Logarithmic growth and Frobenius filtrations for solutions of p-adic differential equations

Bruno Chiarellotto*
Nobuo Tsuzuki†

Version of October 24, 2006

Abstract

For a ∇-module M over the ring $K[\![x]\!]_0$ of bounded functions we define the notion of special and generic log-growth filtrations. Moreover, if M admits also a φ-structure then it is endowed also with generic and special Frobenius slope filtrations. We will show that in the case of M a φ-∇-module of rank 2, the two filtrations coincide. We may then extend such a result also for a module defined on an open set of the projective line. This generalizes previous results of Dwork about the coincidences between the log-growth of the solutions of hypergeometric equations and the Frobenius slopes.

Contents

Introduction

The meaningful connections in the p-adic setting are those with a good radius of convergence for their solutions or horizontal sections. In general this condition is referred as the condition of solvability and it is granted, for example, under the hypotheses of having a Frobenius structure. It has been noticed long time ago that the solutions do not have in

*chiarbru@math.unipd.it
†tsuzuki@math.sci.hiroshima-u.ac.jp

general the same growth in having their good radius of convergence. This has been indicated as the logarithmic growth (log-growth). The first example of this fact was studied by Dwork in considering the Gauss-Manin (hypergeometric) connection arising from the Legendre family of elliptic curves. At the supersingular points (which are not singular points of the connection) it was noted in [14, 8.6] that the solutions have, of course, radius of convergence 1, but they are not bounded (in [6, 5.2.2] it is indicated, without proof, that they have of log-growth equal to $\frac{1}{2}$)! While for ordinary points it was also proved in [14, 8.2] that one solution should be bounded (i.e. of log-growth equal to 0). On the other hand, arising from the geometry, this hypergeometric connection admits Frobenius and the slopes of the Frobenius at the supersingular points (resp. ordinary points) are well known to be exactly $\frac{1}{2}$ (resp. 0 and 1). From this, two lines of investigations could naturally arise. What should be the log-growth behaviour of the solutions varying on the points? And, secondly, in case there is a Frobenius structure what should be the connection between slopes and log-growth? In this article we would like to give some insight on these questions and to show how for the rank two situation (eventually after twisting the Frobenius or by avoiding trivial cases) the slopes behaviour coincides with the log-growth one; hence the coincidence of the slopes and log-growth is a general fact.

The log-growth behaviour was first studied by Dwork in [14], [15], [16], [18], Robba [27] and Christol [6]: they were able to define filtration associated to the log-growth and its associated log-polygon. They defined also generic log-growth and special: in particular at the generic point we always have solutions with log-growth equal to 0. It has been conjectured that the behaviour of the log-growth polygon should obey to the same rules that for the slope polygon for the Frobenius [15, Conjecture 2] (see also [13, Lemma 7.2]). But other than the examples on the hypergeometric differential equations not too much has been known. In this article we would like to study both at the special point and at the generic point the log-growth behaviour for a connection defined on $K[\![x]\!]_0$ (the ring of bounded power series on the open unit disk) endowed with a Frobenius structure. We will give the definitions of both generic and special log-growth filtrations together with their relations with the more usual generic and special Frobenius slopes filtrations. We won't deal in this article with the case of singular point i.e. for example, with the case of a solvable connection defined on the bounded Robba ring (where it exists a generic point, but the special one has a more involved definition). The definition of the log-growth filtration in this case will be an analogous of Kedlaya's definition of Frobenius slope filtration (Newton filtration) which was given in the even more general case that one of the usual (i.e., unbounded) Robba ring. On the other hand the log-growth filtration should be connected with the monodromy filtration, as it as been clearly indicated by Christol and Mebkhout [7] in their definition of monodromy filtration on solvable modules which have the Robba condition (filtration which they build by using the log-growth 0, i.e., the bounded solutions). Moreover we won't in general deal with global behaviour of the log-growth polygon, but we only obtain partial results in case of open subscheme of the projective line.

Here it is the plan of the article. After defining the setting we will introduce the definition of log-growth in paragraph 1. In paragraph 2, we study the way to define a Frobenius slope filtration for a φ-module defined over a field which may not have perfect residue field and not algebraically closed. In paragraph 3 we study the generic log-growth filtration for a module defined over $K[\![x]\!]_0$ (it will be given by differential submodules on $\mathcal{E}$), while in paragraph 4, we introduced also the special log-growth filtration (which it will be given by K-vector spaces). At the paragraph 5 we give an example of unipotent connections and two examples on the type of log-growths one can meet without supposing the existence of Frobenius, i.e., one cannot expect to have rational log-growths. In paragraph 6, we will introduce a Frobenius structure and we will deal with φ-∇-modules on $K[\![x]\!]_0$ and their Frobenius slopes and log-growth filtrations. At the end of paragraph 6 we give a new proof of the specialization theorem of Frobenius structures ([20, Appendix], [21, 2.3.2]). Finally, in the last paragraph, after indicating what should be the cases in which the coincidence cannot happen for trivial reason, we will show the coincidence of the special and generic log-growth and Frobenius slope filtrations for a φ-∇-module over $K[\![x]\!]_0$ of rank 2. In particular we conclude that for rank 2 modules over $K[[x]]_0$ the log-growth filtration increases under specialization (i.e. the associated polygon).

We end with the remark that we conjecture that the statement should be true for any rank and for higher dimension. In particular, for a ∇-module over $K[\![x]\!]_0$ (this time without Frobenius structure) the log-growth filtration should increase under specialization.

We would like to thank Olivier Brinon for his really precious explanations. During the preparation the authors were supported by Research Network "Arithmetic Algebraic Geometry" of the EU (Contract MRTN-CT-2003-504917), MIUR (Italy) project GVA, Japan Society for the Promotion of Science, and Inamori Foundation (Japan).

0.1 Notations. Definitions.

Let us fix the notation as follows;

p : a prime number;

q : a positive power of p;

K : a complete discrete valuation field of mixed characteristics $(0, p)$;

$\mathcal{V}$: the integer ring of K;

k : the residue field of $\mathcal{V}$;

π : a uniformizer of $\mathcal{V}$;

$|\text{-}|$: a p-adic absolute value on a p-adic completetion of a fixed algebraic closure of K, normalized by $|p| = p^{-1}$;

σ : (q-)Frobenius on K, i.e., a lift of q-Frobenius endomorphism ($x \mapsto x^q$ on k). When we discuss φ-modules, we suppose an existence of the lift σ on K.

Let us define a K-algebra $K[\![x]\!]_0$ given by the bounded formal power series with coefficients in K:

$$K[\![x]\!]_0 = \left\{ \sum_{n=0}^{\infty} a_n x^n \;\middle|\; a_n \in K,\ \sup_n |a_n| < \infty \right\},$$

it is a Banach algebra for the Gauss norm $|\text{-}|_0$, defined by

$$\left| \sum_n a_n x^n \right|_0 = \sup_n |a_n|.$$

An endomorphism on $K[\![x]\!]_0$ which satisfies $\sigma|_K = \sigma$ (q-Frobenius on K), $|\sigma(x) - x^q|_0 < 1$ and $\sigma(x) \equiv 0 \,(\mathrm{mod}\, x)$. σ is called a (q-)Frobenius on $K[\![x]\!]_0$.

Let us define a K-algebra $\mathcal{E}$ by

$$\mathcal{E} = \left\{ \sum_{n=-\infty}^{\infty} a_n x^n \;\middle|\; a_n \in K,\ \sup_n |a_n| < \infty,\ \lim_{n \to -\infty} a_n = 0 \right\}.$$

Then $\mathcal{E}$ is a complete discrete valuation field with a residue field $k(\overline{x})$ (where by $\overline{x}$ we indicate the reduction of x) under the Gauss norm $|\sum_n a_n x^n| = \sup_n |a_n|$. We denote by $\mathcal{V}_{\mathcal{E}}$ the valuation ring of $\mathcal{E}$. We have a natural isometric inclusion of K-algebras

$$K[\![x]\!]_0 \to \mathcal{E},$$

which is compatible with extentions of continuous K-derivations. We denote by σ a continuous lift of (q)-Frobenius on $\mathcal{E}$ which is an extension of σ on K.

Let us define a K-algebra E by the completion of the function field $K(x)$ with one variable x over K under the Gauss norm (see the construction in Remark 0.2 below). E is a p-adically complete discrete valuation field with residue field $k(\overline{x})$. The K-derivation $\frac{d}{dx}$ on $K(x)$ uniquely extends to a continuous K-derivation on E. We also denote by σ a continuous lift of (q)-Frobenius on E which is an extension of σ on K.

For any $\overline{a} \in k$ (resp. $\overline{a} = \overline{\infty}$) and a lifting $a \in \mathcal{V}$ (resp. $a = \infty$), there is a natural isometric inclusion

$$j_a : E \to \mathcal{E}_a$$

of K-algebras defined by $x - a \mapsto x - a$ (resp. $\frac{1}{x} \mapsto x_\infty$ if $a = \infty$), where $\mathcal{E}_a$ means that one replaces the indeterminate x by $x - a$ in the definition. (See [6, 2.4.7].) j_a is compatible with derivations. Any Frobenius on E uniquely extends on $\mathcal{E}_a$ and it is a Frobenius on $\mathcal{E}_a$

Remark 0.1 It should be noted that we don't ask for the moment that $K[\![x]\!]_0$ (resp. E) is stabilized by the Frobenius on $\mathcal{E}$. Later, when we will deal with $K[\![x]\!]_0$ (resp. E) endowed with Frobenius structure, then we will take an extention to $\mathcal{E}$ of such a structure.

Remark 0.2 We will focus our theory and definitions on the local case $K[\![x]\!]_0$: there will be a special situation defined over K and a generic one over $\mathcal{E}$ and E. On the other hand, following Kedlaya's construction [23, section 3], one can think $\mathcal{E}$ and E as in the following setting. Notation as before are in force. Let's fix an algebra homomorphism $C_k \to C_{k(\overline{x})}$ of Cohen rings for k and $k(\overline{x})$, respectively, and an algebra homomorphism $C_k \to \mathcal{V}$. We then define the following

$$\Gamma^{k(\overline{x})} = \mathcal{V} \otimes_{C_k} C_{k(\overline{x})}.$$

In particular by choosing instead of $k(\overline{x})$ the field $k((\overline{x}))$ one will find $\Gamma^{k((\overline{x}))}$, which can be described as the ring of series, by choosing a lift x of $\overline{x}$, $\sum_{i \in \mathbf{Z}} a_i x^i$ such that $a_i \in \mathcal{V}$ and $|a_i| \to 0$ if $i \to -\infty$. We have $\Gamma^{k((\overline{x}))}[\frac{1}{p}] = \mathcal{E}$ in Fontaine's notation [19]. It is also possible to consider $\Gamma^{k((\overline{x}))}$ as the completion of the ring $\mathcal{R}^b_{\mathcal{V}}$ of the formal power series $\sum_{i \in \mathbf{Z}} a_i x^i$ with coefficients in $\mathcal{V}$ such that there exists $\eta < 1$ with $|a_i|\eta^i \to 0$ if $i \to -\infty$ for the Gauss norm (in this case such a completion coincides with the π-adic completion). We will refer to $\mathcal{R}^b_{\mathcal{V}}[\frac{1}{p}] = \mathcal{R}^b$ as the "bounded Robba ring" (the notation as in Crew [9], which coincides in Kedlaya's notation with $\Gamma^{k((\overline{x}))}_{\mathrm{conv}}[\frac{1}{p}]$ and in the second author's notation with $\mathcal{E}^\dagger$ [29]).

But then one could have pursued the case of $k(\overline{x})$ by constructing $\Gamma^{k(\overline{x})}$, which is complete for the π-adic topology. By choosing a lifting x of $\overline{x}$ in $\Gamma^{k(\overline{x})}$ then one may define a map

$$\iota : \mathcal{V}[x] \to \Gamma^{k(\overline{x})}$$

and the map is given by sending x in a lifting of $\overline{x}$. We want to see that ι is a dense immersion. It will follow that the completion of the field of rational functions $K(x)$ under the Gauss norm is equal to $\Gamma^{k(\overline{x})}[\frac{1}{p}]$, and hence $\Gamma^{k(\overline{x})}$ is our E. E is the field of analytic elements (see [6, section 2]).

The map ι is injective being injective mod π and $\Gamma^{k(\overline{x})}$ is without π-torsion and $\mathcal{V}[x]$ (x is seen as an indeterminate) is separated for the π-adic topology. On the other hand the map ι can be extended to the localization

$$\iota_{(\pi)} : (\mathcal{V}[x])_\pi \to \Gamma^{k(\overline{x})}$$

($\Gamma^{k(\overline{x})}$ admits π as uniformizer). On the ring $\Gamma^{k(\overline{x})}$ the norm is given by the sup on the coefficients (i.e. the Gauss norm coincide with the π-adic one), hence the completion of $(\mathcal{V}_K[x])_\pi$ by the Gauss norm and the π-adic norm are same. On the other hand such a completion should be $\Gamma^{k(\overline{x})}$ because it is complete for the π-adic topology and it has the same residue field. It follows that $(\mathcal{V}_K[x])_\pi$ is dense in $\Gamma^{k(\overline{x})}$. Hence $K(x)$ is dense in $\Gamma^{k(\overline{x})}[\frac{1}{p}]$, which is complete for the π-adic norm that in $K(x)$ coincides with the Gauss's one. The field of analytic elements of $K(x)$ is $E = \Gamma^{k(\overline{x})}[\frac{1}{p}]$.

0.2 Connection and Frobenius.

Let R be a commutative ring. We denote by $\mathrm{Mat}(r, R)$ (resp. $\mathrm{GL}(r, R)$) the set of matrixes of degree r with entries in R (resp. the general linear group of degree r with entries in R). Let $A = (a_{ij})$ be a matrix over R. For a map τ from R, we put $A^\tau = \tau(A) = (\tau(a_{ij}))$. If $|\text{-}|$ be a semi-norm on R, we put $|A| = \max_{i,j} |a_{ij}|$.

The main objects of this paper are φ-modules, ∇-modules and φ-∇-modules over or $K[\![x]\!]_0$, $\mathcal{E}$ or E.

Let R be either one of the previous rings. Let $d : R \to \Omega_R$ be a continuos K-derivation such that $d\left(\sum_i a_i x^i\right) = \left(\sum_i i a_i x^{i-1}\right) dx$, where $\Omega_R = R dx$. A ∇-module M over R is a free R-module of finite rank endowed with a homomorphism $\nabla : M \to M \otimes_R \Omega_R$ such that $\nabla(m + n) = \nabla(m) + \nabla(n)$ and $\nabla(am) = m \otimes da + a\nabla(m)$ for $a \in R, m, n \in M$. Let $(e_1, \cdots, e_r)$ be a free basis of a ∇-module M. A morphism of ∇-modules is an R-linear homomorphism which commutes with connections. The category of ∇-modules over R is abelian with tensor products, internal homs and a unit object. A matrix G over R defined by

$$\nabla(e_1, \cdots, e_r) = (e_1, \cdots, e_r) G \otimes dx$$

is called a matrix of the connection ∇ with respect to the basis $e_1, \cdots, e_r$. If we change a basis by $(e_1, \cdots, e_r) \to (e_1, \cdots, e_r)P$ ($P \in \mathrm{GL}(r, R)$), then the matrix G' of connection is given by $G' = P^{-1} \frac{d}{dx} G + P^{-1} G P$.

Suppose that R is endowed with a Frobenius endomorphism σ. A φ-module M over R is a free R-module of finite rank endowed with a σ-linear homomorphism $\varphi : M \to M$, called Frobenius, such that $\varphi_\sigma : \sigma^* M \to M$ ($a \otimes m \mapsto a\varphi(m)$ ($a \in R, m \in M$)) is an isomorphism of R-modules. A morphism of φ-modules is an R-linear homomorphism which commutes with Frobenius. The category of φ-modules over R is abelian with tensor products, internal homs and a unit object. Let $(e_1, \cdots, e_r)$ be a free basis of a φ-module M. A matrix A over R defined by

$$\varphi(1 \otimes e_1, \cdots, 1 \otimes e_r) = (e_1, \cdots, e_r) A$$

is called a matrix of the Frobenius φ with respect to the basis $e_1, \cdots, e_r$. A is invertible by definition. If we change a basis by $(e_1, \cdots, e_r) \to (e_1, \cdots, e_r)P$ ($P \in \mathrm{GL}(r, R)$), then the matrix A' of Frobenius is given by $A' = P^{-1}AP^{\sigma}$.

A φ-∇-module M over R is a free R-module of finite rank endowed with a Frobenius $\varphi : M \to M$ and a connection $\nabla : M \to M \otimes_R \Omega_R$ such that φ is horizontal with respect to ∇, that is, the diagram

$$\begin{array}{ccc} \sigma^* M & \xrightarrow{\sigma^*\nabla} & \sigma^*(M \otimes_R \Omega_R) \\ \varphi_\sigma \downarrow & & \downarrow 1 \otimes \varphi \otimes \sigma \\ M & \xrightarrow[\nabla]{} & M \otimes_R \Omega_R \end{array}$$

is commutative. In other word, if A and G are matrixes of Frobenius and of the connection with respect to a basis, then the commutative condition is given by a relation

$$\frac{d}{dx}A + GA = \frac{d\sigma(x)}{dx}AG^{\sigma}.$$

The category of φ-∇-modules over R is abelian with tensor products, internal homs and a unit object. The category of φ-∇-modules over R is independent of the choices of Frobenius σ's on R up to canonical isomorphisms.

