \in \Pi\rangle$, the unipotent radical of P_Π is U^w.

Let us recall that the depth $\mathrm{dp}(\beta)$ of a positive root β is the minimal length of a σ in W for which $\sigma\beta \in -\Phi^+$ ([5]). Then $\mathrm{dp}(\beta) - 1$ is the minimal length of a $\sigma' \in W$ for which $\sigma'\beta$ is a simple root.

Lemma 4.5 *Let $\mathcal{O}$, v_0, $w = \phi(v_0)$ and Π be as in Theorem 4.4 and let $x = \dot{w}t\mathrm{v} \in \mathcal{O} \cap \dot{w}B$. Then $\mathrm{v} \in U^w$. In particular, this holds for $x \in v_0 \cap \dot{w}B$.*

Proof. Let us assume that for a fixed ordering of the positive roots some root γ in $\Phi^+(\Pi)$ occurs in the expression of v and let us assume that γ is of minimal depth in $\Phi(\Pi)$ with this property. By Lemma 3.4 the root γ is not simple. Then, there exists $\sigma \in W_\Pi$ such that $\sigma(\gamma) = \alpha \in \Pi$ and we choose σ of minimal length with this property. Minimality of depth implies that for every root $\gamma' \in \Phi^+(\Pi)$ occurring in v we have $\sigma\gamma' \in \Phi^+$ while $\sigma \in W_\Pi$ implies that for every $\beta \in \Phi^+ \setminus \Phi(\Pi)$ occurring in v we have $\sigma\gamma' \in \Phi^+$. Then, for $\dot{\sigma} \in N(T)$ we would have:

$$\dot{\sigma}x\dot{\sigma}^{-1} = \dot{w}t'\dot{\sigma}\mathrm{v}\dot{\sigma}^{-1} \in \mathcal{O} \cap Bw_\Pi w_0 B \quad \text{for some } t' \in T.$$

The B-orbit represented by $\dot{\sigma}x\dot{\sigma}^{-1}$ would be stationary with $\alpha \in \Pi$ and $\dot{\sigma}\mathrm{v}\dot{\sigma}^{-1} \notin P_\alpha^u$ contradicting Lemma 3.4. $\square$

Let us consider the action of T^w by conjugation on $x = \dot{w}\mathrm{v} \in v_0 \cap \dot{w}U$. A necessary condition for $\dim(v_0 \cap \dot{w}U) = 0$ to hold is that $(T^w)^\circ$ commutes with v.

If $w_0 = -1$, since $w_\Pi(\beta)$ is positive for all roots $\beta \in \Phi^+ \setminus \Phi(\Pi)$ we have

$$\dim T^w = n - \mathrm{rk}(1-w) = n - \mathrm{rk}(1+w_\Pi) = |\Pi|$$

so $(T^w)^\circ$ is generated by $\mathrm{Im}(\alpha^\vee)$ for $\alpha \in \Pi$.

If $w_0 = -\vartheta \neq -1$ an analysis of the diagrams in Corollary 4.2 shows that $(T^w)^\circ$ is generated by $\mathrm{Im}(\alpha^\vee)$ for $\alpha \in \Pi$ and $\mathrm{Im}((\alpha^\vee)(\vartheta\alpha^\vee)^{-1})$ for $\alpha \in \Delta \setminus \Pi$. Therefore we need to show that the roots occurring in v are orthogonal to Π and, if Φ is of type A_n, D_{2n+1} or E_6, that that they are ϑ-invariant.

Lemma 4.6 *Let Φ be simply-laced. Let $\mathcal{O}$, v_0, w, Π, be as in Theorem 4.4 and let $x = \dot{w}t\mathrm{v} \in \mathcal{O} \cap \dot{w}B$. Then all roots occurring in v are orthogonal to Π.*

Proof. If $\Pi = \emptyset$ there is nothing to prove. If $\Pi = \Delta$ then $w = w_0 w_\Pi = 1 = \phi(v_0)$ so $\mathcal{O}$ is central and the statement is evident. We shall assume for the rest of the proof that $\Pi \neq \emptyset, \Delta$.

The basic idea of the proof is to show that if some $\gamma \not\perp \Pi$ would occur in v there would exist $v \in \mathcal{V}$ such that $\phi(v)$ is not an involution contradicting Theorem 2.7. The proof consists in the construction of an element $\tau \in W$ such that:

1. $\tau\gamma = \alpha$ is a simple root;
2. no root occurring in v is made negative by τ;
3. $\tau w\gamma$ is negative.

Putting $\sigma = s_\alpha\tau = \tau s_\gamma$, conditions 1 and 2 guarantee that, for $\dot{\sigma} = \dot{s}_\alpha\dot{\tau} \in N(T)$ we would have:

$$\dot{\sigma}x\dot{\sigma}^{-1} \in \dot{\sigma}\dot{w}\dot{\sigma}^{-1}BX_{-\alpha}B \in B\sigma w_0 w_\Pi \sigma^{-1}Bs_\alpha B.$$

Then, we would have:

$$\sigma w_0 w_\Pi \sigma^{-1}\alpha = s_\alpha\tau w s_\gamma^{-1}\tau^{-1}\alpha = -s_\alpha\tau w\gamma.$$

Moreover, if $\gamma \in \Phi^+$ is not orthogonal to Π, then $w\gamma$ is negative and different from $-\gamma$ for if $w\gamma = -\gamma$ then for every $\alpha \in \Pi$ we have: $(\gamma, \alpha) = (\gamma, w\alpha) = (w^{-1}\gamma, \alpha) = -(\gamma, \alpha) = 0$.

Thus, condition 3 guarantees that $\sigma w_0 w_\Pi \sigma^{-1}\alpha$ is positive and different from α so $\sigma w_0 w_\Pi \sigma^{-1} < \sigma w_0 w_\Pi \sigma^{-1} s_\alpha$. Then if γ occurred in v we would have $\phi(B.\dot{\sigma}x\dot{\sigma}^{-1}) = \sigma w_0 w_\Pi \sigma^{-1} s_\alpha$ which is not an involution.

We shall deal with the different possibilities for Φ and Π separately, using the labeling of Π in Corollary 4.2. We will rule out roots inductively so that the preceding steps will ensure condition 2 to hold. We shall also make use of the following three observations.

I. Let $\Phi' = \Phi(\Delta')$ be a subsystem of Φ on which the actions of w_0 and $w_{\Delta'}$ coincide and let $\Pi \subset \Delta'$. If the occurrence of a $\gamma \in \Phi'$ that is not orthogonal to Π has been excluded for Φ' and Π, then the occurrence of γ is excluded for Φ and Π. Indeed, if $\tau \in W_{\Delta'}$ satisfies conditions 1, 2 and 3 for $\gamma \in \Phi'$, regarding τ as an element of W we see that conditions 1 and 3 are immediate and condition 2 follows from condition 2 in Φ' because an element in $W_{\Delta'}$ cannot make negative a root in $\Phi^+ \setminus \Phi'$. Thus, if Φ' and Π have already been handled, the analysis for Φ and Π will reduce to roots in $\Phi^+ \setminus \Phi'$.

