Representation Theory of groups

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Abstract

It is not necessary to solve all the exercises to pass the exam but Exercise 4 is mandatory. **All answers have to be motivated.**

1. Let G be a finite group and let $g, h \in G$. Show that g is conjugate to h if and only if $\chi(g) = \chi(h)$ for every irreducible complex character of G.

2. Let Q be the following Dynkin quiver of type D_4

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   (a) Give a labeling of the nodes so that if there is a path from i to j then $i < j$.

   (b) Let $V = V_{\alpha_1}$ be the irreducible representation of dimension vector α_1, according to your labeling. Verify that V is surjective at 4.

   (c) Verify that $F_4^+ V$ is surjective at 3, that $F_3^+ F_4^+ V$ is surjective at 2, that $F_2^+ F_3^+ F_4^+ V$ is surjective at 1.

   (d) Provide an explicit description of the representation $W = F_1^+ F_2^+ F_3^+ F_4^+ V$ of Q (i.e., dimension vector and linear maps between the vector spaces).

   (e) Prove that W is indecomposable but not irreducible.

3. Let k be an algebraically closed field. Let A be a finite dimensional semisimple algebra over k and let $Z(A)$ be the center of A

(a) Show that $a \in A$ acts as a scalar on every irreducible representation of A if and only if $a \in Z(A)$ (Hint: use map $\oplus_{i\in I}\rho_i \colon A \to \mathrm{End}(\oplus_{i\in I}V_i)$ where the $\rho_i \colon A \to \mathrm{End}(V_i)$ for $i \in I$ are all the irreducible representations of A).

(b) Let G be a finite group and assume in addition that $k = \mathbb{C}$. For the character χ of a representation ρ of G, let

$$Z(\chi) = \{g \in G \mid |\chi(g)| = \chi(1)\}.$$

Show that $g \in Z(\chi)$ if and only if $\rho(g)$ is a scalar matrix.

(c) Show that $Z(G) = \bigcap_{i\in I} Z(\chi_i)$ where the χ_i are all the complex irreducible characters of G.

4. Let the following table be part of the character table of a finite group G:

G	$Cl(1)$	$Cl(a)$	$Cl(b)$	$Cl(c)$	$Cl(d)$	$Cl(e)$	$Cl(f)$
χ_1		1	1	1	1	1	1
χ_2	1		1	ζ	ζ^2	ζ	ζ^2
χ_3	1	1		ζ^2	ζ	ζ^2	ζ
χ_4	2	-2	0		-1		1
χ_5	2	-2	0	$-\zeta$		ζ	ζ^2
χ_6	2	-2	0	$-\zeta^2$	$-\zeta$		ζ
χ_7	3	3	-1	0	0	0	

where ζ is a primitive third root of unity, $Cl(g)$ denotes the conjugacy class of $g \in G$ and a, b, c, d, e, f are nontrivial elements of G.

(a) Determine $\chi_1(1)$ and the order of G.

(b) Complete the character table of G.

(c) Determine the order of each conjugacy class.

(d) Determine the order of $[G, G]$ and the order of $Z(G)$.

(e) Is G a solvable group?

(f) Show that c cannot be conjugate to its inverse whereas b is conjugate to its inverse.

(g) Let ρ_i for $i = 1, \ldots, 7$ be the representation of G whose character is χ_i. Decompose $\rho_7 \otimes \rho_4$, $\rho_7 \otimes \rho_5$ and $\rho_7 \otimes \rho_6$ into irreducibles.