Since the derivations commute with the natural map $K[\![x]\!]_0 \to \mathcal{E}$, there is a natural functor

$$(\nabla\text{-modules over } K[\![x]\!]_0) \to (\nabla\text{-modules over } \mathcal{E})$$

by extension of scalars. This functor is exact, commutes with tensor products, internal homs and unit objects Since a Frobenius on $K[\![x]\!]_0$ can extend uniquely on $\mathcal{E}$, the same exist for φ-modules and φ-∇-modules.

The isometry $j_a : E \to \mathcal{E}_a$ (see 0.1) also induces natural functors as above.

0.3 Generic disk

Let t be a generic point. We understand t as an element of a large extension Ω as valuation field over K which induces an isometry

$$\iota : \mathcal{E} \to \Omega$$

sending x to t. Hence t have the absolute value 1 in Ω and the reduction of t is algebraically independent over the residue field k of K. We again denote by $\mathcal{E}$ the image $\iota(\mathcal{E})$. We may then consider $\mathcal{E}[\![X - t]\!]_0$: it is an $\mathcal{E}$-Banach algebra under the Gauss norm $|\text{-}|_0$.

One may also see E as a subfield of $\mathcal{E}$ and again identify E with its image in Ω.

Proposition 0.3 *With the notation as above, we have the followings.*

(1) *The map*

$$\tau : \mathcal{E} \to \mathcal{E}[\![X - t]\!] \quad f \mapsto \sum_{n=0}^{\infty} \frac{1}{n!}\iota\left(\frac{d^n f}{dx^n}\right)(X - t)^n$$

is a K-algebra homomorphism which respects the derivations. The image is included in $\mathcal{E}[\![X - t]\!]_0$ and $|f| = |\tau(f)|_0$.

(2) *Let σ be a lift of Frobenius endomorphism as before. Then $\tau(\sigma(x)) - \sigma(t) \in \mathcal{E}[\![X-t]\!]_0$ with $|\tau(\sigma(x)) - \sigma(t)|_0 \le 1$ and $\tau(\sigma(x)) - \sigma(t)|_{X=t} = 0$. If we define an $\mathcal{E}$-algebra homomorphism*

$$\sigma : \mathcal{E}[\![X-t]\!] \to \mathcal{E}[\![X-t]\!] \quad \sum_n a_n (X-t)^n \mapsto \sum_n \sigma(a_n)(\tau(\sigma(x)) - \sigma(t))^n,$$

then the diagram

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{\tau} & \mathcal{E}[\![X-t]\!] \\ \sigma \downarrow & & \downarrow \sigma \\ \mathcal{E} & \xrightarrow[\tau]{} & \mathcal{E}[\![X-t]\!] \end{array}$$

is commutative and $\sigma(\mathcal{E}[\![X-t]\!]_0) \subset \mathcal{E}[\![X-t]\!]_0$. Moreover it respects the derivations.

The same holds for E, where we can take the restriction of the map τ for $\mathcal{E}$.

Proof. See [6, paragraph 2 and 4.6.3]. □

Let M be a ∇-module (resp. a φ-module, resp. a φ-∇-module) over E (resp. $\mathcal{E}$). By Proposition 0.3 a ∇-module (resp. a φ-module, resp. a φ-∇-module) M^τ over $E[\![X-t]\!]_0$ (resp. $\mathcal{E}[\![X-t]\!]_0$) is associated to M. The construction is compatible with any isometry $j_a : E \to \mathcal{E}_a$ in 0.1 (see 3.5).

1 Log-growth of solutions

Let F be a complete discrete valuation field of mixed characteristics $(0,p)$, for our pourposes we may think to K, $\mathcal{E}$ or E, and let $|\text{-}|$ be its valuation.

1.1 Log-Growth.

Let x be an indeterminate over F (in case $F = \mathcal{E}$ or E, it has been understood as a new indeterminate other than that one in $\mathcal{E}$ or E).

Definition 1.1 *For any $\delta \in \mathbb{R}$ with $\delta \ge 0$, an element $f = \sum_i a_i x^i \in F[\![x]\!]$ is said to have log-growth δ (bounded if $\delta = 0$) if*

$$|f|_\delta = \sup_i \frac{|a_i|}{(i+1)^\delta} < \infty.$$

We denote by $F[\![x]\!]_\delta$ the $F[\![x]\!]_0$-module consisting of functions with log-growth δ ($F[\![x]\!]_0$ for the bounded in 0.1*).*

$F[\![x]\!]_\delta$ is an F-Banach space under $|\text{-}|_\delta$. It is clear that for any $\delta_1 \ge \delta_2$, then $F[\![x]\!]_{\delta_2} \subset F[\![x]\!]_{\delta_1}$. Each $F[\![x]\!]_\delta$ is stable under the F-derivation $\frac{d}{dx}$, but not for integration. The integration of element of $F[\![x]\!]_\delta$ is included in $F[\![x]\!]_{\delta+1}$. If we denote by $\mathcal{A}_F(0,1)$ the ring of power series over F which are analytic on the open unit disk with center at 0; then we have an inclusion (for any δ)

$$F[\![x]\!]_\delta \to \mathcal{A}_F(0,1)$$

which respects the derivations.

For a ∇-module M over $F[\![x]\!]_0$ (the connection in any case is relative to x), we define F-vector spaces $\mathrm{Sol}_\delta(M)$ and $H^0_\delta(M)$ by

$$\mathrm{Sol}_\delta(M) = \left\{ f : M \to F[\![x]\!]_\delta \;\middle|\; \begin{array}{l} f \text{ is a } F[\![x]\!]_0\text{-linear homomorphism} \\ f\left(\nabla\left(\dfrac{d}{dx}\right)v\right) = \dfrac{d}{dx}f(v) \text{ for all } v \in M \end{array} \right\}$$
$$H^0_\delta(M) = \left\{ s \in M \otimes_{F[\![x]\!]_0} F[\![x]\!]_\delta \;\middle|\; \nabla\left(\frac{d}{dx}\right)s = 0 \right\}.$$

$\mathrm{Sol}_\delta(M)$ is called the space of solutions of M with log-growth δ and $H^0_\delta(M)$ is called the space of horizontal sections of M of log-growth δ.

Proposition 1.2 *Let $M^\vee$ be the dual ∇-module of M. Then there is natural isomorphism $\mathrm{Sol}_\delta(M) \cong H^0_\delta(M^\vee)$.*

Proof. If G is a matrix which represents the connection of M for a basis, then $-{}^tG$ represents the connection of $M^\vee$ with respect to dual basis. The assertion easily follows from direct computations. □

1.2 Solutions with log-growth.

Let M be a ∇-module over $F[\![x]\!]_0$. We have $\dim_F \mathrm{Sol}_\delta(M), \dim_F H^0_\delta(M) \leq \mathrm{rank}_{F[\![x]\!]_0} M$.

Definition 1.3 *Let M be a ∇-module of rank r over $F[\![x]\!]_0$, we say it has a full set of solutions of log-growth δ if $M \otimes_{F[\![x]\!]_0} F[\![x]\!]_\delta$ is isomorphic as ∇-module to $F[\![x]\!]^r_\delta$.*

Let G be the matrix of the connection and let G_n be a matrix with entries in $F[\![x]\!]_0$ for each nonnegative integer n defined by

$$G_0 = 1, \quad G_{n+1} = \frac{d}{dx}G_n + GG_n. \tag{TE}$$

Then $Y = \displaystyle\sum_{n=0}^{\infty} \frac{G_n(0)}{n!}x^n$ gives a full set of solutions in $F[\![x]\!]$.

Lemma 1.4 *Let M be a ∇-module over $F[\![x]\!]_0$ and let G, G_n be as above. Then M has a full set of solutions in $F[\![x]\!]_\delta$ if and only if $\sup_n \left|\dfrac{G_n(0)}{n!(n+1)^\delta}\right| < \infty$. In particular, if $\sup_n \left|\dfrac{G_n(0)}{n!(n+1)^\delta}\right| < \infty$ holds for one basis, then the same holds for any basis.*

Proposition 1.5 *Let M be a ∇-module over $F[\![x]\!]_0$ of rank r. Then the following conditions are equivalent.*

(i) *M has a full set of solutions in $F[\![x]\!]_\delta$.*

(ii) $\dim_F \mathrm{Sol}_\delta(M) = r$

(iii) $\dim_F H^0_\delta(M) = r$.

PROOF. (i) $\Rightarrow$ (ii) is trivial. In order to prove (ii) $\Rightarrow$ (i), we need that the solution matrix is invertible. By taking the Wronskian, the solution of the determinant ∇-module is a determinant of the solution matrix. Then the assertion follows from Lemma 1.6 below.

(ii) $\Leftrightarrow$ (iii) follows from Proposition 1.2 □

Lemma 1.6 *Let $a \in \mathcal{A}_F(0,1)$, not zero, such that $\frac{a'}{a} \in F[\![x]\!]_0$. Then it is a unit in $F[\![x]\!]_0$.*

PROOF. Since a has no zero in the open unit disk, one can apply [26, Lemma 5.4.11]. □

Corollary 1.7 *Let M and N be ∇-modules over $F[\![x]\!]_0$ such that they have a full set of solutions in $F[\![x]\!]_\delta$. Then $M \otimes N$, $M^\vee$ and any subquotient of M have a full set of solutions in $F[\![x]\!]_\delta$.*

Proposition 1.8 *Let L be an extension of F as complete valuation fields. A ∇-module M over $F[\![x]\!]_0$ has a full set of solutions in $F[\![x]\!]_\delta$ if and only if the extension $M \otimes L[\![x]\!]_0$ has a full set of solutions in $L[\![x]\!]_\delta$.*

PROOF. The assertion easily follows from Lemma 1.4. □

Proposition 1.9 *Let L be an extension of F in a completion of an algebraic closure of F. Then, given a ∇-module M over $F[\![x]\!]_0$, there is a natural identity*

$$\mathrm{Sol}_\lambda(M \otimes L[\![x]\!]_0) = \mathrm{Sol}_\lambda(M) \otimes_F L$$

for any $\lambda \geq 0$.

PROOF. We may assume that L is the completion of an algebraic closure of F. Let G be the automorphism group of L over F, then the G-invariant part of L is equal to F [1]. Since the action of G preserves the valuation, G acts semi-linearly on $\mathrm{Sol}_\lambda(M \otimes L[\![x]\!]_0)$ and the G-invariant part of $\mathrm{Sol}_\lambda(M \otimes L[\![x]\!]_0)$ is contained in $\mathrm{Sol}_\lambda(M)$. Let $\mathrm{Sol}(M_L)$ (resp. $\mathrm{Sol}(M_L)$) be the space of solutions of M in $F[\![x]\!]$ (resp. $L[\![x]\!]$) which is of dimension $\dim M$ and consider the G-invariant part of the short exact sequence

$$0 \to \mathrm{Sol}_\lambda(M \otimes L[\![x]\!]_0) \to \mathrm{Sol}(M_L) \to \mathrm{Sol}(M_L)/\mathrm{Sol}_\lambda(M \otimes L[\![x]\!]_0) \to 0.$$

Because the G-invariant part of $\mathrm{Sol}(M_L)$ is $\mathrm{Sol}(M)$, the dimension of G-invariant part of $\mathrm{Sol}_\lambda(M \otimes L[\![x]\!]_0)$ is equal to the dimension of $\mathrm{Sol}_\lambda(M \otimes L[\![x]\!]_0)$. Hence we have a natural identity. □

2 φ-modules over a p-adically complete discrete valuation field

2.1 Frobenius

We discuss extensions of Frobenius endomorphisms for some complete discrete valuation field F of mixed characteristics $(0,p)$ (again we may think about K, $\mathcal{E}$ or, eventually, E). A q-Frobenius is a continuous endomorphism on F which is a lift of q-th power map on the residue field. The following two propositions may be well-known.

Proposition 2.1 *Suppose that F admits a q-Frobenius σ.*

(1) *Let L be a finite and unramified extension of F. Then there exists a unique lift $\sigma_L : L \to L$ of Frobenius endomorphism such that $\sigma_L|_F = \sigma$. Moreover, if L is Galois over F, then σ_L commutes with the action of Galois group.*

(2) *For any positive integer e, there is a finite Galois extension L of F which satisfies the following conditions:*

   (i) *the absolute ramification index of L is a multiple of e;*

   (ii) *there exists a lift $\sigma_L : L \to L$ of Frobenius endomorphism such that $\sigma_L|_F = \sigma$. Moreover it admits a suitable power σ_L^i, $i \neq 0$, which commutes with the Galois action.*

Proof. (1) easily follows from Hensel's lemma.

(2) Put $L = F(p^{\frac{1}{e}}, \zeta_e)$ where ζ_e be a primitive e-th root of unity. Then L is a finite Galois extension over F. Now we will extend the Frobenius endomorphism. Let F' be the maximal unramified subextension of F in L. Then L/F' is totally ramified. Since the Frobenius endomorphism has already extended on F' by (1), we may assume that $F' = F$. If e' is the p-primary part of e, then $L = F(p^{\frac{1}{e}}, \zeta_{e'})$.

Let us define an action of Frobenius σ on the polynomial ring $F[x]$ by $f(x) = \sum_i a_i x^i \mapsto f^\sigma(x) = \sum_i \sigma(a_i) x^i$. Let $f(x) \in F[x]$ be a monic irreducible polynomial such that $f(\zeta_{e'}) = 0$. Then the polynomial $f^\sigma(x)$ is irreducible because the extension $F(\zeta_{e'})/F$ is totally ramified, hence we can extend to the perfection $\widehat{F}^{\mathrm{perf}}$ of the residue field (see Proposition 2.2 (1) below). Since $f^\sigma(x)$ divides $x^{e'} - 1$, all solutions of $f^\sigma(x) = 0$ are primitive e'-th roots of unity. Take one solution $\zeta'_{e'}$ and define an extension $\sigma' : F(\zeta_{e'}) \to F(\zeta_{e'})$ by $\sigma'(\zeta_{e'}) = \zeta'_{e'}$ and by σ on F. This is a lift of Frobenius endomorphism since $F(\zeta_{e'})$ is totally ramified over F. A similar argument works for the extension L over $F(\zeta_{e'})$. Because all the roots of the equation $x^e = p$ are contained in L.

Being $\sigma_L(\zeta_e)$ is an e-th root of unity, after taking a suitable power of σ_L we will have $\sigma_L(\zeta_e) = \zeta_e$. On the other hand, $\sigma_L(p^{\frac{1}{e}})$ is $\zeta p^{\frac{1}{e}}$, ζ an e-th root of unity, therefore after taking a suitable power we will have $\sigma_L(p^{\frac{1}{e}}) = p^{\frac{1}{e}}$. □

Proposition 2.2 *Suppose that F admits a q-Frobenius. Then there exists an extension L over F as complete discrete valuation fields with same valuation groups such that L admits a q-Frobenius σ_L with $\sigma_L|_F = \sigma$, which satisfies the following conditions:*

(1) *the residue field of L is a perfection of the residue field of F;*

(2) *the residue field of L is a separable closure of the residue field of F;*

(3) *the residue field of L is an algebraic closure of the residue field of F.*

In the case of (2) the Frobenius σ_L is unique and commutes with the action of the automorphism group of L over F.

L denotes by $\widehat{F}^{\mathrm{perf}}$, $\widehat{F}^{\mathrm{ur}}$ and $\widehat{F}^{\mathrm{alg}}$ for the case (1) - (3), respectively, in this paper.

Proof. (1) Put L to be the completion of direct limit of diagram $F \xrightarrow{\sigma} F \xrightarrow{\sigma} F \xrightarrow{\sigma} \cdots$ and define σ_L by σ at each component. Then (L, σ_L) is a desired object.

(2) follows from Proposition 2.1 (1).

(3) follows from previous (1) and (2). □

2.2 Frobenius slope filtration over F

Let F be a complete discrete valuation field of mixed characteristics $(0, p)$, let us denote by $\mathcal{V}_F$ (resp. k_F) the ring of integers of F (resp. the residue field of $\mathcal{V}_F$) and let σ be a q-Frobenius on F. The definition of φ-modules over F is in 0.2 (replace R by F).

Definition 2.3 *Let M be a φ-module over F.*

(1) *M is said to be unit-root if there is a $\mathcal{V}_F$-module Γ of rank $\dim_F M$ such that $\varphi(\Gamma) \subset \Gamma$ and $\varphi(\Gamma)$ generates Γ over $\mathcal{V}_F$.*

(2) *M is said to be pure of slope $\lambda \in \mathbb{R}$ if there is a finite extension L of F and $\gamma \in L$ with $\lambda = -\log_q|\gamma|$ such that there is an extension σ_L of Frobenius on L with $\sigma_L|_F = \sigma$ and $(M \otimes_F L, \gamma^{-1}\varphi)$ is unit-root.*

Let M be a φ-module over F. Then, for any positive integer i, (M, φ^i) becomes a φ-module over F with respect to the q^i-Frobenius σ^i. We simply say that (M, φ^i) is φ^i-module associated to (M, φ).