II. If some γ may not occur in v for every $x = \dot{w}t\mathrm{v} \in \mathcal{O} \cap \dot{w}B$ then the whole W_Π-orbit of γ may not occur in v$'$ for every $x' = \dot{w}t'\mathrm{v}' \in \mathcal{O} \cap \dot{w}B$ because Lemma 4.5 gives $\dot{\omega}\dot{w}t\mathrm{v}\dot{\omega}^{-1} \in \mathcal{O} \cap \dot{w}B$ for every $\omega \in W_\Pi$. In particular, if $\gamma \not\perp \alpha_i$ for some $\alpha_i \in \Pi$ then also $s_i\gamma \not\perp \alpha_i$ and it is enough to show that one of the two roots may not occur in v.

III. Let $\gamma \not\perp \Pi$. If we can find τ with $\ell(\tau) = \mathrm{ht}(\gamma) - 1$ satisfying condition 1, then condition 3 holds automatically for those roots γ for which $w_\Pi\gamma \not< \gamma$. Indeed, if we decompose $w_\Pi = s_{\gamma_1} \cdots s_{\gamma_r}$ as a product of reflections with respect to mutually orthogonal roots in $\Phi(\Pi)$, for every $\gamma \in \Phi$ we have $w_\Pi\gamma = \gamma - \sum_i (\gamma, \gamma_i)\gamma_i$ and we may write $w_\Pi\gamma = \gamma - \sigma_1 + \sigma_2$ where σ_1 and σ_2 are sums of roots in Π with disjoint support. The condition on $w_\Pi\gamma$ is equivalent to $\sigma_2 \neq 0$. The condition on $\ell(\tau)$ means that we are taking $\tau = s_{i_p} \cdots s_{i_1}$ for a sequence of simple roots α, α_{i_p}, $\ldots$, α_{i_1} such that $\alpha_{i_j} + \cdots + \alpha_{i_p} + \alpha \in \Phi^+$ and $\gamma = \alpha_{i_1} + \cdots + \alpha_{i_p} + \alpha$.

Moreover, the γ_j and the σ_i are ϑ-invariant because $-\gamma_j = w_\Pi\gamma_j = w_0\gamma_j$. If $\tau w\gamma = -\tau(\vartheta\gamma - \sigma_1 + \sigma_2)$ were positive, τ would make $\vartheta\gamma - \sigma_1 + \sigma_2$ negative so $\vartheta\gamma - \sigma_1 + \sigma_2 \leq \gamma$ and, by symmetry, we would have:

$$\gamma - 2\sigma_1 + 2\sigma_2 \leq \vartheta\gamma - \sigma_1 + \sigma_2 \leq \gamma.$$

Thus, $2(\sigma_1 - \sigma_2)$ would be a sum of positive roots, contradicting our assumption on the their supports unless $\sigma_2 = 0$.

Up to replacing γ by $w_\Pi\gamma$ applying observation II, we may always make sure that $w_\Pi\gamma \not< \gamma$. For this reason in most of the cases we will be able to find γ and τ satisfying both condition 2 and the assumptions needed for observation III. In the remaining cases, condition 3 will be verified by direct computation.

Type A_n. In this case $\Pi = \{\alpha_l, \alpha_{l+1}, \ldots, \alpha_{n-l+1}\}$ for $1 \leq l \leq \left[\frac{n+1}{2}\right]$. Let $\gamma \not\perp \Pi$ occur in v. Then $\gamma = \gamma_{t,i-1} = \alpha_t + \cdots + \alpha_{i-1}$ for $1 \leq t \leq l-1 \leq i-1 \leq n-l+1$ or $l \leq t \leq n-l+2 \leq i-1 \leq n-1$.

Observation II implies that it is enough to show that $\gamma_{t,l-1}$ for every $t \leq l-1$ and $\gamma_{n-l+2,i-1}$ for $i-1 \geq n-l+2$ may not occur in v. Let $\gamma_{t,l-1}$ be of minimal height among the $\gamma_{s,l-1}$ occurring in v. We consider $\tau = s_{t+1}s_{t+2}\cdots s_{l-1}$ with $\tau = 1$ if $t = l-1$. Then $\tau\gamma = \alpha_t$ so condition 1 is satisfied. The roots made negative by τ are all of the form $\gamma_{p,l-1}$ for $p > t$, so minimality of the height ensures condition 2, and $w_\Pi \gamma_{t,l-1} > \gamma$ with $\ell(\tau) = \mathrm{ht}(\gamma_{t,l-1}) - 1$. The roots of type $\gamma_{n-l+2,i-1}$ are handled symmetrically.

Type D_n. Let us consider Π_1. The positive roots outside $\Phi(\Pi_1)$ that are not orthogonal to Π_1 are:

$$\begin{gathered}
\gamma_{t,i-1} = \alpha_t + \cdots + \alpha_{i-1} \text{ for } 1 \leq t \leq l-1 \leq i-1 \leq n-1;\\
s_n\gamma_{t,n-2} = \alpha_t + \cdots + \alpha_{n-2} + \alpha_n \text{ for } 1 \leq t \leq l-1;\\
s_n\gamma_{t,n-1} = s_n s_{n-1}\gamma_{t,n-2} = \alpha_t + \cdots + \alpha_{n-1} + \alpha_n \text{ for } t \leq l-1;\\
\omega_{t,i} = \alpha_t + \cdots + \alpha_{i-1} + 2(\alpha_i + \cdots + \alpha_{n-2}) + \alpha_{n-1} + \alpha_n, \text{ for } 1 \leq t \leq l-1 \leq i-1 \leq n-3.
\end{gathered}$$

Let $\gamma_{t,l-1}$ be the root of minimal height among the $\gamma_{s,l-1}$ occurring in v. Then $\tau = s_{i+1}\cdots s_{l-1}$ satisfies condition 1 by construction, condition 2 by minimality of the height and condition 3 by observation III. Observation II rules out all other roots, since $\gamma_{i,j-1} = s_{j-1}\cdots s_l \gamma_{i,l-1}$ and $\omega_{t,i} = s_i \cdots s_{n-2}s_n\gamma_{t,n-1}$.