Proposition 2.4 *Let M be a φ-module over F and let i be a positive integer. (M, φ) is pure of slope λ if and only if (M, φ^i) is pure of slope λ.*

Definition 2.5 *Let M be a φ-module over F. An increasing filtration $\{S_\lambda M\}$ of M indexed by $\lambda \in \mathbb{R}$ is called a Frobenius slope filtration of M if the filtration satisfies the conditions*

(i) $\cup_\lambda S_\lambda M = M$ *and* $\cap_\lambda S_\lambda M = 0$*;*

(ii) *$S_\lambda M$ is a sub φ-module over F;*

(iii) *for any λ, if we define $S_{\lambda-} M = \cup_{\nu<\lambda} S_\nu M$, then $S_\lambda M / S_{\lambda-} M$ is a φ-module pure of slope λ.*

Since M is of finite dimension, there are finite rational numbers $\lambda_1 < \lambda_2 < \cdots < \lambda_n$ such that the slope filtration jumps at λ_i, that is, $S_{\lambda_i} M \neq S_{\lambda_i -} M$. λ_i's are called slopes of M and λ_1 is called the first slope. We define the Newton polygon of the Frobenius slope filtration as usual.

Theorem 2.6 *Let M be a φ-module over F. There exists a unique Frobenius slope filtration $\{S_\lambda M\}_{\lambda \in \mathbb{R}}$ of M. Moreover, if k_F is perfect, then the filtration $S_\lambda M$ is degenerated, i.e., M is isomorphic to $\oplus_\lambda S_\lambda M / S_{\lambda-} M$. This decomposition is unique.*

Moreover, the Frobenius slope filtration is compatible under tensor products, duals and for any extension of F in the category of complete discrete valuation fields of characteristics $(0, p)$ with a compatible Frobenius. Any morphism of φ-modules is strict with respect to the Frobenius slope filtrations.

Proof of the unicity. The uniqueness easily follows from the fact that any morphism between pure φ-modules with different slopes is a zero map [12, IV, 3 (b), Proposition]. $\square$

If the residue field k_F is algebraically closed, then the slope filtration is a filtration associated to the Frobenius slope decomposition of Dieudonné-Manin ([12, IV, 4]).

To prove the existence of Frobenius slope filtration, we give several assertions.

Proposition 2.7 *Let M be a φ-module over F and let i be a positive integer. Suppose that (M, φ^i) has the Frobenius slope filtration $\{S_\lambda M\}$. Then the filtration is stable under the Frobenius φ, that is, $\varphi(S_\lambda M) \subset S_\lambda M$. In particular, $\{S_\lambda M\}$ is the Frobenius filtration of (M, φ) and the notion of Frobenius slope is independent of the choice of the power of the Frobenius σ.*

Proof. Let λ_1 be the first slope and let N be an F-subspace generated by $\varphi(S_{\lambda_1} M)$. Then N is stable under φ^i, and N is a sub φ^i-module of M. Moreover, it is pure of slope λ_1. Hence $N = S_{\lambda_1} M$ and $S_{\lambda_1} M$ is stable under φ. N is a φ-submodule of M and is of pure of slope λ_1 by Proposition 2.4. Taking the quotient M/N, we conclude by induction. □

Proposition 2.8 *Let L/F be a finite Galois extension and let σ_L be a Frobenius of L with $\sigma_L|_F = \sigma$ such that σ_L commutes with the action of Galois group. Let M be a φ-module over F and let $\{S_\lambda(M \otimes_F L)\}$ be the Frobenius slope filtration of $M \otimes_F L$. Then M admits the Frobenius slope filtration $\{S_\lambda M\}$ such that $S_\lambda(M \otimes_F L) = (S_\lambda M) \otimes_F L$ for all λ.*

Proof. We have only to prove the filtration is stable under the action of Galois group G. Indeed, $S_\lambda M$ is the G-invariant part $(S_\lambda(M \otimes_F L))^G$. Since the action of G commutes with σ_L, the semilinear action of G commutes with the Frobenius φ_L on $M \otimes_F L$. Let λ_1 be the first slope of $M \otimes_F L$. Since the absolute value on L is stable under the action of G, $\tau(S_{\lambda_1}(M \otimes_F L))$ is a sub φ-module pure of slope λ_1 for any $\tau \in G$. By the unicity of the filtration $S_{\lambda_1}(M \otimes_F L)$ is stable under the action of G. Taking the quotient we conclude that the filtration $\{S_\lambda(M \otimes_F L)\}$ is stable under the action of G. □

Lemma 2.9 *Let M be a φ-module over F. Suppose that the residue field k_F of F is infinite. Then there exists a cyclic vector of M over F, that is, there exists an element $e \in M$ such that $e, \varphi(e), \varphi^2(e), \cdots$ generate M over F.*

Proof. Let $s = \max_{x \in M} \dim_F \langle x, \varphi(x), \varphi^2(x), \cdots \rangle$. If $s = \dim_F M$, there is nothing to prove. Suppose $s < \dim_F M$. Let $x \in M$ be an element such that $\dim_F \langle x, \varphi(x), \cdots \rangle = s$. Since M is generated by $\varphi^s(M)$, there is an element $y \in M$ such that $x, \varphi(x), \cdots, \varphi^{s-1}(x), \varphi^s(y)$ are linearly independent over F. Let us fix a basis of $\wedge^{s+1} M$ including $v = x \wedge \varphi(x) \wedge \cdots \wedge \varphi^{s-1}(x) \wedge \varphi^s(y)$. Then there is a polynomial $f(t_0, t_1, \cdots, t_s)$ in $F[t_0, t_1, \cdots, t_s]$ such that

(i) f is at most of degree 1 on each t_i $(0 \leq i \leq s)$ and the coefficient of the monomial t_s is 1;

(ii) $(x + ay) \wedge \varphi(x + ay) \wedge \cdots \wedge \varphi^s(x + ay)$ goes to $f(a, \sigma(a), \cdots, \sigma^s(a))$ for any $a \in F$ by the projection $\wedge^{s+1} M \to F$ to the v-component.

The maximality of s implies $f(a, \sigma(a), \cdots, \sigma^s(a)) = 0$ for all $a \in F$. However it is impossible by the next lemma. □

Lemma 2.10 *Let $f(t_0, t_1, \cdots, t_s) \in F[t_0, t_1, \cdots, t_s]$ with $f(x) \notin F$ such that f is at most of degree 1 on each t_i. Suppose that k_F is infinite. Then there exists an element $b \in F$ such that $f(b, \sigma(b), \cdots, \sigma^s(b)) \neq 0$.*

Proof. We may assume that the maximum of the absolute values of the coefficients of f is 1 and the absolute value of the constant term of f is less than or equal to 1. Then $f(t, \sigma(t), \cdots, \sigma^s(t))$ modulo the maximal ideal of $\mathcal{V}_F$ becomes a nonconstant polynomial in $k_F[t]$ because $\sigma(t) = t^q$ on k_F. Hence there is an element $b \in F$ such that $f(b, \sigma(b), \cdots, \sigma^s(b)) \neq 0$. □

Lemma 2.11 *Let* $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$ *be a square matrix of degree n with entries in $\mathcal{V}_F$ such that* (i) *A_{11} is invertible of degree m over $\mathcal{V}_F$ and* (ii) $|A_{12}|, |A_{22}| < 1$. *Then there is a matrix* $Y = \begin{pmatrix} I_m & 0 \\ W & I_{n-m} \end{pmatrix}$ *with $|W| \leq |A_{21}|$ such that* $Y^{-1}AY^{\sigma} = \begin{pmatrix} A'_{11} & A'_{12} \\ 0 & A'_{22} \end{pmatrix}$. *Here I_m is the identity matrix of degree m, and A'_{11} and A'_{22} are square matrixes of degree m and $n-m$, respectively.*

Proof. Since F is complete (hence $\mathcal{V}_F$), one can construct such a matrix Y inductively on modulo powers of the maximal ideal of $\mathcal{V}_F$. □

Lemma 2.12 *Suppose that the residue field k_F is perfect, then for any short exact sequence of φ-module $0 \to M_1 \to M_2 \to M_3 \to 0$ in which we have a slope decomposition on M_1 and M_3 with different slopes. Then M_2 is a direct sum as a φ-module.*

Proof. The assertion easily follows from the fact the Frobenius σ on F is a homeomorphism if k_F is perfect. (See also [12, IV, 4, Lemma 1].) □

Proof of the existence of the filtration of Theorem 2.6. Suppose that the residue field k_F of F is infinite. We will prove the existence of Frobenius slope filtration by induction on the dimension $\dim_F M$. Let e be a cyclic vector of M by Lemma 2.9.

Let us define a matrix $B \in \mathrm{GL}(r, F)$ ($r = \dim_F M$) by

$$\varphi(\varphi^{r-1}(e), \cdots, \varphi(e), e) = (\varphi^{r-1}(e), \cdots, \varphi(e), e)B, \quad B = \begin{pmatrix} a_1 & 1 & & & \\ a_2 & & 1 & & \\ \vdots & & & \ddots & \\ a_{r-1} & & & & 1 \\ a_r & & & & \end{pmatrix}$$

and let λ_1 be the first slope of the Newton polygon of the polynomial $1 + a_1x + \cdots + a_rx^r$ with a multiplicity s.

Let us choose a finite Galois extension L/F which satisfies the conditions

(i) there exist elements $\beta, \gamma \in L$ such that $|\gamma| < 1$, $|\beta^{-s}a_s| = 1$ and $|\beta^{-j}\gamma^{s-j}a_j| \leq 1$ for all $j > s$;

(ii) there is a lift σ_L of Frobenius endomorphism with $\sigma_L|_F = \sigma^i$ such that σ_L commutes with the action of Galois group.

Such L, β and γ exist by Proposition 2.1 and by the fact the the multiplicity of the smallest slope of the polynomial is s. Applying Propositions 2.7 and 2.8, we have only to prove the existence of the sub φ-module N of $M \otimes_F L$ such that N is of dimension s and pure of slope λ_1 and all slopes of $(M \otimes_F L)/N$ are strictly greater than λ_1 ($(M \otimes_F L)/N$ admits the slope filtration by induction hypothesis).

Put $P = \begin{pmatrix} 1 & & & & & & \\ & \beta & & & & & \\ & & \ddots & & & & \\ & & & \beta^{s-1} & & & \\ & & & & \beta^s\gamma & & \\ & & & & & \ddots & \\ & & & & & & \beta^{r-1}\gamma^{r-s} \end{pmatrix}$. Let us consider a matrix

$$
\begin{aligned}
B' &= P^{-1}(\beta^{-1}B)P^{\sigma} \\
&= \begin{pmatrix}
\beta^{-1}a_1 & \beta^{-1}\sigma_L(\beta) & & & & \\
\beta^{-2}a_2 & & & & & \\
\vdots & & \ddots & & & \\
\beta^{-s}a_s & & & \beta^{-s}\sigma(\beta^s\gamma) & & \\
\beta^{-s-1}\gamma^{-1}a_{s+1} & & & & & \\
\vdots & & & & \ddots & \\
\beta^{-r+1}\gamma^{s-r+1}a_{r-1} & & & & & \beta^{-r+1}\gamma^{s-r+1}\sigma_L(\beta^{r-1}\gamma^{r-s}) \\
\beta^{-r}\gamma^{s-r}a_r & & & & &
\end{pmatrix}.
\end{aligned}
$$

By the assumption of slopes all entries in B' are contains in the integer ring $\mathcal{V}_L$ of L and, if we put $m = s$, then B' satisfies the hypothesis of Lemma 2.11. Then there is a matrix $Y = \begin{pmatrix} I_s & 0 \\ W & I_{r-s} \end{pmatrix}$ with entries in $\mathcal{V}_L$ such that

$$Y^{-1}B'Y^{\sigma} = \begin{pmatrix} B'_{11} & B'_{12} \\ 0 & B'_{22} \end{pmatrix},$$

where B'_{11} and B'_{22} are square matrixes of degree s and $r-s$, respectively. Moreover, B'_{11} is the invertible matrix over $\mathcal{V}_L$ and all entries of B'_{22} is contained in the valuation ideal of $\mathcal{V}_L$. Hence, the first s components of $(\varphi^{r-1}(e), \cdots, \varphi(e), e)PY$ generate the desired submodule N pure of slope λ_1 and all the slopes of its quotient are greater than λ_1 (all the absolute values of the entries of B'_{22} are strictly smallest than 1 and the Frobenius associated to B'_{22} is a contraction). Therefore, we have proved the existence of the slope filtration in the case where the residue field of F is infinite. If k_F is perfect, then one can construct a splitting by Lemma 2.12.

The remain case is k_F is finite: we will prove the existence of the decomposition by induction on the rank of M. We then consider a cyclic sub φ-module of M. If it is equal to M we use the previous method. If it is not equal to M then we may apply the inductions hypotheses and we may suppose to have slope decomposition for it and for the quotient of M by it. By Lemma 2.12 we conclude.

The rest follows from the uniqueness of Frobenius slope filtrations □

Proposition 2.13 *Let M be a φ-module over F and let e be a cyclic vector which satisfies $\varphi^n(e)+a_1\varphi^{n-1}(e)+\cdots+a_ne = 0\,(n = \dim_F M)$. Then the Newton polygon of the Frobenius slope filtration coincides with the Newton polygon of the polinomial $1 + a_1x + \cdots + a_nx^n$.*

Proof. We may assume that the residue field of F is algebraically closed by Theorem 2.6 and Proposition 2.2. Then the assertion is a classical result [12, IV, 4, Lemmas 2, 3]. □

Remark 2.14 For φ-modules over F, the Harder-Narasimhan filtration [23, 3.5] coincides with the Frobenius slope filtration.

3 Log-growth filtration over $\mathcal{E}$ and E

3.1 The setting

We refer to 0.2 for the definition of ∇-modules. We then consider M a ∇-module over $\mathcal{E}$, but one can think also to replace $\mathcal{E}$ by E. By 0.3 a ∇-module M^τ over $\mathcal{E}[\![X-t]\!]_0$ is associated to M. We denote again by $\mathcal{A}_{\mathcal{E}}(t,1)$ the ring of power series over $\mathcal{E}$ which are analytic on the open unit disk with center at t.

Definition 3.1 *Given a ∇-module M over $\mathcal{E}$, we say that M is solvable if the associated ∇-module M^τ over $\mathcal{E}[\![X-t]\!]_0$ has a full set of solutions in $\mathcal{A}_{\mathcal{E}}(t,1)$*

3.2 Topology of log-growth

In this section we recall the definition of bounded filtration of a ∇-modules over $\mathcal{E}$ defined by Christol and Robba (where, actually, it was defined over E) and define the filtration of logarithmic growth for ∇-modules over $\mathcal{E}$ and E. All theorems over E were proved in [6], [27] and [28].

Let λ be a nonnegative number and let $||\text{-}||_\lambda$ be the operator norm of $\mathcal{E}\left[\frac{d}{dx}\right]$ acting on $\mathcal{E}[\![x-t]\!]_\lambda$, that is,

$$||u||_\lambda = \sup_{a\in\mathcal{E}[\![x-t]\!]_\lambda, a\neq 0} \frac{|u(a)|_\lambda}{|a|_\lambda}.$$

It is proved in [27, 1.11] that such a norm for $u=\sum a_i \frac{d^i}{dx^i}$ is given by

$$||u||_\lambda = \max_i\,(i+1)^\lambda |i!\,a_i|.$$

Let M be a ∇-module over $\mathcal{E}$ of rank r and let e be a cyclic vector of M with respect to $\nabla(\frac{d}{dx})$. Such a cyclic vector always exists by [11]. Since $\mathcal{E}\left[\frac{d}{dx}\right]$ is Euclidean, there is an element $L\in\mathcal{E}\left[\frac{d}{dx}\right]$ of degree r such that

$$M \cong \mathcal{E}\left[\frac{d}{dx}\right] \Big/ \mathcal{E}\left[\frac{d}{dx}\right] L$$

as $\mathcal{E}\left[\frac{d}{dx}\right]$-modules. We define a semi-norm $||\text{-}||_{M,e,\lambda}$ of log-growth λ on M by the induced one from the norm $||\text{-}||_\lambda$ on $\mathcal{E}\left[\frac{d}{dx}\right]$.

Theorem 3.2 ([27, 2.6, 3.5], [6, 4.3.3, 4.3.4]) *Let M^λ be a subset of M which is defined by $M^\lambda=\{m\in M\,|\,||m||_{M,e,\lambda}=0\}$, in other words, M^λ is a closure of $\{0\}$ for the topology associated to $||\text{-}||_{M,e,\lambda}$. Then we have the followings.*

(1) *M^λ is a ∇-submodule of M over $\mathcal{E}$.*

(2) *If $\lambda\le\mu$, then $M^\mu\subset M^\lambda$.*

(3) *M/M^λ is the maximum quotient of M such that M/M^λ has a full set of solutions in $\mathcal{E}[\![X-t]\!]_\lambda$. In particular, the topology of M induced by the semi-norm $||\text{-}||_{M,e,\lambda}$ is independent of the choice of the cyclic vector e.*

(4) *If M is solvable, then $M^\lambda\neq M$.*

Proof. The following proof is due to section 4.3 in [6].