Let us assume now that $\Pi = \Pi_2$. The above argument shows that all roots occurring in v are orthogonal to Π_1. Indeed, conditions 1 and 2 on τ are independent of Π while for condition 3 a direct computation shows that $\tau w \gamma_{t,l-1} = \tau s_\alpha w_0 w_{\Pi_1}(\gamma_{t,l-1})$ with $\alpha = \alpha_t$ or α_{t-1} is a negative root. Then the roots that might occur in v and are not orthogonal to isolated roots are:

$$\begin{gathered}
\gamma_{t,i-1} = \alpha_t + \cdots + \alpha_{i-1} \text{ for } 1 \leq t \leq i-1 \leq l-2;\\
\omega_{t,i} = \alpha_t + \cdots + \alpha_{i-1} + 2(\alpha_i + \cdots + \alpha_{n-2}) + \alpha_{n-1} + \alpha_n\\
\text{for } 1 \leq t \leq i-1 \leq l-2 \text{ and } i > t+1 \text{ if } \alpha_t \in \Pi.
\end{gathered}$$

For the first set of roots we might use observation II and assume that $\alpha_t \notin \Pi$ so that $w_\Pi\gamma_{t,i-1} \not\leq \gamma_{t,i-1}$. Then $\tau = s_{t+1}\cdots s_{i-1}$ together with a minimality argument for fixed i rules it out. For the second set of roots we might assume that $\alpha_i \notin \Pi$ so that $w_\Pi\omega_{t,i} \not\leq \omega_{t,i}$ and we might use $\tau = s_n s_{n-1}(s_{n-3}\cdots s_{i-1})(s_{n-2}\cdots s_i)(s_{i-2}\cdots s_t)$. Condition 1 holds by construction, condition 2 follows from the inductive procedure and condition 3 follows from observation III.

If $\Pi = \Pi_3$ the roots that are not orthogonal to Π that might occur in v are:

$$\begin{gathered}
\gamma_{t,i-1} = \alpha_t + \cdots + \alpha_{i-1} \text{ for } 1 \leq t \leq i-1 \leq n-1;\\
s_n\gamma_{t,n-2} \text{ for } 1 \leq t \leq n-1;\\
\alpha_n;\\
s_{n-1}s_n\gamma_{t,n-2} \text{ for } t \leq n-2;\\
\omega_{t,i} = \alpha_t + \cdots + \alpha_{i-1} + 2(\alpha_i + \cdots + \alpha_{n-2}) + \alpha_{n-1} + \alpha_n\\
\text{for } 1 \leq t \leq i-1 \leq n-3 \text{ with } i > t+1 \text{ if } \alpha_t \in \Pi.
\end{gathered}$$

For the $\gamma_{t,i-1}$ we might assume that $\alpha_t \notin \Pi$ and use $\tau = s_{t+1}\cdots s_{i-1}$ together a minimality argument for fixed i. For the second set of roots we might use $\tau = s_{n-2}\cdots s_t$. For α_n we use $\tau = 1$. For the $s_n s_{n-1}\gamma_{t,n-2}$ we might assume that $\alpha_t \notin \Pi$ and use $\tau = s_{t+1}\cdots s_{n-2}s_{n-1}s_n$. For the last set of roots we might assume that $\alpha_t \notin \Pi$ and take $\tau = s_{n-1}s_n(s_{n-3}\cdots s_{i-1})(s_{n-2}\cdots s_i)(s_{i-2}\cdots s_t)$.

If $\Pi = \Pi_4$ the roots that are not orthogonal to Π are those listed for the previous case, except from α_n. We need to consider the first, second and last set of roots. The above arguments and Weyl group elements work also in this case. The case $\Pi = \Pi_5$ is handled symmetrically.

Type E_6. In this case Π is either Π_1 or Π_2 as in Corollary 4.2 and we will apply observation I to $\Delta' = \Pi_1$.

Let $\Pi = \Pi_1$. The positive roots γ that are not orthogonal to α_6 and that might occur in v are: $\mu_1 = \alpha_2 + \alpha_4 + \alpha_5$; $s_3\mu_1$; $s_4s_3\mu_1$; $s_1s_3\mu_1$; $s_4s_1s_3\mu_1$; $s_3s_4s_1s_3\mu_1$ and their images through s_6. By observation II it is enough to rule out μ_1 and this is achieved by using $\tau = s_4s_5$, where condition 2 holds because $\tau \in W_\Pi$ and $\mathrm{v} \in U^w$. Therefore all roots occurring in v are orthogonal to α_6. The root α_1 is handled symmetrically.

The admissible roots γ that are orthogonal to α_1 and α_6 and are not orthogonal to α_4 are: α_2; $\gamma = \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6$ and their images through s_4. We rule out α_2 with $\tau = 1$ while the remaining roots are ruled out using observation II since $s_3\gamma \not\perp \alpha_1$ may not occur in v.

The only root in $\Phi^+ \setminus \Phi(\Pi_1)$ that is orthogonal to $\alpha_1,\ \alpha_4$ and α_6 is the highest root in Φ, which is orthogonal to Π, whence the statement in this case.

Let $\Pi = \Pi_2$. The roots that might occur in v and are not orthogonal to α_3 are: $\nu_1 = \alpha_2 + \alpha_4$; $s_5\nu_1$; $\nu_2 = s_6s_5\nu_1$; $\nu_3 = \alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5$; $\nu_4 = s_6\nu_3$; $s_5\nu_4$ and their images through s_3. We only need to consider $\nu_1,\ \nu_2,\ \nu_3$ and ν_4. We may use, respectively, $\tau_1 = s_4$; $\tau_2 = s_4s_5s_6$; $\tau_3 = s_4s_3s_1s_5s_4$ and $\tau_4 = s_1s_4s_2s_5s_4s_6$, where the conditions are easily verified. Thus, all roots occurring in v are orthogonal to α_3 and, by symmetry, to α_5. The roots in $\Phi^+ \setminus \Phi^+(\Pi_1)$ that are not orthogonal to α_4 are: $s_4\nu_1$; $s_4s_5s_3\nu_1$; $s_4s_3s_5\nu_4$ and $s_4\nu_4$ and they may not occur in v by observation II.

Type E_7. In this case Π is either Π_1 of type D_6; Π_2 of type D_4; the union of Π_2 with $\{\alpha_7\}$; or $\Pi_3 = \{\alpha_2, \alpha_5,\ \alpha_7\}$. We shall make frequent use of observation I with $\Delta' = \Pi_1$.

Let $\Pi = \Pi_1$. The roots in $\Phi^+ \setminus \Phi(\Pi_1)$ that are not orthogonal to α_3 are: α_1; $\mu_1 = s_4s_5s_2s_4s_3\alpha_1$; $\mu_2 = s_6\mu_1$; $\mu_3 = s_5\mu_2$; $\mu_4 = s_7\mu_2$; $\mu_5 = s_5\mu_4$; $\mu_6 = s_6\mu_5$; $\mu_7 = s_4s_5s_2s_4s_3\mu_6$ and their images through s_3. As they all lie in the W_Π-orbit of α_1, which is erased by $\tau = 1$, all roots occurring in v are orthogonal to α_3.