(1) Since $||\text{-}||_{M,e,\lambda}$ is a semi-norm such that $||P(m)||_{M,e,\lambda} \leq ||P||_\lambda ||m||_{M,e,\lambda}$ for $P \in \mathcal{E}\left[\frac{d}{dx}\right]$ and $m \in M$, M^λ is an $E\left[\frac{d}{dx}\right]$-submodule, and hence is a ∇-submodule over $\mathcal{E}$.

(2) Since $||m||_{M,e,\lambda} \leq ||m||_{M,e,\mu}$, we have $M^\mu \subset M^\lambda$.

(3) Suppose that N is a ∇-submodule over $\mathcal{E}$ such that M/N has a full set of solutions in $\mathcal{E}[\![X-t]\!]_\lambda$. Since any $\mathcal{E}\left[\frac{d}{dx}\right]$-homomorphism $\mathcal{E}\left[\frac{d}{dx}\right] \to \mathcal{E}[\![X-t]\!]_\lambda$ is continuous with respect to the norm $||\text{-}||_\lambda$ on $\mathcal{E}\left[\frac{d}{dx}\right]$, the kernel of a solution $M \to \mathcal{E}[\![X-t]\!]_\lambda$ is closed. Denote by I the intersection of kernels of solutions of M in $\mathcal{E}[\![X-t]\!]_\lambda$, then $N \subset I$ and we have a short exact sequence of solutions with respect to the exact sequence $0 \to I/N \to M/N \to M/I \to 0$ of $\mathcal{E}[\![X-t]\!]_\lambda$-solvable modules by Corollary 1.7. Since any solution of M/N killes I/N, we have $\mathrm{Sol}_\lambda(I/N) = 0$ and $N = I$. Hence, $M^\lambda \subset N$.

The rest is to prove that, if the induced semi-norm $||\text{-}||_{M,e,\lambda}$ is a norm on M, then M has a full set of solutions in $\mathcal{E}[\![X-t]\!]_\lambda$. (We apply it for the quotient M/M^λ.) Let us fix a basis $e_1, \cdots, e_r$ of M over $\mathcal{E}$ and let $||\text{-}||$ be a norm on M defined by $||\sum_i a_i e_i|| = \max_i |a_i|$. Let $Y = \sum_{m=0}^\infty \frac{G_n}{n!}(X-t)^n$ be a solution matrix of M in $\mathcal{A}_\mathcal{E}(t,1)$ with respect to the basis $e_1, \cdots, e_r$ (see (TE) in 1.2). Since $\mathcal{E}$ is complete and M is of finite dimension over $\mathcal{E}$, two norms $||\text{-}||_{M,e,\lambda}$ and $||\text{-}||$ are equivalent to each other (see [24, XII, 2.2]). Hence there is a constant $C_1, C_2 > 0$ such that $C_1||m|||_{M,e,\lambda} \leq ||m|| \leq C_2||m|||_{M,e,\lambda}$ for any $m \in M$. Then we have

$$\begin{aligned} \left|\frac{G_n}{n!}\right| &= \max_i \left|\left|\frac{1}{n!}\frac{d^n}{dx^n}e_i\right|\right| \leq C_2 \max_i \left|\left|\frac{1}{n!}\frac{d^n}{dx^n}e_i\right|\right|_{M,e\lambda} \\ &\leq C_2 \max_i \left|\left|\frac{1}{n!}\frac{d^n}{dx^n}\right|\right|_\lambda ||e_i||_{M,e,\lambda} \leq \frac{C_2}{C_1}(n+1)^\lambda. \end{aligned}$$

Here, for the second inequality, we regard $(M, ||\text{-}|||_{M,e,\lambda})$ as an $(\mathcal{E}[\frac{d}{dx}], ||\text{-}|||_\lambda)$-module. Therefore, all entries of Y are contained in $\mathcal{E}[\![X-t]\!]_\lambda$ and M has a full set of solutions in $\mathcal{E}[\![X-t]\!]_\lambda$.

(4) It is sufficient to prove $M^0 \neq M$ by (2). Let $s : M \to \mathcal{A}_\mathcal{E}(t,1)$ be an $\mathcal{E}\left[\frac{d}{dx}\right]$-homomorphism such that $s(e) \neq 0$. Such an s always exists since M is solvable and e is a cyclic vector. Suppose that $M = M^0$. Then, for any $\epsilon > 0$, there is an element $P_\epsilon \in \mathcal{E}\left[\frac{d}{dx}\right]$ such that $P_\epsilon(e) = 0$ and $||1 - P_\epsilon||_0 < \epsilon$. If we put $1 - P_\epsilon = \sum_{i=0}^l a_i \frac{d^i}{dx^i}$, then $|i!a_i| < \epsilon$.

For any $0 < \rho < 1$, let us define a norm $||\text{-}||(\rho)$ on $\mathcal{A}_\mathcal{E}(t,1)$ by

$$\left|\left|\sum_n u_n(X-t)^n\right|\right|(\rho) = \sup_n |u_n|\rho^n.$$

Since $\left|\left|\frac{d^i}{dx^i}(X-t)^n\right|\right|(\rho) = \left|\frac{n!}{(n-i)!}\right|\rho^{n-i} \leq |i!|\rho^{n-i}$, we have $\left|\left|\frac{d^i}{dx^i}u\right|\right|(\rho) \leq |i!|\rho^{-i}\,||u||\,(\rho)$. Hence,

$$1 = \frac{||s((1-P_\epsilon)(e))||(\rho)}{||s(e)||(\rho)} = \frac{||(1-P_\epsilon)(s(e))||(\rho)}{||s(e)||(\rho)} \leq \max_i |i!a_i|\rho^{-i} \leq \epsilon\rho^{-l} \to \epsilon\,(\text{as}\,\rho \to 1).$$

Since our ϵ is arbitrary, this gives a contradiction. Therefore, $M^0 \neq M$. □

Remark 3.3 Consider an isometric immersion $\mathcal{E} \to \Omega$ of complete fields (hence $\mathcal{E}$ is closed in Ω) which is also, by composing with the immersion of E in $\mathcal{E}$, an isometric immersion $E \to \Omega$. For an operator $L \in E[\frac{d}{dx}]$ (resp. $L \in \mathcal{E}[\frac{d}{dx}]$), Robba's Theorems 2.5 and 2.6 in [27] deal with the topology of $E[\frac{d}{dx}]/E[\frac{d}{dx}]L$ (resp. $\mathcal{E}[\frac{d}{dx}]/\mathcal{E}[\frac{d}{dx}]L$) as operators on the Banach algebra $\Omega[\![X-t]\!]_\lambda$. And then what it is actually written is the fact that the closure of $\{0\}$ gives a factorisation of L in $E[\frac{d}{dx}]$, $L = S_1 \circ R$, (resp. one of L in $\mathcal{E}[\frac{d}{dx}]$, $L = S_2 \circ Q$) such that R (resp. Q) is completely solvable in $\Omega[\![X-t]\!]_\lambda$ and its solutions there coincide with the solutions of L in $\Omega[\![X-t]\!]_\lambda$. Hence R and Q are the same being monic. See also [28, 5.4] : the same holds for any closed field of $\mathcal{E}$.

One can also find an example of an operator with coefficients $K(x)$ such that the previous decomposition doesn't exist in $K(x)$ in [28]. Indeed, the field $K(x)$ is not closed in $\mathcal{E}$, hence in Ω.

By Theorem 3.2 we have

Theorem 3.4 ([6, 4.3.5]) *Let M be a solvable ∇-module over $\mathcal{E}$. Then there is a unique filtration*

$$M = M_{-1} \supsetneq M_0 \supsetneq M_1 \supsetneq \cdots \supsetneq M_s = 0$$

where M_{j-1}/M_j is the maximum quotient of M_{j-1} such that M_{j-1}/M_j is solvable in $\mathcal{E}[\![X-t]\!]_0$.

Proof. Put M_0 to be the closure of $\{0\}$ for the topology induced by $||\text{-}||_0$ on M. And apply again the same tecnique replacing M with M_0. Since M is of finite dimension, $M = M_{-1} \supsetneq M_0 \supsetneq \cdots$ will stop after finite steps and the filtration is our desired one. □

Since any element in $\mathcal{E}[\![X-t]\!]_\lambda$ is integrable in $\mathcal{E}[\![X-t]\!]_{\lambda+1}$, we have

Corollary 3.5 ([6, 4.3.6]) *With the same notation as before, for any $i \leq s$ the quotient module M/M_i is completely solvable in $\mathcal{E}[\![X-t]\!]_i$. In particular, there is a nonnegative number λ such that $M^\lambda = 0$.*

Remark 3.6 As a matter of fact, even if our ∇-module M over $\mathcal{E}$ is not solvable on $\mathcal{A}_{\mathcal{E}}(t,1)$, but it has some non zero solution on such a ring, then it admits an analogous filtration which, in particular, doesn't reach 0. In such a case we will always have a bounded solution (i.e. in $\mathcal{E}[\![X-t]\!]_0$) [6, 4.3.5].

3.3 Log-growth filtration as ∇-module over $\mathcal{E}$ and E

Definition 3.7 (Log-growth filtration as ∇-modules) *After Theorem 3.2 we indicate the decreasing filtration $\{M^\lambda\}_{\lambda\in\mathbb{R}}$ of solvable ∇-module M over $\mathcal{E}$ (resp. E), where we put $M^\lambda = M$ for $\lambda < 0$, as the log-growth filtration of M.*

Let M be a soluble ∇-module over $\mathcal{E}$ (resp. E) and let $\{M^\lambda\}$ be the log-growth filtration of M. For a real number λ, we define

$$\begin{aligned} M^{\lambda-} &= \bigcap_{\mu<\lambda} M^\mu \\ M^{\lambda+} &= \bigcup_{\mu>\lambda} M^\mu. \end{aligned}$$

Both $M^{\lambda-}$ and $M^{\lambda+}$ are sub ∇-modules over $\mathcal{E}$ (resp. E). Since M is of finite dimension over $\mathcal{E}$ (resp. E) and $M^\lambda = 0$ for sufficient large λ, there are real numbers $0 = \lambda_1 < \lambda_2 < \cdots < \lambda_r$ such that

$$M = M^{\lambda_1-} \supsetneq M^{\lambda_1+} = M^{\lambda_2-} \supsetneq \cdots \supsetneq M^{\lambda_{r-1}+} = M^{\lambda_r-} \supsetneq M^{\lambda_r+} = 0$$

for $M \neq 0$. We call $\lambda_1 < \lambda_2 < \cdots < \lambda_r$ the breaks of log-growth filtration. We define the polygon of the log-growth filtration by the convex hull of the points $(0,0)$ and $(\dim M - \dim M^{\lambda_i +}, \sum_{j \leq i} \lambda_j(\dim M^{\lambda_j -} - \dim M^{\lambda_j +}))$ $(i = 1, \cdots, r)$.

The breaks of log-growth of ∇-modules is not always rational. We give two examples of log-growth with an arbitrary real number in section 5. Moreover, the examples say that both $M^{\lambda -} = M^{\lambda} \supsetneq M^{\lambda +}$ and $M^{\lambda +} = M^{\lambda} \subsetneq M^{\lambda -}$ occur. In the two cases, the two ∇-modules do not admit Frobenius over $\mathcal{E}$. We expect all breaks are rational and $M^{\lambda +} = M^{\lambda}$ if M admits a Frobenius structure (see Conjecture 4.6 (1)).

3.4 Action of Frobenius on log-growth filtration

We consider now the compatibility between the Frobenius and log-growth filtration over $\mathcal{E}$ and E.

Proposition 3.8 *Consider a φ-∇-module M over $\mathcal{E}$ (resp. E). Then each element of the log-growth filtration as a ∇-module, M^{λ}, is φ-∇-module.*

Proof. M^{λ} has the property that all the solutions of $(M/M^{\lambda})^{\tau}$ have log-growth less than or equal to λ, and it is the smallest submodule with this property. If φ is the Frobenius isomorphism, it is horizontal and it doesn't change the log-growth [6, 4.6.4], then

$$F^{-1} : M^{\lambda} \to (M^{\sigma})^{\lambda}$$

induces an isomorphism. We may then consider the exact sequences:

$$0 \to (M^{\lambda})^{\sigma} \to M^{\sigma} \to \left(\frac{M}{(M)^{\lambda}}\right)^{\sigma} \to 0.$$

and

$$0 \to (M^{\sigma})^{\lambda} \to M^{\sigma} \to \frac{M^{\sigma}}{(M^{\sigma})^{\lambda}} \to 0.$$

All the solution of $(\frac{M}{(M)^{\lambda}})^{\sigma}$ have log-growth less than or equal to λ, from the property of $(M^{\sigma})^{\lambda}$ we have that there exists a surjection

$$\frac{M^{\sigma}}{(M^{\sigma})^{\lambda}} \to \left(\frac{M}{(M)^{\lambda}}\right)^{\sigma} = \frac{M^{\sigma}}{(M^{\lambda})^{\sigma}}$$

Hence an injection

$$(M^{\sigma})^{\lambda} \to (M^{\lambda})^{\sigma},$$

which, composed with $F^{-1} : M^{\lambda} \to (M^{\sigma})^{\lambda}$, gives an injection of modules with the same rank, hence an isomorphism:

$$M^{\lambda} \to (M^{\sigma})^{\lambda} \to (M^{\lambda})^{\sigma}$$

giving the Frobenius isomorphism. □

3.5 Log-growth over E and $\mathcal{E}_a$

Let $\overline{a}$ be either an element of k or $\overline{\infty}$ and let a be either a lift of $\overline{a}$ in $\mathcal{V}$ or $a = \infty$. The map j_a (0.1) induces a natural commutative diagram

$$\begin{array}{ccc} E & \xrightarrow{\tau} & E[\![X-t]\!]_0 \\ j_a \downarrow & & \downarrow \tilde{j}_a \\ \mathcal{E}_a & \xrightarrow[\tau_a]{} & \mathcal{E}_a[\![X-t]\!]_0 \end{array}$$

which is compatible under derivations. Here $\tilde{j}_a(\sum_n a_n(X-t)^n) = \sum_n j_a(a_n)(X-t)^n$. Note that j_a induces an isometry $E[\![X-t]\!]_\lambda \to \mathcal{E}_a[\![X-t]\!]_\lambda$ for any λ.

Let M be a solvable ∇-module over E. Then the above comutative diagram induces a natural isomorphism

$$M^\tau \otimes \mathcal{E}_a[\![X-t]\!]_0 \cong (M \otimes \mathcal{E}_a)^{\tau_a}$$

of ∇-modules (resp. φ-modules, resp. φ-∇-modules) over $\mathcal{E}_a[\![X-t]\!]_0$.

Proposition 3.9 ([28, 5.4] (see Remark 3.3)) *With the notation above, the log-growth filtration of M over E induces the log-growth filtration of $M \otimes \mathcal{E}_a$ by extention to $\mathcal{E}_a$, i.e., $M^\lambda \otimes \mathcal{E}_a = (M \otimes \mathcal{E}_a)^\lambda$ for any λ.*

Proof. Since a cyclic vector e of M is also a cyclic vector of $M \otimes \mathcal{E}_a$, $M \to M \otimes \mathcal{E}_a$ is continuous with respect to the topologies induced by the semi-norms. Hence, $M^\lambda \subset (M \otimes \mathcal{E})^\lambda$ and $\dim_E M^\lambda \leq \dim_{\mathcal{E}_a} (M \otimes \mathcal{E}_a)^\lambda$. On the other hand, $\dim_E M/M^\lambda \leq \dim_{\mathcal{E}_a} (M \otimes \mathcal{E}_a)/(M \otimes \mathcal{E}_a)^\lambda$ by the maximum property of solutions (Theorem 3.2 (3)). Therefore, we have $M^\lambda \otimes \mathcal{E}_a = (M \otimes \mathcal{E}_a)^\lambda$. □

Remark 3.10 One could have started with a convergent isocrystal $\mathcal{M}$ on an open scheme X of the projective line $\mathbb{P}^1_k$. Then it has a realization as a projective module $M_{]X[}$ over the tube $]X[$ of X. The ring of analytic functions $\mathcal{A}$ on the affinoid $]X[$ coincides with the ring of analytic elements on the same affinoid, i.e., the uniform limit of rational functions without poles on $]X[$ [26]. We can then take the completion of the field of the fraction of $\mathcal{A}$ for the Gauss norm, obtaining E: we will have a solvable ∇-module M over E [4, 3.1] (see also [2]). On the other hand we can also specialize M_T to any residue class $\overline{a} \in X$ (see [8, III.7.2] for the general statement): we will obtain a solvable ∇-module M_a which is actually defined over $K[\![x-a]\!]_0$ (by the maximum module principle on $]X[$). The associated generic fiber $M_a \otimes \mathcal{E}_a$ coincides with the module $M \otimes \mathcal{E}_a$ which we have defined before the remark.

4 Generic and special log-growth filtrations

Let us define log-growth filtration for ∇-modules over the ring of bounded functions $K[\![x]\!]_0$: both at the generic and special points. We will say that a ∇-module M over $K[\![x]\!]_0$ is solvable if the associated ∇-module $M \otimes \mathcal{E}$ over $\mathcal{E}$ is solvable (see 3.1).

4.1 Generic log-growth filtration

Definition 4.1 *Let M be a solvable ∇-module over $K[\![x]\!]_0$, its generic log-growth filtration will be the decreasing log-growth filtration $\{(M \otimes \mathcal{E})^\lambda\}_{\lambda \in \mathbb{R}}$ on $M \otimes \mathcal{E}$ as defined in Definition 3.7.*

We call the polygon of the log-growth filtration in 3.3 the generic log-growth polygon.

4.2 Special log-growth filtration

Let M be a solvable ∇-module over $K[\![x]\!]_0$. Then, we know, by the transfer theorem [6, section 5], that our module has a full set of solutions in $\mathcal{A}_K(0,1)$: this follows by specialization at 0 of the Taylor expansion (TE) in paragraph 1.2 of the solutions of M at the generic disk. Moreover we have also (transfer theorem):

Proposition 4.2 [6, 5.1.8] *Consider M, such that the associated $M \otimes \mathcal{E}$ has a full set of solutions in $\mathcal{E}[\![X-t]\!]_\lambda$, then M has a full set of solutions in $K[\![x]\!]_\lambda$.*

For a ∇-module M over $K[\![x]\!]_0$ we define

$$\mathrm{Sol}(M) = \mathrm{Sol}(M, \mathcal{A}_K(0,1))$$

as the K-vector space of the solutions of M in $\mathcal{A}_K(0,1)$. Of course, if M is solvable, then $\dim_K \mathrm{Sol}(M) = \mathrm{rank}\, M$. We then introduce an increasing filtration of $\mathrm{Sol}(M)$ by K-vector spaces $\{\mathrm{Sol}_\lambda(M)\}_{\lambda \in \mathbb{R}}$ (defined in 1.1) with $\mathrm{Sol}_\lambda(M) = 0$ for $\lambda < 0$

We now define by means of Propositions 1.5 and 4.2 the K-vector space

$$V(M) = H^0(M \otimes \mathcal{A}_K(0,1))$$

(the K-vector spaces of horizontal sections of the extention of (M, ∇) to $M \otimes \mathcal{A}_K(0,1)$), by solvability we know that $\dim_K V(M) = \mathrm{rank}\, M$. We then have

Proposition 4.3 *In the previous hypotheses there is a natural perfect duality*

$$V(M) \times \mathrm{Sol}(M) \to K$$

given by $(s, f) \mapsto f(s)$.

We are able, by using the perfect pairing in Proposition 4.3, to define

Definition 4.4 *The special log-growth filtration of M is the decreasing filtration $\{V(M)^\lambda\}_{\lambda \in \mathbb{R}}$ on $V(M)$ given by*

$$V(M)^\lambda := \mathrm{Sol}_\lambda(M)^\perp.$$

Here $(\text{-})^\perp$ is the orthogonal subspace with respect to the pairing.

By definition if $\lambda < 0$ we have $V(M)^\lambda = V(M)$ and, under the same assumptions as in Proposition 4.2, $V(M)^\lambda = 0$. In particular we again define

$$V(M)^{\lambda-} = \cap_{\rho<\lambda} V(M)^\rho.$$

and

$$V(M)^{\lambda+} = \cup_{\rho>\lambda} V(M)^\rho.$$

We define the polygon of the special log-growth filtration similarly in 3.3 and call it the special log-growth polygon.

Remark 4.5 Following Remark 3.10, for X open set in the projective line over k, we are able to define the special log-growth filtration for a convergent isocrystal at any closed point of X.

4.3 Specialization of log-growth filtrations.

Dwork conjectured that the log-growth filtration has a similar property to the slopes of Frobenius structures (the specialization theorem of Frobenius slopes polygon [20, Appendix] [21, 2.3.2], see 6.4). He conjectured the specialization conjecture for log-growth filtrations.

Conjecture 4.6 *Let M be a ∇-module over $K[\![x]\!]_0$.*

(1) *Suppose that M admits a Frobenius structure. Then $(M\otimes\mathcal{E})^{\lambda} = (M\otimes\mathcal{E})^{\lambda+}$ (resp. $V(M)^{\lambda} = V(M)^{\lambda+}$) for any λ and all breaks of generic (resp. special) log-growth filtration are rational.*

(2) ([15, Conjecture 2]) *The special log-growth polygon is above the generic log-growth polygon.*

We compare the codimensions of the first step of the graduations for the generic and special log-growth filtrations. In the case of Frobenius slopes the similar result implies the specialization theorem. However, since the log-growth filtration is not compatible for subquotients and tensor products, the proposition below does not implies the specialization conjecture above.

Proposition 4.7 ([15, Corollary 2], [27, 3.5]) *Let M be a ∇-module over $K[\![x]\!]_0$ which is solvable. Then we have*

$$\dim_K V(M)/V(M)^0 \leq \dim_{\mathcal{E}} M\otimes\mathcal{E}/(M\otimes\mathcal{E})^0.$$

Proof. Since the dual $M^{\vee}$ of M has a sub ∇-module of rank $\dim_K V(M)/V(M)^0$ which is trivial over $K[\![x]\!]_0$ by Proposition 1.2, $(M\otimes\mathcal{E})^{\tau}$ has at least $\dim_K V(M)/V(M)^0$ bounded solutions which are linearly independent. □

4.4 Special log-growth filtration and Frobenius.

Proposition 4.8 *Consider M a φ-∇-module over $K[\![x]\!]_0$. Then the special log-growth filtration is stable under Frobenius action.*

Proof. Since the Frobenius on $\mathcal{A}_K(0,1)$ stabilizes the K-Banach space $K[\![x]\!]_{\lambda}$, the K-vector space $\mathrm{Sol}_{\lambda}(M)$ is stable under the Frobenius in $\mathrm{Sol}(M)$. Therefore, by the compatibility of the perfect pairing 4.3 with the Frobenius action, we conclude that $V(M)^{\lambda}$ is also stable under Frobenius. □

5 Examples

For a real number u, we denote by $\lfloor u \rfloor$ the greatest integer which is less than or equal to u.

5.1 Unipotent case

Example 5.1 First we study the log-filtration for the unipotent case. Let M be a ∇-module over E of rank r which is defined by

$$\nabla(e_0, \cdots, e_{r-1}) = (e_0 \cdots, e_{r-1}) \begin{pmatrix} 0 & \frac{1}{x} & 0 & \cdots & 0 \\ & 0 & \frac{1}{x} & \cdots & 0 \\ & & \ddots & \ddots & \vdots \\ & & & 0 & \frac{1}{x} \\ & & & & 0 \end{pmatrix} dx,$$

thats is, M is a unipotent ∇-module of rank r. Actually such a module arises from a convergent isocrystal on $\mathbb{P}^1_k \setminus \{0, \infty\}$, hence it is a solvable ∇-module over E. It is known to have a Frobenius structure [5, section 5], [25, 4.3] and [29, 4.2.3]. We will prove that the log-growth filtration of M is given by

$$M^\lambda = \begin{cases} M & \text{if } \lambda < 0 \\ \langle e_{i+1}, \cdots, e_{r-1} \rangle & \text{if } i \leq \lambda < i+1 \, (i = 0, 1, \cdots, r-2) \\ 0 & \text{if } \lambda \geq r-1, \end{cases}$$

where $\langle e_{i+1}, \cdots, e_{r-1} \rangle$ is a sub ∇-module of M over E which is generated by $e_{i+1}, \cdots, e_{r-1}$.

We define integers $\{c_{n,i}\}_{n,i \geq 1}$ by

$$\prod_{j=0}^{n-1} (x - j) = \sum_{i=1}^{n} c_{n,i} x^i.$$

and, for $i \geq 1$, define series $l_i(X, t) \in E[\![X - t]\!]$ by

$$l_i(X, t) = \sum_{n=1}^{\infty} \frac{c_{n,i}}{n!} (X - t)^n.$$

Then the solution matrix of M is given by

$$Y = \begin{pmatrix} 1 & l_1(X,t) & l_2(X,t) & \cdots & l_{r-1}(X,t) \\ & 1 & l_1(X,t) & \cdots & l_{r-2}(X,t) \\ & & \ddots & \ddots & \vdots \\ & & & 1 & l_1(X,t) \\ & & & & 1 \end{pmatrix}.$$

What we need is to prove the following proposition since M/M^λ is the maximul quotient such that M/M^λ has a full set of solution in $E[\![X - t]\!]_\lambda$ by Theorem 3.2.

Lemma 5.2 *$l_i(X, t)$ is exactly of log-growth i.*

PROOF. Applying Corollary 3.5 to the filtration by the unipotency, $l_i(X, t)$ belongs to $E[\![X - t]\!]_i$. Hence, we have only to prove $l_i(X, t)$ does not belong to $E[\![X - t]\!]_\lambda$ for $\lambda < i$. In other words, it is enough to find a subsequence $c_{n_1,i}, c_{n_2,i}, \cdots$ such that $\left\{ \frac{|c_{n_j,i}/n_j!|}{(n_j+1)^\lambda} \right\}_{j \geq 1}$ is unbounded for any $0 \leq \lambda < i$.

Let us consider the polynomial $\prod_{j=0}^{n-1}(x-j)$ and the vertex of its Newton polygon. The cardinality of the set of numbers among $1, 2, \cdots, n-1$ which are divided by p^j is $\lfloor \frac{n-1}{p^j} \rfloor$ and the p-adic order of the product of all the elements of such a set is $j\lfloor \frac{n-1}{p^j} \rfloor + \lfloor \frac{n-1}{p^{j+1}} \rfloor + \lfloor \frac{n-1}{p^{j+2}} \rfloor + \cdots$. Hence, the p-adic order of the $(\lfloor \frac{n-1}{p^j} \rfloor + 1)$-th coefficient of $\prod_{j=0}^{n-1}(x-j)$ is

$$\left(\lfloor \frac{n-1}{p} \rfloor + \lfloor \frac{n-1}{p^2} \rfloor + \cdots\right) - \left(j\lfloor \frac{n-1}{p^j} \rfloor + \lfloor \frac{n-1}{p^{j+1}} \rfloor + \lfloor \frac{n-1}{p^{j+2}} \rfloor + \cdots\right).$$

Note that the first term is the p-adic order of $(n-1)!$. Let us take a sequence $\{n_j\}$ which is defined by $n_j = ip^j$. Since $\lfloor \frac{n_j-1}{p^j} \rfloor + 1 = i$ for any sufficient large j, the p-adic order of $c_{n_j,i}$ is as above and we have

$$\left|\frac{c_{n_j,i}}{n_j!}\right| = p^{j\lfloor \frac{n_j-1}{p^j} \rfloor + \lfloor \frac{n_j-1}{p^{j+1}} \rfloor + \lfloor \frac{n_j-1}{p^{j+2}} \rfloor + \cdots + \mathrm{ord}_p(n_j)} \geq p^{(i-1)j+j} = p^{ij},$$

where $\mathrm{ord}_p(n_j) \geq j$ is the p-adic order of n_j. We conclude that the subsequence $\{c_{n_j,i}\}$ satisfies our desired property. □

Let $\overline{a} \neq 0$ be an element in k and let a be a lift of a in $\mathcal{V}$. Then, by Remark 3.10, the unipotent isocrystal on $\mathbb{P}^1_k \setminus \{0, \infty\}$ of rank r induces a solvable ∇-module $M_{\overline{a}}$ over $K[\![x-a]\!]_0$. The solution space $\mathrm{Sol}(M_{\overline{a}}) = \mathrm{Sol}(M_{\overline{a}}, \mathcal{A}_K(a,1))$ and the space $V(M_{\overline{a}}) = H^0(M_{\overline{a}} \otimes \mathcal{A}_K(a,1))$ of horizontal sections are as follows:

$$\begin{array}{rcl} \mathrm{Sol}(M_{\overline{a}}) & = & \left\langle \begin{array}{l} (1, l_1(x,a), \cdots, l_{r-1}(x,a)), (0, 1, l_1(x,a), \cdots, l_{r-2}(x,a)), \\ \cdots, (0, \cdots, 1, l_1(x,a)), (0, \cdots, 0, 1) \end{array} \right\rangle \\ V(M_{\overline{a}}) & = & \left\langle \begin{array}{l} e_0, -l_1(x,a)e_0 + e_1, \cdots, \\ (-1)^{r-1}l_{r-1}(x,a)e_0 + (-1)^{r-2}l_{r-2}(x,a)e_1 + \cdots + e_{r-1} \end{array} \right\rangle \end{array}$$

where $(a_0, \cdots, a_{r-1})$ means the map defined by $(e_0, \cdots, e_{r-1}) \mapsto (a_0, \cdots, a_{r-1})$ and $l_i(x,a)$ is a specialization of $l_i(X,t)$ above. The increasing filtration $\{\mathrm{Sol}_\lambda(M_{\overline{a}})\}$ of the solution space $\mathrm{Sol}(M_{\overline{a}})$ is given by

$$\mathrm{Sol}_\lambda(M_{\overline{a}}) = \begin{cases} 0 & \text{if } \lambda < 0 \\ \langle (0, \cdots, 0, 1, l_1(x,a), \cdots, l_i(x,a)), \cdots, (0, \cdots, 0, 1) \rangle & \text{if } i \leq \lambda < i+1 \\ \mathrm{Sol}(M_{\overline{a}}) & \text{if } \lambda \geq r-1 \end{cases}$$

$(i = 0, 1, \cdots, r-2)$ by the calculation above. Taking the orthogonal subspace, we have the special log-growth filtration

$$V(M_{\overline{a}})^\lambda = \begin{cases} V(M_{\overline{a}}) & \text{if } \lambda < 0 \\ \left\langle e_0, \cdots, (-1)^{r-2-i}l_{r-2-i}(x,a)e_0 + \cdots + e_{r-2-i} \right\rangle & \text{if } i \leq \lambda < i+1 \\ 0 & \text{if } \lambda \geq r-1. \end{cases}$$

□

5.2 Examples without Frobenius

Example 5.3 Let δ be a real number with $0 < \delta < 1$. Let us define a sequence $\{a_l\}_{l \geq 0}$ in K by

$$a_l = p^{\lfloor (1-\delta)l \rfloor}.$$

Then $|a_l| \to 0\,(l \to \infty)$. Let us define a ∇-module M over $K[\![x]\!]_0$ of rank 2 with a basis e_1, e_2 by

$$\nabla(e_1, e_2) = (e_1, e_2) \begin{pmatrix} 0 & \sum_{l=0}^{\infty} a_l x^{p^l - 1} \\ 0 & 0 \end{pmatrix} dx.$$

Since the integration $\sum_{l=0}^{\infty} \frac{a_l}{p^l} x^{p^l}$ of $\sum_{l=0}^{\infty} a_l x^{p^l - 1}$ is not contained in $K[\![x]\!]_0$, M is not trivial neither over $K[\![x]\!]_0$ nor over $\mathcal{E}$.

The solution matrix Y of M at the generic point t is

$$\begin{aligned} Y &= \begin{pmatrix} 1 & y(X,t) \\ 0 & 1 \end{pmatrix} \\ y(X,t) &= \sum_{n=0}^{\infty} \left\{ \sum_{p^l \geq n+1} \begin{pmatrix} p^l - 1 \\ n \end{pmatrix} \frac{a_l t^{p^l - 1 - n}}{n+1} \right\} (X - t)^{n+1} \in K[\![t]\!][\![X - t]\!]. \end{aligned}$$

Since the sequence $\{|a_l|\}$ is decreasing, we have

$$\left| \sum_{p^l \geq n+1} \begin{pmatrix} p^l - 1 \\ n \end{pmatrix} \frac{a_l t^{p^l - 1 - n}}{n+1} \right| \leq \begin{cases} \left| \dfrac{a_{\lfloor \log_p(n+1) \rfloor + 1}}{n+1} \right| & \text{if } n+1 \text{ is not a power of } p, \\ \left| \dfrac{a_m}{p^m} \right| & \text{if } n+1 = p^m. \end{cases}$$

Moreover, if $n+1 = p^m$, the equality of the formula above holds in the case where $|a_m| > |a_{m+1}|$. For any $\epsilon \geq 0$, we have

$$\frac{\left|\frac{a_m}{p^m}\right|}{(p^m + 1)^\epsilon} \geq \frac{\left|\frac{a_{\lfloor \log_p(n+1) \rfloor + 1}}{n+1}\right|}{(n+2)^\epsilon}$$

if $p^m \leq n+1 < p^{m+1}$. Since $\dfrac{\left|\frac{a_m}{p^m}\right|}{(p^m + 1)^\epsilon} = \dfrac{p^{(1-\epsilon)m - \lfloor (1-\delta)m \rfloor}}{(1 + 1/p^m)^\epsilon} = \dfrac{p^{-\epsilon m - \lfloor -\delta m \rfloor}}{(1 + 1/p^m)^\epsilon}$, we have

$$\frac{p^{(\delta - \epsilon)m}}{(1 + 1/p^m)^\epsilon} \leq \frac{\left|\frac{a_m}{p^m}\right|}{(p^m + 1)^\epsilon} < p^{(\delta - \epsilon)m + 1},$$

and, hence the sequence $\dfrac{\left|\frac{a_m}{p^m}\right|}{(p^m + 1)^\epsilon}$ is bounded if and only if $\epsilon \geq \delta$. Since the number of m such that $|a_m| > |a_{m+1}|$ is infinite, $y(X,t)$ is exactly of log-growth δ. Therefor, we have

$$\begin{aligned} (M \otimes \mathcal{E})^\lambda &= \mathcal{E} e_1 & &\text{if } 0 \leq \lambda < \delta, \\ (M \otimes \mathcal{E})^\lambda &= 0 & &\text{if } \lambda \geq \delta \end{aligned}$$

and $M \otimes \mathcal{E} = (M \otimes \mathcal{E})^{0-} \supsetneq (M \otimes \mathcal{E})^0 = (M \otimes \mathcal{E})^{\delta -} \supsetneq (M \otimes \mathcal{E})^\delta = (M \otimes \mathcal{E})^{\delta +} = 0$.