The possibilities for $\gamma \not\perp \alpha_2$ are: $\gamma_1 = s_4s_3\alpha_1$; $\gamma_2 = s_5\gamma_1$; $\gamma_3 = s_6\gamma_2$; $\gamma_4 = s_7\gamma_3$; $\gamma_5 = s_4s_5s_3s_4s_2\gamma_3$; $\gamma_6 = s_4s_5s_3s_4s_2\gamma_4$; $\gamma_7 = s_6\gamma_6$; $\gamma_8 = s_5\gamma_7$ and their images through s_2. They all lie in the W_Π-orbit of α_1, hence these roots may not occur in v.

The only positive root that is orthogonal to α_2 and α_3 and lies is $\Phi \setminus \Phi(\Pi_1)$ is the highest root in $\Phi(E_7)$, whence the statement for $\Pi = \Pi_1$.

Let Π be either Π_2 or $\Pi_2 \cup \{\alpha_7\}$. The possible occurring roots that are not orthogonal to α_3 are those listed when analyzing $\Pi = \Pi_1$. We need to consider α_1; μ_2; μ_6 and μ_4 (the last one only if $\alpha_7 \notin \Pi$). They are excluded by using $\tau = 1$; $\tau_2 = s_4s_5s_6s_2s_4s_1$; $\tau_6 = s_4s_2s_5s_6s_7s_4s_5s_6s_1$ and $\tau_4 = s_6s_5s_4s_3s_2s_1s_4$ where conditions are easily checked making use of observations I, II and III.

The possible roots not orthogonal to α_2 are those listed when discussing the case $\Pi = \Pi_1$. We only need to consider γ_3; γ_4 (only if $\alpha_7 \notin \Pi$) and γ_7. We rule out γ_3 with $\sigma_3 = s_3s_4s_5s_6$; γ_4 with $\sigma_4 = s_6s_5s_4s_3s_1$ and γ_7 with $\sigma_7 = s_1s_4s_2s_5s_4s_3s_6s_5s_7s_6s_4$.

The only root in $\Phi^+ \setminus \Phi^+(\Pi_1)$ that is orthogonal to both α_2 and α_3 is the highest root, whence the statement in this case.

Let $\Pi = \Pi_3$. The possible occurring roots that are not orthogonal to α_7 are: $\beta_1 = \alpha_1 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6$; $\beta_2 = s_2\beta_1$; $\beta_3 = s_4\beta_2$; $\beta_4 = s_3\beta_3$; $\beta_5 = s_5\beta_3$; $\beta_6 = s_5\beta_4$; $\beta_7 = s_4\beta_6$; $\beta_8 = s_2\beta_7$ and their images through s_7 so it is enough to consider β_i for $i = 1, 3, 4, 7$. They are ruled out by using $\omega_1 = s_3s_4s_5s_6$; $\omega_3 = \omega_1s_2s_4$; $\omega_4 = \omega_3s_3$ and $\omega_7 = s_4s_3s_1s_5s_6s_4s_5s_3s_4$. In order to verify condition 2 for ω_7 we need to show that $\alpha_1 + \alpha_3 + \alpha_4$ may not occur in v and this is achieved by using $\tau = s_3s_4$. The remaining verifications are standard.

The possible roots ν that are not orthogonal to α_2, are: $\nu_1 = \alpha_1 + \alpha_3 + \alpha_4$; $\nu_2 = s_5\nu_1$; $\nu_3 = s_6s_7\beta_7$; $\nu_4 = s_5\nu_3$ and their images through s_2. The root ν_1 has already been ruled out, ν_3 is ruled out by using $\tau = s_3s_2s_1s_5s_4s_6s_5s_7s_6s_3s_4$, whereas for the other roots we use observation II.

The positive roots in $\Phi \setminus \Phi(\Pi_1)$ that are orthogonal to α_2 and α_7 are also orthogonal to α_5, concluding the proof for type E_7.

Type E_8. In this case Π is either Π_0 of type E_7, Π_1 of type D_6 or Π_2 of type D_4. We shall make use of observation I applied to $\Delta' = \Pi_0$.

Let $\Pi = \Pi_0$. The possible roots occurring in v that are not orthogonal to α_3 are: $\gamma_1 = \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8$; $\gamma_2 = s_2\gamma_1$; $\gamma_3 = s_4s_1s_3\gamma_2$; $\gamma_4 = s_5\gamma_3$; $\gamma_5 = s_6\gamma_4$; $\gamma_6 = s_7\gamma_5$; $\gamma_7 = s_4s_5s_6s_2s_4s_3s_5\gamma_3$; $\gamma_8 = s_7\gamma_7$; $\gamma_9 = s_6\gamma_8$; $\gamma_{10} = s_5\gamma_9$; $\gamma_{11} = s_1s_4s_3\gamma_{10}$; $\gamma_{12} = s_2\gamma_{11}$ and their images through s_3. The root γ_1 is excluded by $\tau = s_7s_6s_5s_4$ and for the remaining ones we use observation II.

The possible occurring roots that are not orthogonal to α_2 are: $\beta_1 = s_1s_3\gamma_1$; $\beta_2 = s_4s_5s_3s_4s_2\beta_1$ $\beta_3 = s_6\beta_2$; $\beta_4 = s_7\beta_3$; $\beta_5 = s_5\beta_4$; $\beta_6 = s_6\beta_5$; $\beta_7 = s_5\beta_3$; $\beta_8 = s_4s_5s_6s_7s_3\gamma_7$ and their images through s_2. All these roots lie in the W_Π-orbit of γ_1 so they might not occur in v.

Next we consider occurrence of roots that are not orthogonal to α_5. They are: $\nu_1 = s_5s_4\gamma_1$; $\nu_2 = s_4s_2s_3\gamma_1$; $\nu_3 = s_1s_3s_6s_7\gamma_7$; $\nu_4 = s_4s_2s_3s_4s_5\nu_3$ and their images through s_5. They all lie in the W_Π-orbit of γ_1 so they might not occur in v.

The possible occurring roots that are not orthogonal to α_6 are $\pi_1 = s_6\nu_1$; $\pi_2 = s_6\nu_3$ and they cannot occur. The possible roots that are not orthogonal to α_7 are then α_8 and $\pi_3 = s_7\pi_2$ and using $\tau = 1$ for α_8 and observation II for π_3 we see that they cannot occur in v.

Thus the roots occurring in v are orthogonal to α_j for $j = 2, 3, 5, 6, 7$. The only positive root that does not lie in $\Phi(E_7)$ and that is orthogonal to these simple roots is the highest root in Φ, whence the statement for $\Pi = \Pi_0$.