Let us now consider the log-growth filtration at special point. Taking $t = 0$, we have $V(M) = H^0(M \otimes \mathcal{A}_K(0,1)) = Ke_1 + K(e_2 - y(x,0)e_2)$. By the above estimate we have $\mathrm{Sol}_\lambda(M) = K(0,1)$ for $0 \leq \lambda < \delta$ and $\mathrm{Sol}_\lambda(M) = K(0,1) + K(1, y(x,0))$ for $\lambda \geq \delta$. Hence,

$$V(M)^\lambda = \begin{cases} V(M) & \text{if } \lambda < 0, \\ Ke_1 & \text{if } 0 \leq \lambda < \delta, \\ 0 & \text{if } \lambda \geq \delta. \end{cases}$$

Let us prove that M does not admit a Frobenius structure neither over $K[\![x]\!]_0$ nor over $\mathcal{E}$. Let σ be a Frobenius on $\mathcal{E}$ such that $\sigma(x) = x^q$. Suppose that there exists a Frobenius on M which is represened by the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ over $\mathcal{E}$. Then it satisfies the relation

$$\frac{d}{dx}\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} 0 & \sum_{l=0}^{\infty} a_l x^{p^l-1} \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = qx^{q-1}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 & \sum_{l=0}^{\infty} a_l x^{p^l-1} \\ 0 & 0 \end{pmatrix}^{\sigma}.$$

Then $\frac{d}{dx}c = 0$, and c is a constant. if $c \neq 0$, then it is impossible because a should be the integration of $-\sum a_l x^{p^l-1}$. Hence, $c = 0$, and a and d are nonzero constants. Then we have

$$\frac{d}{dx}b + d\sum_{l=0}^{\infty} a_l x^{p^l-1} = qa\sum_{l=0}^{\infty} a_l x^{qp^l-1}.$$

When $q \neq p$, it is impossible. Suppose that $q = p$. Since $0 < \delta < 1$, both sets of $\{l \,|\, |a_l| = |a_{l+1}|\}$ and $\{l \,|\, |a_l| \neq |a_{l+1}|\}$ are infinite. Hence, a can not be in $\mathcal{E}$. Therefore, M does not admit a Frobenius structure. □

Example 5.4 Let δ be a real number with $0 \leq \delta < 1$. Let us define a sequence $\{b_l\}_{l\geq 1}$ in K by

$$b_l = p^{\lfloor (1-\delta)l - \log_p(l)\rfloor}.$$

Then $|b_l| \to 0\,(l \to \infty)$. Let us define a ∇-module M over $K[\![x]\!]_0$ of rank 2 with a basis e_1, e_2 by

$$\nabla(e_1, e_2) = (e_1, e_2)\begin{pmatrix} 0 & \sum_{l=1}^{\infty} b_l x^{p^l-1} \\ 0 & 0 \end{pmatrix} dx.$$

Since the integration $\sum_{l=1}^{\infty} \frac{b_l}{p^l} x^{p^l}$ of $\sum_{l=1}^{\infty} b_l x^{p^l-1}$ does not contain in $K[\![x]\!]_0$, M is trivial neither over $K[\![x]\!]_0$ nor over $\mathcal{E}$. Moreover, M does not admit a Frobenius structure as in the previous example.

The solution matrix Y of M at the generic point t is

$$\begin{aligned} Y &= \begin{pmatrix} 1 & y(X,t) \\ 0 & 1 \end{pmatrix} \\ y(X,t) &= \sum_{n=0}^{\infty}\left\{\sum_{p^l \geq n+1}\begin{pmatrix} p^l - 1 \\ n \end{pmatrix}\frac{b_l t^{p^l-1-n}}{n+1}\right\}(X-t)^{n+1} \in K[\![t]\!][\![X-t]\!]. \end{aligned}$$

Since $\dfrac{\left|\frac{b_m}{p^m}\right|}{(p^m+1)^{\epsilon}} = \dfrac{p^{(1-\epsilon)m - \lfloor (1-\delta)m - \log_p(m)\rfloor}}{(1+1/p^m)^{\epsilon}} = \dfrac{p^{-\epsilon m - \lfloor -\delta m - \log_p(m)\rfloor}}{(1+1/p^m)^{\epsilon}}$, we have

$$\frac{p^{(\delta-\epsilon)m + \log_p(m)}}{(1+1/p^m)^{\epsilon}} \leq \frac{\left|\frac{b_m}{p^m}\right|}{(p^m+1)^{\epsilon}} < p^{(\delta-\epsilon)m+\log_p(m)+1}.$$

Hence, the sequence $\dfrac{\left|\frac{b_m}{p^m}\right|}{(p^m+1)^\epsilon}$ is bounded if and only if $\epsilon > \delta$. By the similar reason as in Example 5.3 $y(X,t)$ is of log-growth $\leq \lambda$ for any $\lambda > \delta$ and $y(X,y)$ is not of log-growth δ. Therefor, we have

$$\begin{array}{rcll} (M\otimes\mathcal{E})^\lambda & = & \mathcal{E}e_1 & \text{if } 0 \leq \lambda \leq \delta, \\ (M\otimes\mathcal{E})^\lambda & = & 0 & \text{if } \lambda > \delta \end{array}$$

and $M\otimes\mathcal{E} = (M\otimes\mathcal{E})^{0-} \supsetneq (M\otimes\mathcal{E})^{0} = (M\otimes\mathcal{E})^{\delta-} = (M\otimes\mathcal{E})^{\delta} \supsetneq (M\otimes\mathcal{E})^{\delta+} = 0$.

Specializing $t = 0$, we have a log-growth filtration at special point:

$$V(M)^\lambda = \begin{cases} V(M) & \text{if } \lambda < 0, \\ Ke_1 & \text{if } 0 \leq \lambda \leq \delta, \\ 0 & \text{if } \lambda > \delta. \end{cases}$$

□

6 Generic and special Frobenius slope filtrations

According to our needs we will consider modules which will have only φ-structure or ∇-structure or both at the same time.

6.1 Generic Frobenius slope filtration.

For a φ-modules over the ring of bounded functions $K[\![x]\!]_0$, after Theorem 2.6, by replacing F by $\mathcal{E}$ (or eventually by E) we are able to give

Definition 6.1 *Let M a φ-module over $K[\![x]\!]_0$. Then for its associated module $M\otimes\mathcal{E}$ which is a φ-module over $\mathcal{E}$ we have a Frobenius slope filtration $\{S_\lambda(M\otimes\mathcal{E})\}_{\lambda\in\mathbb{R}}$. We called it the generic slope filtration.*

Proposition 6.2 *Given a φ-∇-module over $\mathcal{E}$, then the Frobenius slope filtration given in Theorem* 2.6 *is stable under ∇.*

Proof. We use only the fact that, dealing with vector spaces over normed fields, if we add to an invertible operator a contractible one, we obtain an injective operator. (See also [22, 6.12] for a similar statement.) □

6.2 Solvability and Frobenius Structure

Proposition 6.3 *Let M be a unit-root φ-∇-module over $\mathcal{E}$, then M^τ has a full set of solutions in $\mathcal{E}[\![X-t]\!]_0$.*

Again, one could replace $\mathcal{E}$ with E. In order to prove Proposition 6.3, we denote by $\mathcal{V}_\mathcal{E}$ the ring of integers of $\mathcal{E}$ and we need the following (but in fact it can be proved in more general situation)

Lemma 6.4 *Let $A = \sum_{i=0}^{\infty} A_i(X-t)^i$ be an invertible matrix of order r over $\mathcal{V}_\mathcal{E}[\![X-t]\!]$. Then there is an invertible matrix $Y \in 1+(X-t)\mathrm{Mat}(r,\mathcal{V}_\mathcal{E}[\![X-t]\!])$ such that*

$$AY^\sigma = YA_0.$$

In particular, if $A \in 1+\pi^n(X-t)\mathrm{Mat}(r, \mathcal{V}_{\mathcal{E}}[\![X-t]\!])$*, then* $Y \in 1+\pi^n(X-t)\mathrm{Mat}(r, \mathcal{V}_{\mathcal{E}}[\![X-t]\!])$*. Here* π *is a uniformizer of* K*.*

Proof. Let $Y = \sum_i Y_i(X-t)^i$ be a solution matrix. Since $\sigma(X-t) \equiv (X-t)^q \pmod{\pi}$ and $\sigma(X-t) \in (X-t)\mathcal{V}_{\mathcal{E}}[\![X-t]\!]$, we have that $Y_i A_0$ is equal to $A_i +$ (a linear sum of $Y_0^\sigma, \cdots, Y_{i-1}^\sigma$ with coefficients in $\mathrm{Mat}(r, \mathcal{V}_{\mathcal{E}}[\![X-t]\!])$)$+\pi u Y_i^\sigma$, for some $u \in \mathcal{V}_{\mathcal{E}}$ (depending on i) inductively. Since A_0 is invertible over $\mathcal{V}_{\mathcal{E}}$, we solve it over $\mathcal{V}_{\mathcal{E}}$ step by step. □

Proof of Proposition 6.3. Being M a unit-root φ-module over $\mathcal{E}$ we can find a unit-root integral structure in M for the Frobenius, which will imply an unit-root integral structure for M^τ (i.e. with coefficients in $\mathcal{V}_{\mathcal{E}}[\![X-t]\!]_0$). Then from Lemma 6.4 we may suppose that the Frobenius structure is actually given by a constant invertible matrix in $\mathcal{V}_{\mathcal{E}}$. We conclude that the connection in such a basis is trivial. □

Corollary 6.5 *Let* M *be a* φ*-*∇*-module over* $\mathcal{E}$ *such that it is pure of slope* λ*. Then* M^τ *has a full set of solutions in* $\mathcal{E}[\![X-t]\!]_0$*.*

We notice that the solvability is closed under extentions, because $\mathcal{A}_{\mathcal{E}}(t,1)$ is closed under integrations. We then have connected with the Frobenius slope filtration (which is stable under ∇) by taking iterated extentions.

Theorem 6.6 *Let* M *be a* φ*-*∇*-module over* $\mathcal{E}$*. Then,* M *is solvable.*

6.3 Special Frobenius slope filtration.

Let us now consider a φ-∇-module M over the ring of bounded functions $K[\![x]\!]_0$: we will introduce its special Frobenius filtration (at special point). By Theorem 6.6 we know that such a module is solvable: by transfer Theorem 4.2 the associated ∇-module $M \otimes \mathcal{A}_K(0,1)$ is trivial. It follows by the horizontality, that the Frobenius of M induces a Frobenius structure on $V(M)$. Moreover, the composite of natural maps $V(M) \to M \otimes \mathcal{A}_K(0,1) \to M \otimes K$ gives an isomorphism of φ-modules over K.

Definition 6.7 *Given a* φ*-*∇*-module* M *over the ring of bounded functions* $K[\![x]\!]_0$*, we will define the special Frobenius slope filtration as* $\{S_\lambda(V(M))\}_{\lambda\in\mathbb{R}}$ *as given in* 2.6*.*

6.4 The specialization theorem.

We will prove the existence of Newton filtration [21, 2.6.2] and the specialization theorem ([20, Appendix], [21, 2.3.2]) over $K[\![x]\!]_0$ in our setting.

We do not suppose the same multiplicity of the first Frobenius slopes both at the generic and the special fibers in our version of the existence of Newton filtration.

Theorem 6.8 *Let* M *be a* φ*-*∇*-module over* $K[\![x]\!]_0$ *such that the first special slope of* M *is* λ *and the first generic slope is greater than or equal to* λ*. Then there exists a* φ*-*∇*-submodule* N *of* M *over* $K[\![x]\!]_0$ *of rank* $\dim_K S_\lambda(V(M))$ *such that* N *is pure of slope* λ *both at the generic and special fibers. In particular, we have* $\dim_K S_\lambda(V(M)) \leq \dim_{\mathcal{E}} S_\lambda(M \otimes \mathcal{E})$*.*

In order to prove Theorem 6.8, we prepare several assertions.

Lemma 6.9 *Let $y = \sum_n y_n x^n$ be an element in $\mathcal{A}_K(0,1)$. Suppose that y is not contained in $K[\![x]\!]_0$. For any nonnegative integer L, there is an integer $m > L$ such that $|y_n| < |y_m|$ for $n < m$ and $|y_n| \le |y_m|$ for any $n \le m + L$.*

PROOF. If $L = 0$, the conclusion always holds. Suppose that $L > 0$ and such an m does not exist. Take an integer $r > L$ such that $|y_n| < |y_r|$ for $n < r$. Since $y \notin K[\![x]\!]_0$, we have $|y_n| \to \infty$ as $n \to \infty$, hence such an r exists. Put $\rho = |\pi|^{-\frac{1}{L}}$, where π is a uniformizer of K. T The hypothesis of the nonexistence of such an m implies $\max_{l \le sL+r} |y_l| \ge |y_r|\rho^{sL}$ for any nonnegative integer s. Then, for any n with $sL + r \le n < (s+1)L + r$, we have

$$\max_{l \le n} |y_l| \ge \max_{l \le sL+r} |y_l| \ge |y_r|\rho^{sL} = |y_r|\rho^{sL-n}\rho^n \ge |y_r|\rho^{-L-r}\rho^n.$$

Hence the radius of convergence of y is less than or equal to $\rho^{-1} < 1$. This is a contradiction. □

Proposition 6.10 *Let $a_0, \cdots, a_r$ be elements of $K[\![x]\!]_0$ such that $a_0 \pmod{x} \neq 0$ and $|a_i|_0 \le |a_0|_0$. Suppose $y \in \mathcal{A}_K(0,1)$ satisfies the equality*

$$a_0 y + a_1 y^\sigma + \cdots + a_r y^{\sigma^r} = 0.$$

Then y is contained in $K[\![x]\!]_0$.

PROOF. Note that our Frobenius σ satisfies $|\sigma(x) - x^q|_0 < 1$ and $\sigma(x) \equiv 0 \pmod{x}$. Suppose that y is not contained in $K[\![x]\!]_0$. Let us put $a_0 = \sum_n a_{0,n}x^n$ and let L be the least nonnegative integer such that $|a_{0,L}| = |a_0|_0$. Then there is an integer $m > L$ such that $|y_n| < |y_m|$ for any $n < m$ and $|y_n| \le |y_m|$ for any $n \le m+L$ by Lemma 6.9. Then the absolute value of the coefficient of x^{m+L} in $a_0 y$ is $|a_{0,L}y_m|$. On the contrary, the absolute value of the coefficient of x^{m+L} in $a_1 y^\sigma + \cdots + a_r y^{\sigma^r}$ is less than $|a_{0,L}y_m|$ since $\frac{m+L}{q} < m$ by $m > L$ and $q \ge 2$. This is impossible. Therefore y is contained in $K[\![x]\!]_0$. □

Lemma 6.11 *Let $\mathrm{Frac}(K[\![x]\!]_0)$ be the field of fractions of $K[\![x]\!]_0$ and let*

$$\mathcal{R} = \left\{ \sum_{n \in \mathbb{Z}} a_n x^n \;\middle|\; \begin{array}{l} a_n \in K \ \sum_{n \ge 0} a_n x^n \in \mathcal{A}_K(0,1), \\ \exists\, \rho < 1 \,\text{such that}\, |a_n|\rho^n \to 0 \ (n \to -\infty) \end{array} \right\}$$

be the Robba ring with the natural inclusion $\mathcal{A}_K(0,1) \subset \mathcal{R}$. Then $\mathrm{Frac}(K[\![x]\!]_0) \subset \mathcal{R}$ and $\mathcal{A}_K(0,1) \cap \mathrm{Frac}(K[\![x]\!]_0) = K[\![x]\!]_0$.

PROOF. The denominator of each element of $\mathrm{Frac}(K[\![x]\!]_0)$ is 1 or a polynomial all of whose roots α satisfy $|\alpha| < 1$ by Weierstrass preparation theorem. Hence, any element of $\mathrm{Frac}(K[\![x]\!]_0)$ converges on a annulus $\eta < |x| < 1$ and is contained in $\mathcal{R}$. Since any element of $\mathcal{A}_K(0,1)$ has no pole on the unit disk, we have $\mathcal{A}_K(0,1) \cap \mathrm{Frac}(K[\![x]\!]_0) = K[\![x]\!]_0$. □

PROOF OF THEOREM 6.8. Let A be the Frobenius matrix of M ($r = \mathrm{rank}\, M$) and let B be the Frobenius matrix over K with respect to the induced Frobenius on $S_\lambda(V(M))$ ($s = \dim S_\lambda(V(M))$). Then there is an $r \times s$ matrix Y with entries in $\mathcal{A}_K(0,1)$ such that $AY^\sigma = YB$ and $\mathrm{rank}(Y \pmod{x}) = s$ by the solvability of M (Theorem 6.6). We have only to prove that all entries of Y are contained in $K[\![x]\!]_0$. Indeed, if so, $S_\lambda(V(M)) \otimes_K K[\![x]\!]_0$ gives a $K[\![x]\!]_0$-submodule N of M such that N is stable under the connection and the Frobenius. Since the Frobenius matrix of N is B, N has our desired property.

We may assume that the residue field k of K is algebraically closed by Proposition 2.2 and there exists an element $\beta \in K$ with $|\beta| = q^{-\lambda}$ by Proposition 2.1. Then we may assume that $\lambda = 0$ and B is the identity matrix of degree s.

Let $K[\![x]\!]_{0,(x)}$ be the localization of $K[\![x]\!]_0$ by the prime ideal generated by x. Then the Frobenius endomorphism σ on $K[\![x]\!]_0$ can extend on $K[\![x]\!]_{0,(x)}$ since $\sigma(x) \equiv 0 \,(\mathrm{mod}\, x)$. Take a lift e of cyclic vector of $M \otimes K$ in M. Then $e, \varphi(e), \cdots, \varphi^{r-1}(e)$ forms a basis of $M \otimes K[\![x]\!]_{0,(x)}$ by Nakayama's lemma. After changing a basis by $e, \varphi(e), \cdots, \varphi^{r-1}(e)$, if we denote the new $r \times s$ matrix by $\widetilde{Y}$, each column $\widetilde{Y}_i \in \left(\mathcal{A}_K(0,1) \otimes_{K[\![x]\!]_0} K[\![x]\!]_{0,(x)}\right)^r$ $(1 \leq i \leq s)$ of $\widetilde{Y}$ satisfies a Frobenius equation

$$\begin{pmatrix} b_1 & 1 & & \\ \vdots & & \ddots & \\ b_{r-1} & & & 1 \\ b_r & & & \end{pmatrix} \widetilde{Y}_i^{\sigma} = \widetilde{Y}_i$$

with $b_i \in K[\![x]\!]_{0,(x)}$ and $|b_i| \leq 1$ for all i by the assumption of slopes of $M \otimes \mathcal{E}$. Here $|\text{-}|$ is the norm of $\mathcal{E}$. The top entry $\widetilde{y} = \frac{\widetilde{w}}{\widetilde{v}}$ $(\widetilde{w} \in \mathcal{A}_K(0,1), \widetilde{v} \in K[x], \widetilde{v} \not\equiv 0 \,(\mathrm{mod}\, x)$ of $\widetilde{Y}_i$ satisfies a Frobenius equation

$$c_0\widetilde{w} + c_1\widetilde{w}^{\sigma} + \cdots + c_r\widetilde{w}^{\sigma^r} = 0$$

for some $c_i \in K[\![x]\!]_0$ with $c_0 \not\equiv 0 \,(\mathrm{mod}\, x)$ and $|c_i|_0 \leq |c_0|_0 \,(1 \leq i \leq r)$. It follows from Proposition 6.10 above that $\widetilde{w}$ is contained in $K[\![x]\!]_0$, so that $\widetilde{y}$ is contained in $\mathrm{Frac}(K[\![x]\!]_0)$. Hence, all entries of Y are contained both in $\mathrm{Frac}(K[\![x]\!]_0)$ and in $\mathcal{A}_K(0,1)$. Therefore, all entries of Y are contained in $K[\![x]\!]_0$ by Lemma 6.11. □

Theorem 6.12 (The specialization theorem) *Let M be a φ-∇-module over $K[\![x]\!]_0$. Then the special Newton polygon of Frobenius slopes of M is above the generic Newton polygon of Frobenius slopes of M.*

Proof. We may assume that the residue field k of K is algebraically closed and all the generic and special slopes are contained in the valuation group $\log_q |K^{\times}|$ by Propositions 2.1 and 2.2. We recall the standard argument of taking exterior powers and reduce the theorem to the case of the first slope [20, Appendix]. Let $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_r$ and $\mu_1 \leq \mu_2 \leq \cdots \leq \mu_r$ be the generic slopes and the special slopes of M, respectively. What we need to prove is that (i) $\lambda_1 + \cdots + \lambda_i \leq \mu_1 + \cdots + \mu_i$ for all $i \leq r$ and (ii) $\lambda_1 + \cdots + \lambda_r = \mu_1 + \cdots + \mu_r$. If one takes the i-th exterior power, then the first generic and special slopes are $\lambda_1 + \cdots + \lambda_i$ and $\mu_1 + \cdots + \mu_i$. Taking the highest exterior power, the equality (ii) holds since the Frobenius is an isomorphism. Hence the assertion follows from Theorem 6.8. □

7 Comparison between log-growth filtration and Frobenius filtration. The rank 2 case

7.1 The main result

Our aim is to compare log-growth and Frobenius slope filtrations for a φ-∇-module over $K[\![x]\!]_0$ both at the generic and special points and study their behaviour. It is also clear that a direct identification cannot be made. Think, for example, to a trivial ∇-module, endowed with constant Frobenius structure which allows different slopes. By the transfer theorem (Proposition 4.2) we have

Proposition 7.1 *Let M be a ∇-module over $K[\![x]\!]_0$ such that $(M \otimes \mathcal{E})^0 = 0$, then M has a full set of solutions in $K[\![x]\!]_0$ (i.e. it is trivial; i.e. $V(M)^0 = 0$).*

Before giving our principal result we need some additional notation. Let M be a φ-∇-module over $K[\![x]\!]_0$ and $M^\vee$ its dual as φ-∇-module over $K[\![x]\!]_0$; we have a natural morphism of φ-∇-module over $K[\![x]\!]_0$:

$$M \otimes M^\vee \to K[\![x]\!]_0$$

which induces a perfect pairing of φ-∇-modules over $\mathcal{E}$

$$(M \otimes \mathcal{E}) \otimes (M^\vee \otimes \mathcal{E}) \to \mathcal{E}$$

as well as the perfect pairing (see 4.3) of φ-modules over K

$$V(M) \otimes V(M^\vee) \to K.$$

We will denote always by the symbol $(-)^\perp$ the orthogonal space.

We focus now on rank 2, φ-∇-module defined over $K[\![x]\!]_0$ but not trivial as ∇-modules. We have:

Theorem 7.2 *Let M be a φ-∇-module over $K[\![x]\!]_0$ of rank 2, suppose that $(M \otimes \mathcal{E})^0 \neq 0$ and the first Frobenius slope of $M \otimes \mathcal{E}$ is zero; then*

(1) *for any $\lambda \in \mathbb{R}$, we have*

$$(S_\lambda(M \otimes \mathcal{E}))^\perp = (M^\vee \otimes \mathcal{E})^\lambda;$$

(2) *for any $\lambda \in \mathbb{R}$, we have*

$$(S_\lambda(V(M)))^\perp = V(M^\vee)^\lambda.$$

We will give the proof in the sections 7.3 and 7.4.

Remark 7.3 In general the first generic Frobenius slope will be equal to λ_1, then we could refine the previous statement with $(S_\lambda(M \otimes \mathcal{E}))^\perp = (M^\vee \otimes \mathcal{E})^{\lambda-\lambda_1}$ and $(S_\lambda(V(M)))^\perp = V(M^\vee)^{\lambda-\lambda_1}$. To reduce this statement to the previous one we will need to make a finite extention of the base field K in order to twist the Frobenius, on the other hand the filtrations are induced by those in K by scalar extention.

Corollary 7.4 *For a φ-∇-module over $K[\![x]\!]_0$ of rank 2 (we do not assume $(M \otimes \mathcal{E})^0 \neq 0$), the special log-polygon is above the generic log-polygon.*

Proof. Since the special Newton polygon of Frobenius is above the generic Newton polygon of Frobenius (Theorem 6.12), the assertion follows from Theorem 7.4 and the lemma below. □

Lemma 7.5 *Let M be a vector space of finite dimension over a field and let $\{M^\lambda\}_{\lambda \in \mathbb{R}}$ (resp. $\{S^\lambda\}_{\lambda \in \mathbb{R}}$) be a decreasing (resp. increasing) filtration of M with $\cup_\lambda M^\lambda = M$ and $\cap_\lambda M^\lambda = 0$ (resp. $\cup_\lambda S_\lambda = M$ and $\cap_\lambda S_\lambda = 0$). Suppose that $\dim M - \dim M^\lambda = \dim S_\lambda$ for all λ. Then the Newton polygon associated to $\{M^\lambda\}$ coincides with the Newton polygon associated to $\{S_\lambda\}$.*

Remark 7.6 As it has been stated, Theorem 7.2 is a local result. Consider now the case of M being a convergent F-isocrystal defined on an open set X of $\mathbb{P}^1_k$. Then, by following Remarks 3.10 and 4.5, for any point of X, one is able to define a log-growth filtration and a Frobenius slope filtration. In particular it makes sense to suppose that our module has non trivial log-growth filtration at the generic point. Then by Theorem 7.2, if M has rank 2, one concludes that the log-growth filtration and the Frobenius slope filtration can be recovered one from the other at any point of X (even at the generic) and the special log-growth polygon is above the generic log-growth polygon. However, we stress the fact that we don't have any information at the points in the infinity $\mathbb{P}^1_k \setminus X$ for an overconvergent F-isocrystal on X.

7.2 The behaviors of log-growth

Let F be a p-adically complete discrete valuation field with a valuation $|\text{-}|$ normalized by $|p| = p^{-1}$, let π_F be a uniformizer of F and let us put $|\pi_F| = q^{-\alpha}$. Let σ be a q-Frobenius on F (again, we will refer for such a field to K, $\mathcal{E}$, E). Let $\mathcal{A}_F(0,1)$ be a ring of analytic functions on the open unit disk over F with a parameter x, and let σ be an extension of Frobenius to $\mathcal{A}_F(0,1)$ such that $\sigma(x) = x^q$.

Proposition 7.7 *Let $a, b, c \in F[\![x]\!]_0$ such that $a \,(\mathrm{mod}\, x) \neq 0$ and $|c|_0 < |b|_0$. Suppose $y \in \mathcal{A}_F(0,1), y \neq 0$ satisfies the equality*

$$ay + by^\sigma + cy^{\sigma^2} = 0.$$

Then, $y \in F[\![x]\!]_0$ if and only if $|a|_0 = |b|_0$.

Proof. The fact $y \in F[\![x]\!]_0$ implies $|a|_0 = |b|_0$ using the absolute value $|\text{-}|_0$. The converse follows from Proposition 6.10. □

Proposition 7.8 *Let $a, b, c \in F[\![x]\!]_0 \subset \mathcal{A}_F(0,1)$ such that $a \,(\mathrm{mod}\, x) \neq 0$ and $|c|_0 \leq |b|_0$. Suppose $y \in \mathcal{A}_F(0,1), y \notin F[\![x]\!]_0$ satisfies the equality*

$$ay + by^\sigma + cy^{\sigma^2} = 0.$$

Suppose $|b|_0/|a|_0 = q^\mu$ with $\mu > 0$. Then y is exactly of log-growth μ, that is, $y \in F[\![x]\!]_\mu$ and $y \notin F[\![x]\!]_\lambda$ for any $\lambda < \mu$.

In order to prove Proposition 7.8 we fix notation. We may assume that $|a|_0 = 1$ and $a = \sum_n a_n x^n$ is a monic polynomial of degree $r \geq 0$ in $F[x]$ (i.e., $a_r = 1$) such that $|a_n| < 1$ for $n < r$ if we put $a = \sum_{i=0}^r a_i x^i$. Indeed, we have $a = ua'$ such that a' is a monic polynomial in $F[x]$ as above, and u is a unit in $F[\![x]\!]_0$ by Weierstrass preparation theorem. By dividing the equation by u, then our situation satisfies the assumption. If $b = 0$, then $c = 0$ and $y = 0$. Hence, $b \neq 0$. Let us put $b = \sum_{n=0}^\infty b_n x^n$ with $|b_s| = |b|_0 = q^\mu$ and $|b_n| < |b_s|$ for any $n < s$.

Let us denote by $y = \sum_n y_n x^n$. For the coefficient of x^n in y, we have

$$\sum_{i=0}^r a_i y_{n-i} + \sum_{0 \leq i \leq \frac{n}{q}} b_{n-iq} y_i^\sigma + \sum_{0 \leq i \leq \frac{n}{q^2}} c_{n-iq^2} y_i^{\sigma^2} = 0, \tag{$*$}$$

where $c = \sum_n c_n x^n$.

First we study y as an element in $F[\![x]\!]$ in the case where $r > 0$.

Lemma 7.9 *Suppose that $r > 0$ and suppose that there is an integer m which satisfies two conditions:*

(a) $m > \frac{r}{q-1}\left(1 + 2\frac{\mu}{\alpha}\right)$;

(b) $|y_i| < |y_m|$ *for* $i < m$;

(c) $|y_i| < q^{-\mu}|y_m|$ *for* $i \leq \left(m + \left(1 + \frac{\mu}{\alpha}\right) r\right)/q$.

Then the radius of convergence of y is less than or equal to $q^{-\frac{\mu}{r}}$. In particular, y does not belong to $\mathcal{A}_F(0,1)$.

Proof. In the equality $(*)$ for $n = m + r$ the absolute values of the second and third terms are less than $|y_m|$ by the condition (c) since $\frac{n}{q} = \frac{m+r}{q} \leq \left(m + \left(1 + \frac{\mu}{\alpha}\right) r\right)/q$. On the contrary, the absolute value of $a_r y_m$ in the first term is $|y_m|$. The equality implies that there is an integer m_1 with $m < m_1 \leq m + r$ such that $|y_{m_1}| > |y_m|$ by the assumption on a.

Again we apply the same argument to the equality $(*)$ for $n = m_1 + r$. Then there is an integer m_2 with $m_1 < m_2 \leq m_1 + r$ such that $|y_{m_2}| > |y_{m_1}|$. By the condition (c) we can apply the same argument at least $\frac{\mu}{\alpha}$ times. Note that the valuation group $|F^\times|$ is generated by $q^{-\alpha}$. Hence there is an integer m' with $m < m' \leq m + \frac{\mu}{\alpha} r$ such that $|y_{m'}| \geq q^{\mu}|y_m|$. Take the least integer m' with the condition, then we have $|y_i| < |y_{m'}|$ for all $i < m'$.

Since

$$m - \frac{m' + \left(1 + \frac{\mu}{\alpha}\right) r}{q} \geq m - \frac{m + \frac{\mu}{\alpha} r + \left(1 + \frac{\mu}{\alpha}\right) r}{q} = \frac{(q-1)m - \left(1 + 2\frac{\mu}{\alpha}\right) r}{q} > 0$$

by the condition (a), we have $|y_i| < q^{-\mu}|y_{m'}|$ for $i \leq \left(m' + \left(1 + \frac{\mu}{\alpha}\right) r\right)/q < m$ by the condition (b). Hence, the integer m' satisfies the same conditions (a), (b) and (c) when we replace m by m'.

Applying the argument above inductively, we have a sequence of positive integers $n_0 = m < n_1 < n_2 < \cdots$ with $n_i \leq m + \frac{\mu}{\alpha} i r$ such that $|y_{n_i}| \geq q^{\mu i}|y_m|$. Therefore, the radius of convergence of y is less than or equal to $q^{-\frac{\mu}{r}}$ and y is not contained in $\mathcal{A}_F(0,1)$. □

For a nonnegative integer m, let us consider three conditions (the notation are coherent with the previous ones):

(i) $m > \frac{r}{q-1}\left(1 + 2\frac{\mu}{\alpha}\right) + s$;

(ii) $|y_n| < |y_m|$ for $n < m$;

(iii) $|y_n| \leq |y_m|$ for $n \leq m + \frac{r}{q-1}\left(1 + \frac{\mu}{\alpha}\right) + s$.

By Lemma 6.9 we have

Lemma 7.10 *There exists an integer m_0 which satisfies the conditions* (i) - (iii) *above.*

Lemma 7.11 *Let m_0 be an integer which satisfies the conditions* (i) - (iii)*. Then the integer $m_1 = m_0 q + s - r$ also satisfies the inequality $m_1 > m_0$ and the conditions* (i) - (iii)*. Moreover, we have $|y_{m_1}| = q^{\mu}|y_{m_0}|$.*

We prepare several assertions.