Let $\Pi = \Pi_1$ and let us consider the occurrence in v of roots that are not orthogonal to α_3. They have been listed when dealing with $\Pi = \Pi_0$ and we need to consider only γ_1, γ_3 and γ_{11}. We exclude them by using, respectively $\tau = s_7s_6s_5s_4$; $\tau = s_7s_6s_5s_4s_3s_2s_1s_4$ and $\tau = s_8s_6s_5s_4s_3s_2s_1s_4s_3s_5s_4s_6s_5s_2s_4s_3s_1s_7s_6s_5s_4$.

The possible occurring roots that are not orthogonal to α_2 have been listed when dealing with $\Pi = \Pi_0$ and we only have to consider β_1. This root is ruled out by using $\tau = s_7s_6s_5s_4s_3s_1$. Similarly, in order to show that all roots occurring in v are orthogonal to α_5 we only need to exclude ν_3. However, $\nu_3 = s_5s_4\gamma_{11}$ so it may not occur in v. Observation II together with the discussion for $\Pi = \Pi_0$ imply that all roots occurring in v are also orthogonal to α_6 and α_7. As before, it follows that all roots occurring in v are orthogonal to Π_1.

Let $\Pi = \Pi_2$. In order to show that all roots occurring in v are orthogonal to α_3 we have to rule out: γ_1; γ_3; γ_5; γ_6; γ_9 and γ_{11} with notation as before. We may use, respectively: $\tau_1 = s_7s_6s_5s_4$; $\tau_3 = s_6s_5s_4s_3s_2s_1s_4$; $\tau_5 = s_7s_6s_5s_4s_3s_2s_1s_4s_5s_6$; $\tau_6 = s_4s_5s_6s_7s_8s_2s_4s_3s_1s_5s_4s_6s_5s_7s_6$; $\tau_9 = s_5s_6s_7s_8s_3s_4s_5s_1s_3s_4s_6s_5s_2s_4s_7s_6$ and $\tau_{11} = s_8s_6s_5s_4s_3s_2s_4s_1s_3s_5s_6s_4s_5s_2s_4s_3s_7s_6s_5s_4s_1$. Even if $w_\Pi\gamma_6 = \gamma_6 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5 > \gamma_6$ we cannot apply observation III in this case because there is no τ with $\ell(\tau) = \mathrm{ht}(\gamma_6) - 1$ satisfying condition 2. For this case we verify condition 3 by direct computation.

Since all roots that might occur in v and that are not orthogonal to α_2 lie in the W_Π-orbit of some γ_i, they are excluded. Similarly, the possible occurring roots that are not orthogonal to α_5 all lie in the W_Π-orbit of some previously excluded root.

No root γ in $\Phi^+ \setminus \Phi(\Pi_0)$ for which $\gamma \not\leq \alpha_4$ is orthogonal to α_2, α_3 and α_5, thus we have the statement in type E_8, concluding the proof. $\square$

Lemmas 3.4 and 4.6 together with Chevalley's commutator formula ([27, Proposition 9.5.3]) imply that the elements in U_w and in $\mathrm{Im}(\alpha^\vee)$ for $\alpha \in \Pi$ commute with $x = \dot{w}\mathrm{v} \in v_0 \cap \dot{w}U$. The descriptions of $(T^w)^\circ$ and of the fiber $\pi_0^{-1}(u\dot{w}B)$ yield thus Theorem 4.4 in type A_1, D_{2n}, E_7 and E_8. For the remaining simply-laced cases there is some extra work to be done.

Lemma 4.7 *Let Φ be of type A_n, for $n \geq 2$, D_{2m+1}, or E_6. Let $\mathcal{O}, v_0, w, x = \dot{w}t\mathrm{v}$ and Π be as in Lemma 4.6. Then all roots occurring in v are ϑ-invariant.*

Proof. By Lemma 4.6 we have $\gamma \perp \Pi$ but for this analysis we shall consider the case $\Pi = \emptyset$. In any case $w_\Pi\gamma = \gamma$ for every γ occurring in v. We will use the same strategy and notation as in the proof of Lemma 4.6. We may use that if $\vartheta\gamma \neq \gamma$ then $w\gamma = -\vartheta w_\Pi \gamma = -\vartheta\gamma \neq -\gamma$ and that observation I still applies. Moreover, as $w_\Pi\gamma = \gamma$ we modify the argument in observation 3 in order to obtain $\gamma \leq \vartheta\gamma \leq \gamma$ for a contradiction.

Type A_n. Let γ be a root occurring in v which is not ϑ-invariant. Then $\gamma_{j,t} = \alpha_j + \cdots + \alpha_t$ with either: $j \leq t \leq l-2$ or $n-l+3 \leq j \leq t$ or $j \leq l-1 \leq n-l+2 \leq t$. For all ranges for t and j we may choose $\gamma_{j,t}$ of minimal height among the $\gamma_{s,t}$ occurring in v and use $\tau = s_{j+1}\cdots s_t$ together with observation III. This argument works also if $\Pi = \emptyset$.

Type D_{2m+1}. The positive roots that are not ϑ-invariant are α_{2m}, α_{2m+1}, or of the form $\gamma_{j,q} = \sum_{p=j}^{n-2}\alpha_p + \alpha_q$ for $1 \leq j \leq 2m-1$ and $q = 2m, 2m+1$. None of these roots is orthogonal to Π_1, Π_2 nor Π_3. If $\Pi = \emptyset$ we exclude α_{2m} and α_{2m+1} with $\tau = 1$. Then we consider γ of minimal height among the $\gamma_{j,q}$ occurring in v and we rule it out by using $\tau = s_{j+1}\cdots s_{2m-1}s_q$.

Type E_6. If γ is a positive root occurring in v which is not ϑ-invariant, observation I with $\Delta' = \Pi_1$ shows that either γ or $\vartheta\gamma$ is one of the following roots: $\beta_1 = \alpha_2 + \alpha_3 + \alpha_4$; $\beta_2 = s_1\beta_1$; $\beta_3 = s_5\beta_2$; $\beta_4 = s_4\beta_3$; $\beta_5 = s_3\beta_4$ and $\beta_6 = s_6\beta_5$. We may rule out the listed roots by using $\tau_1 = s_4s_3$; $\tau_2 = \tau_1 s_1$; $\tau_3 = \tau_2 s_5$; $\tau_4 = s_4s_2s_5s_4s_3s_1$; $\tau_5 = s_4s_2s_3s_1s_4s_3$ and $\tau_6 = \tau_5 s_6$, respectively. The root β_4 needs to be considered only when $\Pi = \emptyset$. In this case condition 2 is not compatible with the assumption on $\ell(\tau_4)$ introduced in observation III so condition 3 has to be verified directly. The image of these roots through ϑ can be handled symmetrically. $\square$

Combining Lemmas 3.4, 4.6 and 4.7, the descriptions of $(T^w)^\circ$ and $\pi_0^{-1}(\dot{w}B)$, and Chevalley's commutator formula we obtain the proof of Theorem 4.4 if Φ is simply-laced. We will deal now with the multiply-laced types.

Lemma 4.8 *Let Φ be multiply-laced. Let $\mathcal{O}$, v_0, w, Π be as in Theorem 4.4 and let $x = \dot{w}t\mathrm{v} \in \dot{w}B \cap \mathcal{O}$. Then the roots occurring in v are orthogonal to Π.*

Proof. As in Lemma 4.6 we need only to consider $\Pi \neq \emptyset, \Delta$. We shall use the same strategy and observations I and II will still be of use. The labeling of the possible Π is as in Corollary 4.2.

Type B_n. In this case Π is either Π_1 or Π_2 and w_Π is either w_{Π_1} or the product of the reflections corresponding to those isolated simple roots with w_{Π_1}.

Let $\Pi = \Pi_1$. The roots γ in $\Phi^+ \setminus \Phi^+(\Pi_1)$ that are not orthogonal to Π_1 are the following:

$$\mu_{t,i-1} = \alpha_t + \cdots + \alpha_{i-1} \text{ with } t < l \text{ and } i \geq l;$$
$$\nu_{t,i} = \alpha_t + \cdots + \alpha_{i-1} + 2(\alpha_i + \cdots + \alpha_n) \text{ with } t < l \text{ and } i \leq l.$$

Observation II implies that it is enough to exclude $\mu_{t,l-1}$ for every $t \leq l-1$. Let $\mu_{t,l-1}$ be of minimal height among the $\mu_{s,l-1}$ occurring in v. Then $\tau = s_{t+1} \cdots s_{l-1}$ rules it out. Condition 3 is easily verified and it holds also if $\Pi = \Pi_2$.

Let now $\Pi = \Pi_2$ and let us assume that γ occurs in v and is not orthogonal to some root in $\Pi \setminus \Pi_1$. Then γ is one of the following roots:

$$\beta_{t,i-1} = \alpha_t + \cdots + \alpha_{i-1} \text{ with } 1 \leq t \leq i-1 < l-1 \text{ and } t \neq i-1 \text{ for } t \text{ odd};$$
$$\delta_t = \alpha_t + \cdots + \alpha_n \text{ for } t < l;$$
$$\gamma_{t,i} = \alpha_t + \cdots + \alpha_{i-1} + 2(\alpha_i + \cdots + \alpha_n) \text{ with } 1 \leq t \leq i-1 < l-1 \text{ and } t \neq i-1 \text{ for } t \text{ odd}.$$

In order to rule out $\beta_{t,i-1}$ we apply observation II and assume that $\alpha_i \in \Pi$. Then we consider $\beta_{t,i-1}$ of minimal height among the roots of type $\beta_{s,i-1}$ occurring in v and we rule it out by using $\tau = s_{t+1} \cdots s_{i-1}$. We rule out the roots of type δ_t by using $\tau = s_{n-1} \cdots s_t$. For the last set of roots we may assume that $\alpha_i \notin \Pi$ by observation II and then $\gamma_{t,i}$ is ruled out by $\tau = (s_{n-2} \cdots s_{i-1})(s_n \cdots s_i)(s_{i-1} \cdots s_t)$, concluding the proof in type B_n.

Type C_n. In this case Π is either Π_1 or Π_2. The roots in $\Phi^+ \setminus \Phi(\Pi_1)$ that are not orthogonal to Π_1 are of the form:

$$\mu_{t,i-1} = \alpha_t + \cdots + \alpha_{i-1} \text{ with } t < l \text{ and } i \geq l;$$
$$\nu_{t,i} = \alpha_t + \cdots + \alpha_{i-1} + 2(\alpha_i + \cdots + \alpha_{n-1}) + \alpha_n \text{ with } t < l \text{ and } i \leq l.$$

It is enough to rule out the $\mu_{t,l-1}$. This is achieved by using the usual minimality argument and $\tau = s_{t+1} \cdots s_{l-1}$. The required conditions hold for both choices of Π hence all roots occurring in v are orthogonal to Π_1.

If $\Pi = \Pi_2$ the possible occurring roots that are not orthogonal to Π are of the following form:

$$\mu_{t,i-1} = \alpha_t + \cdots + \alpha_{i-1} \text{ with } 1 \leq t \leq i-1 \leq l-1;$$
$$\omega_i = 2(\alpha_i + \cdots + \alpha_{n-1}) + \alpha_n \text{ for } 1 \leq i \leq l-1;$$
$$\nu_{t,i} = \alpha_t + \cdots + \alpha_{i-1} + 2(\alpha_i + \cdots + \alpha_{n-1}) + \alpha_n$$
$$\text{with } 1 \leq t \leq i-1 \leq l-1 \text{ and } t < i-1 \text{ for } t \text{ odd}.$$

Let us consider the first set of roots. We consider the root of minimal height of type $\mu_{t,i-1}$. By observation II we might assume that $\alpha_t \notin \Pi$. Then we may use $\tau = s_{t+1} \cdots s_{i-1}$ in order to rule it out. For the second set of roots, by observation II we may assume that $\alpha_i \notin \Pi$ and we may use $\tau = s_{n-1} \cdots s_i$. The last set of roots is ruled out by using $\tau = (s_{n-1} \cdots s_t)(s_{n-1} \cdots s_i)$.

Type F_4. In this case Π is either Π_1 of type B_3, Π_2 of type C_3 or Π_3 of type B_2. We shall apply observation I with $\Delta' = \Pi_1$ or Π_2.

If $\Pi = \Pi_1$ a direct computation shows that the roots that might occur in v and are not orthogonal to Π lie either in the W_Π-orbit of α_4 or in the W_Π-orbit of $\alpha_2 + 2\alpha_3 + 2\alpha_4$. The first one is ruled out by using $\tau = 1$ whereas for the second one we may use $\tau = s_3 s_4$.

If $\Pi = \Pi_2$ the roots that might occur in v and are not orthogonal to Π lie in the W_Π-orbit of α_1 or in the W_Π-orbit of $\alpha_1 + \alpha_2 + \alpha_3$. We rule out these roots by using $\tau = 1$ and $\tau = s_2 s_1$, respectively.

If $\Pi = \Pi_3$ the roots that might occur in v and are not orthogonal to Π lie in the W_Π-orbit of $\alpha_1+\alpha_2+\alpha_3+\alpha_4$ or in the W_Π-orbit of $\alpha_1+\alpha_2+2\alpha_3+2\alpha_4$. We show that these roots might not occur by using $\tau = s_3s_2s_1$ and $\tau = s_1s_3s_4$, respectively.

Type G_2. In this case Π consists of a simple root.

If $\Pi = \{\alpha_1\}$ the positive roots that do not lie in Π and are not orthogonal to α_1 are α_2; $s_1\alpha_2$; $\alpha_1+\alpha_2$ and $s_1(\alpha_1+\alpha_2)$. We rule out α_2 by using $\tau = 1$ and $\alpha_1+\alpha_2$ by using $\tau = s_2$.

If $\Pi = \{\alpha_2\}$ the positive roots that do not lie in Π and are not orthogonal to α_2 are α_1; $s_2\alpha_1$; $3\alpha_1+\alpha_2$ and $s_2(3\alpha_1+\alpha_2)$. We rule out α_1 by using $\tau = 1$ and $3\alpha_1+\alpha_2$ by using $\tau = s_1$. $\square$

If Φ is multiply-laced then $w_0 = -1$ so $(T^w)^\circ = \langle \mathrm{Im}(\alpha^\vee),\ \alpha \in \Pi\rangle$ commutes with $x = \dot{w}\mathrm{v} \in v_0$. However, there might be mutually orthogonal roots α and $\gamma \in \Phi^+$ for which $\alpha+\gamma \in \Phi$ so that $x_\alpha(h)$ and $x_\gamma(h')$ do not commute. By [4, Chapitre VI, §1.3] this might happen only if Φ is doubly-laced and α and γ are short roots. Therefore if Φ is of type G_2, Lemma 4.8 implies that $(T^w)^\circ U_w$ commutes with $x = \dot{w}\mathrm{v} \in v_0$ and Theorem 4.4 is proved. For the doubly-laced types there is still some work to be done.

Lemma 4.9 *Let Φ be doubly-laced. Let $\mathcal{O}$, v_0, w, and Π be as in Theorem 4.4. Let $x = \dot{w}\mathrm{v} \in \dot{w}B \cap \mathcal{O}$. Then x is centralized by $(T^w)^\circ U_w$.*

Proof. If $\Pi = \emptyset$ then $U_w = U_{w_0}$ is trivial and so is $(T^w) = (T^{w_0})^\circ$ thus there is nothing to prove. Similarly, the statement is clear if $\Pi = \Delta$ so that $w = 1$ and x is central.

By Lemmas 3.4 and 4.8 it is enough to show that X_α centralizes v for every $\alpha \in \Pi$. This is true unless α is short and there occurs a short root γ in v, orthogonal to α and such that $\alpha+\gamma \in \Phi$. We shall analyze the different cases separately using terminology and notation introduced in Corollary 4.2 and Lemma 4.8.

Type B_n. The only short root in Π is $\alpha = \alpha_n$. If there occurs γ in v with $\gamma \perp \alpha_n$ and $\gamma+\alpha_n \in \Phi$ then γ is one of the roots $\gamma_i = \alpha_i + \cdots + \alpha_n$ for $1 \leq i \leq l-1$ and we necessarily have $\Pi = \Pi_1$. Let us choose an ordering of the positive roots which is non-decreasing with respect to the height and let us write v as a product of elements in root subgroups taken in this order. Then $x = \dot{w}t\mathrm{v}_1 x_{\gamma_{l-1}}(a_{l-1})\mathrm{v}_2 \cdots \mathrm{v}_{l-1}x_{\gamma_1}(a_1)\mathrm{v}_l$ for some $a_j \in k$ and some $\mathrm{v}_j \in U$ commuting with X_{α_n}. Conjugation by $x_{\alpha_n}(1)$, Lemma 3.4 and Chevalley's commutator formula give

$$x_{\alpha_n}(1)xx_{\alpha_n}(-1) = \dot{w}t\mathrm{v}_1 x_{\gamma_{l-1}}(a_{l-1})x_{\gamma_{l-1}+\alpha_n}(a'_{l-1})\mathrm{v}_2 \cdots \mathrm{v}_{l-1}x_{\gamma_1}(a_1)x_{\gamma_1+\alpha_n}(a'_1)\mathrm{v}_l \in \mathcal{O} \cap \dot{w}B.$$

If $a_j \neq 0$ for some j we would have $a'_j \neq 0$, contradicting Lemma 4.8.

Type C_n. Let $\alpha \in \Pi_1$ be a short root. There is only one positive root γ such that $\gamma \perp \alpha$ and $\alpha+\gamma \in \Phi$ and it is not orthogonal to Π_1 so the statement holds for $\Pi = \Pi_1$.

Let $\Pi = \Pi_2$ and let $\alpha_i \in \Pi_2 \setminus \Pi_1$. The only positive root γ such that $\gamma \perp \alpha_i$ and $\alpha_i + \gamma \in \Phi$ is $\alpha_i + 2(\alpha_{i+1}+\cdots+\alpha_{n-1})+\alpha_n$. If such a root would occur in v we would have $x = \dot{w}t\mathrm{v}_1x_\gamma(a)\mathrm{v}_2$ with $\mathrm{v}_1, \mathrm{v}_2 \in U$ commuting with X_{α_i} and $a \in k^\cdot$. Conjugation by $x_{\alpha_i}(1)$, Lemma 3.4 and Chevalley's commutator formula give:

$$x_{\alpha_i}(1)xx_{\alpha_3}(-1) = \dot{w}t\mathrm{v}_1x_\gamma(a)x_{\gamma+\alpha_i}(a')\mathrm{v}_2 \in \mathcal{O} \cap \dot{w}B$$

with $a' \neq 0$, contradicting Lemma 4.8.

Type F_4. Let $\Pi=\Pi_1$. The only short root in Π_1 is α_3 and the only positive root orthogonal to Π_1 for which $\alpha_3+\gamma\in\Phi$ is $\gamma=\alpha_1+2\alpha_2+3\alpha_3+2\alpha_4$. If $x=\dot{w}t\mathrm{v}$ could be written as $\dot{w}t\mathrm{v}_1x_\gamma(a)\mathrm{v}_2$ for some $a\in k^{\cdot}$ and for some $\mathrm{v}_i\in U$ commuting with X_{α_3}, conjugation with $x_{\alpha_3}(1)$ arguing as in type C_n would lead to a contradiction.

Let $\Pi=\Pi_2$. There are no positive roots γ orthogonal to Π for which $\gamma+\alpha_3$ or $\gamma+\alpha_4$ lies in Φ so the result holds in this case.

Let $\Pi=\Pi_3$. The only short root in Π_3 is α_3 and the roots γ orthogonal to Π for which $\gamma+\alpha_3\in\Phi$ are: $\alpha_1+\alpha_2+\alpha_3$ and $\alpha_1+2\alpha_2+3\alpha_3+2\alpha_4$. Conjugation by $x_{\alpha_3}(1)$ and Lemma 4.8 show that none might occur in v. □

Lemma 4.9 implies that $\dim(v_0\cap\dot{w}U)=0$ for Φ doubly-laced, conlcuding the proof of Theorem 4.4 □

Corollary 4.10 *Let $\mathcal{O}$ be a conjugacy class in a simple algebraic group over an algebraically closed field k of good odd characteristic. Then $\mathcal{O}$ is spherical if and only if there is a B-orbit v in $\mathcal{O}$ for which* $\dim\mathcal{O}=\ell(\phi(v))+\mathrm{rk}(1-\phi(v))$.

Proof. One direction is Theorem 4.4, the other direction is [10, Theorem 5] whose proof holds also in positive characteristic. □

Corollary 4.11 *Let $\mathcal{O},\ v_0,\ w$ be as in Lemma 4.8. Then $v_0=\mathcal{O}\cap BwB$.*

Proof. Let $v\in\mathcal{V}$ be such that $\phi(v)=\phi(v_0)=w$. By [10, Theorem 5] we have $\dim v=\dim\mathcal{O}$ therefore $v=v_0$. □

Remark 4.12 If $\mathcal{O}$ is a symmetric conjugacy class over an algebraically closed field of odd or zero characteristic Theorem 4.4 follows from [25, Proposition 3.9, Theorem 4.6, Theorem 7.1] and Corollary 4.11 follows from [25, Theorem 7.11, Lemma 7.12, Theorem 7.13]. If $\mathcal{O}$ is a spherical conjugacy class over an algebraically closed field of characteristic zero Theorem 4.4 is [10, Theorem 1] and Corollary 4.11 is [10, Corollary 26].

Corollary 4.13 *Let $\mathcal{O},\ v_0,\ w$ be as in Theorem 4.4. For every $x\in v_0\cap\dot{w}B$ we have $U\cap G_x=U_w$ and $T_x^\circ=(T^w)^\circ$ so that $\dim U.x=\ell(w)$ and $\dim T.x=\mathrm{rk}(1-w)$.*

Proof. By Lemmas 4.6, 4.7, 4.8 and 4.9 for every $x\in\dot{w}U\cap v_0$ we have $U\cap G_x\supset U_w$ and $T_x^\circ\supset(T^w)^\circ$ so that $B_x^\circ\supset(T^w)^\circ U_w$. For dimensional reasons all inclusions are equalities. □

Remark 4.14 Another direct consequence of Theorem 4.4 is a generalization of [22, Proposition 6.3]. Let k_0 be the number of even exponents of $\mathrm{Lie}(G)$. Then for every spherical conjugacy class we have

$$\dim\mathcal{O}\leq\ell(w_0)+\mathrm{rk}(1-w_0)=\dim B-(n-\mathrm{rk}(1-w_0))=\dim B-k_0.$$

We end this section with some further consequences of the above results.

Let us recall that a standard parabolic subgroup can be naturally attached to $v\in\mathcal{V}$ ([16, §2]):

$$P(v)=\{g\in G\mid g.v=v\}.$$

Let $L(v)$ denote its Levi component containing T and let $\Delta(v)$ be the corresponding subset of Δ: this is the so-called set of simple roots of v.

Proposition 4.15 *Let $\mathcal{O}$, v_0, w and Π be as in Theorem 4.4. Then $\Delta(v_0) = \Pi$.*

Proof. Let $\alpha \in \Pi$ and let $x \in v_0 \cap \dot{w}B$. Arguing as in Lemmas 4.6, 4.8 and 4.9 we see that $X_{-\alpha}$ commutes with x. Then

$$X_{-\alpha}.v_0 = X_{-\alpha}B.x \subset P_{\alpha}^{u}.(X_{-\alpha}.(X_{\alpha}.x)) = P_{\alpha}^{u}.x = v_0$$

so $\alpha \in \Delta(v_0)$ and $\Pi \subset \Delta(v_0)$.

By [8, Lemma 1(ii)] whose the argument works also in positive characteristic the derived subgroup $[L(v_0), L(v_0)]$ of $L(v_0)$ fixes a point in v_0. Thus, if α lies in $\Delta(v_0)$ there is $y = u\dot{w}\mathrm{v} \in v_0$ for which $X_\alpha \in U \cap G_y$ and therefore $u^{-1}X_\alpha u \in U \cap G_x$ for $x \in \dot{w}U \cap v_0$. By Corollary 4.13 we have $u^{-1}X_\alpha u \subset U_w = \langle X_\gamma \mid \gamma \in \Phi(\Pi)\rangle$. This is possible only if $\alpha \in \Pi$. □

Remark 4.16 In characteristic zero Proposition 4.15 follows from [8, Page 289] and [21, Corollary 3].

We shall consider an application of the above results to the analysis of the G-module decomposition of the ring $k[\mathcal{O}]$ of regular functions on a spherical conjugacy class $\mathcal{O}$. It is well-known that such a G-module is multiplicity-free ([15], [9]).

Theorem 4.17 *Let $\mathcal{O}$, v_0, w, Π as in Theorem 4.4. The weights occurring in the G-module decomposition of $k[\mathcal{O}]$ are self-adjoint and lie in $P^+ \cap Q \cap \mathrm{Ker}(1 + w)$.*

Proof. By Corollary 4.13 for every $x \in v_0 \cap \dot{w}B$ we have $(B_x)^\circ = (T_x)^\circ U_x$. Besides, a conjugacy class $\mathcal{O}$ is locally closed in G so we may apply the arguments in the proof of [21, Corollary 2 (iii)] and [21, §6] to see that weights occurring in the G-module decomposition of $k[\mathcal{O}]$ lie in $\mathrm{Ann}(T_x) = \{\lambda \in P \mid \lambda(t) = 1,\ \forall t \in T_x\} \subset \mathrm{Ann}(T_x^\circ) = \mathrm{Ann}((T^w)^\circ)$. It follows from Lemmas 4.6, 4.7, 4.8 and the description of $(T^w)^\circ$ that for $\lambda \in \mathrm{Ann}(T_x)$ we have $(\lambda, \alpha) = (\lambda, \vartheta\alpha)$ for $\alpha \in \Delta \setminus \Pi$ and $0 = (\lambda, \alpha) = (\lambda, -\vartheta\alpha) = (\lambda, \vartheta\alpha)$ for $\alpha \in \Pi$. Hence $\vartheta\lambda = \lambda$ and we have the first statement. For the second statement the inclusion in P^+ is obvious, the inclusion in Q follows from the fact that the ring of regular functions on $\mathcal{O}$ is a G_{ad}-module. Moreover, $\Delta(v_0) = \Pi$ by Proposition 4.15 and $\lambda \perp \Delta(v_0)$ by [8, Lemma 1(ii)], where the proof holds also in positive characteristic. Then the first statement implies that $-\lambda = w_0\lambda = ww_\Pi\lambda = w\lambda$. □

Remark 4.18 The problem of the G-module decomposition of spherical nilpotent orbits has been already addressed in [1], [20] and [23]. The analysis of $k[G/K]$ for a symmetric variety is to be found in [32].

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