Claim 1. $\left|\sum_{0\le i\le \frac{n}{q^2}} c_{n-iq^2}\, y_i^{\sigma^2}\right| < q^\mu |y_{m_0}|$ *for* $n \le \left(m_1 + \frac{r}{q-1}\left(1+\frac{\mu}{\alpha}\right) + s\right) + r.$

Proof of claim 1. Since

$$\frac{n}{q^2} \le \frac{\left(m_1 + \frac{r}{q-1}\left(1+\frac{\mu}{\alpha}\right) + s\right) + r}{q^2} = \frac{m_0}{q} + \frac{r}{q^2(q-1)}\left(1+\frac{\mu}{\alpha}\right) + \frac{2s}{q^2} < m_0$$

by the condition (i), we have $|y_i| < |y_{m_0}|$ for $i \le \frac{n}{q^2}$ by the condition (ii). The assertion holds since $|c|_0 \le q^\mu$. □

Claim 2. (1) $\left|\sum_{0\le i\le \frac{n}{q}} b_{n-iq}\, y_i^{\sigma}\right| \le q^\mu |y_{m_0}|$ *for* $n \le \left(m_1 + \frac{r}{q-1}\left(1+\frac{\mu}{\alpha}\right) + s\right) + r.$

(2) *If* $n = m_1 + r$, *the equality holds for the inequality in* (1).

(3) $\left|\sum_{0\le i\le \frac{n}{q}} b_{n-iq}\, y_i^{\sigma}\right| < q^\mu |y_{m_0}|$ *for* $n < m_1 + r$.

Proof of claim 2. (1) Since

$$\frac{n}{q} \le \frac{\left(m_1 + \frac{r}{q-1}\left(1+\frac{\mu}{\alpha}\right) + s\right) + r}{q} = m_0 + \frac{r}{q(q-1)}\left(1+\frac{\mu}{\alpha}\right) + \frac{2s}{q} \le m_0 + \frac{r}{q-1}\left(1+\frac{\mu}{\alpha}\right) + s$$

by $q \ge 2$, we have $|y_i| \le |y_{m_0}|$ for $i \le \frac{n}{q}$ by the condition (iii). The inequality holds since $|b|_0 = q^\mu$.

(2) Suppose $n = m_1 + r$. If $i < m_0$, then $|b_{n-iq} y_i| < q^\mu |y_{m_0}|$. If $i > m_0$, then $n - iq < s$. Hence, $|b_{n-iq}| < q^\mu$ and $|b_{n-iq} y_i| < q^\mu |y_{m_0}|$ by the hypothesis on b by the proof of (1). Therefore, $\left|\sum_{0\le i\le \frac{n}{q}} b_{n-iq}\, y_i^{\sigma}\right| = |b_s y_{m_0}| = q^\mu |y_{m_0}|$.

(3) If $i < m_0$, then $|b_{n-iq} y_i| < q^\mu |y_{m_0}|$ by the condition (ii). If $i \ge m_0$, then $n - iq < s$ and $|b_{n-iq} y_i| < q^\mu |y_{m_0}|$ by the proof of (1). Hence, the inequality holds. □

Claim 3. (1) $|y_n| \le q^\mu |y_{m_0}|$ *for* $n \le \max\{m_1 + \frac{r}{q-1}\left(1+\frac{\mu}{\alpha}\right) + s, m_1 + r\}$.

(2) $|y_n| < q^\mu |y_{m_0}|$ *for* $n < m_1$.

Proof. Suppose $r > 0$.

(1) Suppose that there is an integer $n \le \max\{m_1 + \frac{r}{q-1}\left(1+\frac{\mu}{\alpha}\right) + s, m_1 + r\}$ such that $|y_n| > q^\mu |y_{m_0}|$. We may assume that n is the least integer with the inequality. Then $|y_i| < |y_n|$ for $i < n$. If the condition (c) in Lemma 7.9 for $m = n$ holds, then y is not contained in $\mathcal{A}_F(0,1)$. Hence, it is impossible. Therefore, we have only to check the condition (c) under the hypothesis. If $n \le m_1 + \frac{r}{q-1}\left(1+\frac{\mu}{\alpha}\right) + s$, then

$$\begin{aligned} \frac{n+\left(1+\frac{\mu}{\alpha}\right)r}{q} &\le \frac{m_1 + \frac{r}{q-1}\left(1+\frac{\mu}{\alpha}\right) + s + \left(1+\frac{\mu}{\alpha}\right)r}{q} = m_0 + \frac{\frac{\left(1+q\frac{\mu}{\alpha}\right)r}{q-1} + 2s}{q} \\ &\le m_0 + \frac{r}{q-1}\left(\frac{1}{q}+\frac{\mu}{\alpha}\right) + s \le m_0 + \frac{r}{q-1}\left(1+\frac{\mu}{\alpha}\right) + s. \end{aligned}$$

If $n \le m_1 + r$, then

$$\frac{n + \left(1+\frac{\mu}{\alpha}\right)r}{q} \le \frac{m_1 + r + \left(1+\frac{\mu}{\alpha}\right)r}{q} = m_0 + \frac{\left(1+\frac{\mu}{\alpha}\right)r + s}{q} \le m_0 + \frac{r}{q-1}\left(1+\frac{\mu}{\alpha}\right) + s.$$

Hence, we have $|y_i| \le |y_{m_0}| < q^{-\mu}|y_n|$ for $i \le \frac{n+\left(1+\frac{\mu}{\alpha}\right)r}{q}$. Therefore, the condition (c) holds.

(2) Suppose that there is an integer $l < m_1$ such that $|y_l| \geq q^{\mu}|y_{m_0}|$. The equality $(*)$ for $n = l + r$ implies that there is an integer l' with $l < l' \leq l + r$ such that $|y_{l'}| > q^{\mu}|y_{m_0}|$ (hence, $l' > m_0$) by the claims 1 and 2. Since $l' < m_1 + r$, we have a contradiction to (1).

Suppose $r = 0$. The inequality (1) (resp. (2)) holds by the relation $(*)$ and the claim 1 and claim 2 (1) (resp. 2 (3)). $\square$

Proof of Lemma 7.11. By the condition (i) on m_0, we have $m_1 > m_0$. By the claim 3 we have inequalities $|y_n| < q^{\mu}|y_{m_0}|$ for $n < m_1$ and $|y_n| \leq q^{\mu}|y_{m_0}|$ for $n \leq m_1 + \frac{r}{q-1}\left(1 + \frac{\mu}{\alpha}\right) + s$. Hence, we have only to prove $|y_{m_1}| = q^{\mu}|y_{m_0}|$.

Let us consider the equality $(*)$ for $n = m_1 + r$. The the absolute values of the second and third terms are equal to $q^{\mu}|y_{m_0}|$ and less than $q^{\mu}|y_{m_0}|$, respectively. On the other hand, $a_r y_{m_1}$ has the greatest absolute value in the first term by the claim 3 (1) and the hypothesis on a. Hence, $|y_{m_1}| = q^{\mu}|y_{m_0}|$.

This completes the proof of Lemma 7.11. $\square$

Now we continue the proof of Proposition 7.8. Let m_0 be an integer with the three conditions (i) - (iii) and let us define $m_j = m_{j-1}q + s - r$ for any integer $j \geq 1$. Then $m_j = (m_0 + \frac{s-r}{q-1})q^j - \frac{s-r}{q-1}$ and $\{m_j\}_{j \geq 0}$ is a strictly increasing sequence. Let n be an integer such that $m_j \leq n < m_{j+1}$ for an integer $j \geq 0$. Then $|y_n| < |y_{m_{j+1}}| = q^{\mu(j+1)}|y_{m_0}|$. We have an estimate

$$\frac{|y_n|}{(n+1)^{\mu}} < \frac{|y_{m_{j+1}}|}{(m_j+1)^{\mu}} = \frac{q^{\mu(j+1)}|y_{m_0}|}{((m_0 + \frac{s-r}{q-1})q^j - \frac{s-r}{q-1} + 1)^{\mu}} \leq \frac{2^{\mu}q^{\mu}|y_{m_0}|}{(m_0 + \frac{s-r}{q-1})^{\mu}},$$

whose right hand side is independent of j (we use $m_0 + \frac{s-r}{q-1} - \frac{s-r}{q^j(q-1)} + \frac{1}{q^j} \geq \frac{1}{2}\left(m_0 + \frac{s-r}{q-1}\right)$ for the last inequality). Hence, y is of log-growth μ. But, if we take λ with $0 \leq \lambda < \mu$, then we have

$$\frac{|y_{m_j}|}{(m_j+1)^{\lambda}} = \frac{q^{\mu j}|y_{m_0}|}{((m_0 + \frac{s-r}{q-1})q^j - \frac{s-r}{q-1} + 1)^{\lambda}} \to \infty \quad \text{as } j \to \infty.$$

Therefore, y does not belong to $F[\![x]\!]_{\lambda}$ for $\lambda < \mu$ and y is exactly of log-growth μ. $\square$

7.3 Proof of Theorem 7.2 (the generic case)

We have supposed that at the generic point we always have one Frobenius slope equal to 0 and the other Frobenius slope $\lambda_2 > 0$. Indeed, we have supposed that $(M \otimes \mathcal{E})^0 \neq 0$ and moreover, Proposition 6.3, we are not in the unit-roots case. It turns out that there exists a basis e_1, e_2 such that the matrix which represents the Frobenius can be choosen as

$$\begin{pmatrix} a & 1 \\ b & 0 \end{pmatrix} \in \mathrm{GL}(2, \mathcal{E})$$

where $|a| = 1$ and $|b| = q^{-\lambda_2}$ by Lemma 2.9 and Proposition 2.13 (e_2 is a cyclic vector).

Via τ we may consider the associate φ-∇-module over $\mathcal{E}[\![X-t]\!]_0$ and we indicate again the matrix $\begin{pmatrix} a & 1 \\ b & 0 \end{pmatrix}^{\tau}$ by $\begin{pmatrix} a & 1 \\ b & 0 \end{pmatrix}$. Let us change the Frobenius endomorphism on $\mathcal{E}[\![X-t]\!]_0$ by the Frobenius endomorphism σ with $\sigma(X-t) = (X-t)^q$. We may then take a full set of horizontal sections $(e_1, e_2)\begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix}$ of $M^\tau \otimes \mathcal{A}_{\mathcal{E}}(t,1)$ such that

$$\begin{pmatrix} a & 1 \\ b & 0 \end{pmatrix}\begin{pmatrix} y_{11}^\sigma & y_{12}^\sigma \\ y_{21}^\sigma & y_{22}^\sigma \end{pmatrix} = \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix}\begin{pmatrix} \gamma_1 & \gamma_{12} \\ 0 & \gamma_2 \end{pmatrix}$$

for $\gamma_1, \gamma_2, \gamma_{12} \in \mathcal{E}$ with $|\gamma_1| = 1$ and $|\gamma_2| = q^{-\lambda_2}$ by Theorem 2.6. We may then take the perfection $\mathcal{E}^{\mathrm{perf}}$ of the field $\mathcal{E}$ and a scalar extension $\mathcal{E}[\![X-t]\!]_0 \to \mathcal{E}^{\mathrm{perf}}[\![X-t]\!]_0$, and after this we may then, by a base change horizontal sections by $\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$ in $\mathrm{GL}_2(\mathcal{E}^{\mathrm{perf}})$, to suppose that our horizontal sections will satisfy

$$\begin{pmatrix} a & 1 \\ b & 0 \end{pmatrix}\begin{pmatrix} z_{11}^\sigma & z_{12}^\sigma \\ z_{21}^\sigma & z_{22}^\sigma \end{pmatrix} = \begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix}\begin{pmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \end{pmatrix}.$$

We recall that the columns are our horizontal sections, hence for the first column solution we have the two equations

$$az_{11}^\sigma + z_{21}^\sigma = \gamma_1 z_{11} \quad bz_{11}^\sigma = \gamma_1 z_{21}$$

Obtaining $z_{21}^\sigma = \frac{b^\sigma}{\gamma_1^\sigma} z_{11}^{\sigma^2}$, hence

$$az_{11}^\sigma + \frac{b^\sigma}{\gamma_1^\sigma} z_{11}^{\sigma^2} = \gamma_1 z_{11}. \tag{$\sharp$}$$

For the second column:

$$az_{12}^\sigma + z_{22}^\sigma = \gamma_2 z_{12} \quad bz_{12}^\sigma = \gamma_2 z_{22}$$

Obtaining $z_{22}^\sigma = \frac{b^\sigma}{\gamma_2^\sigma} z_{12}^{\sigma^2}$, hence

$$az_{12}^\sigma + \frac{b^\sigma}{\gamma_2^\sigma} z_{12}^{\sigma^2} = \gamma_2 z_{12}. \tag{$\flat$}$$

Now we apply the previous section 7.2 to determine the log-growth where $F = \mathcal{E}^{\mathrm{perf}}$ and $X - t$ is x. Since $\left|\frac{a}{\gamma_1}\right|_0 = 1$ and $\left|\frac{b^\sigma}{\gamma_1\gamma_1^\sigma}\right|_0 < 1$, the equality ($\sharp$) implies $z_{11} \in \mathcal{E}^{\mathrm{perf}}[\![X-t]\!]_0$ by Proposition 7.7, and hence $z_{21} \in \mathcal{E}^{\mathrm{perf}}[\![X-t]\!]_0$. By the hypothesis that $(M \otimes \mathcal{E})^0 \neq 0$, we have $z_{12} \notin \mathcal{E}^{\mathrm{perf}}[\![X-t]\!]_0$. Since $\left|\frac{a}{\gamma_2}\right|_0 = q^{\lambda_2}$ and $\left|\frac{b^\sigma}{\gamma_2\gamma_2^\sigma}\right|_0 = q^{\lambda_2} = \left|\frac{a}{\gamma_2}\right|_0$, the equality ($\flat$) implies $z_{12} \in \mathcal{E}^{\mathrm{perf}}[\![X-t]\!]_{\lambda_2}$ exactly by Proposition 7.8, and hence $z_{22} \in \mathcal{E}^{\mathrm{perf}}[\![X-t]\!]_{\lambda_2}$ exactly. By the shape of the base change matrix, y_{11}, y_{21} are bounded, y_{22} is exactly of log-growth λ_2 and y_{12} has log-growth $\leq \lambda_2$. Note that the log-growth is stable under the action of Frobenius.

Let $e_1^\vee, e_2^\vee$ be the dual basis of the dual φ-∇-module $M^\vee \otimes \mathcal{E}$ with respect to e_1, e_2. Then two $\mathcal{E}$-linear homomorphisms defined by $(e_1^\vee, e_2^\vee) \mapsto (y_{11}, y_{21})$ and by $(e_1^\vee, e_2^\vee) \mapsto (y_{12}, y_{22})$ give a basis of solutions of $M^\vee \otimes \mathcal{E}$ in $\mathcal{A}_{\mathcal{E}}(t,1)$. Hence, we have

$$\dim_{\mathcal{E}} (M^\vee \otimes \mathcal{E})^\lambda = \begin{cases} 2 & \text{if } \lambda < 0 \\ 1 & \text{if } 0 \leq \lambda < \lambda_2 \\ 0 & \text{if } \lambda \geq \lambda_2. \end{cases}$$

On the other hand, since $S_{-\lambda_2}(M^\vee \otimes \mathcal{E}) = (S_0(M \otimes \mathcal{E}))^\perp$ is a φ-∇-submodule of $M^\vee \otimes \mathcal{E}$ of rank 1, the quotient $\left(M^\vee \otimes \mathcal{E}/(S_0(M \otimes \mathcal{E}))^\perp\right)^\tau$ has a full set of solution in $\mathcal{E}[\![X-t]\!]_0$ by Corollary 6.5. Hence, we have

$$(M^\vee \otimes \mathcal{E})^0 = (S_0(M \otimes \mathcal{E}))^\perp.$$

by the universal property of the subspace $(M^\vee \otimes \mathcal{E})^0$ (Theorem 3.2). Comparing the dimension, we have

$$(S_\lambda(M \otimes \mathcal{E}))^\perp = (M^\vee \otimes \mathcal{E})^\lambda$$

for any λ. This completes the proof of Theorem 7.2 in the generic case. □

Remark 7.12 Our proof of Theorem 7.2 works not only for φ-∇-modules defined over $K[\![x]\!]_0$ but also for φ-∇-modules over $\mathcal{E}$.

Corollary 7.13 *Let M be a φ-∇-module over $\mathcal{E}$ (resp. E) of rank 2 with Frobenius slopes $\lambda_1 < \lambda_2$. If $\lambda_2 - \lambda_1 > 1$, then M^τ has a full set of solutions in $\mathcal{E}[\![X-t]\!]_0$ (resp. $E[\![X-t]\!]_0$).*

Proof. We may assume that $\lambda_1 = 0$. If M^τ does not have a full set of solutions in $\mathcal{E}[\![X-t]\!]_0$, $(M^\vee)^{\lambda_2} = (S_{\lambda_2}(M))^\perp \neq 0$ by Theorem 7.2 and Remark 7.12. However, if $\lambda_2 > 1$, then $(M^\vee)^{\lambda_2} = 0$ by Corollary 3.5. Threrefore, M^τ has a full set of solutions in $\mathcal{E}[\![X-t]\!]_0$ by Corollary 1.7. □

7.4 Proof of Theorem 7.2 (the special case)

Suppose that M has a full set of solutions in $K[\![x]\!]_0$. Then $(M \otimes \mathcal{E})^\tau$ also has a full set of solutions in $\mathcal{E}[\![X-t]\!]_0$. Hence we may assume that M does not have a full set of solutions in $K[\![x]\!]_0$.

We may assume that the residue field k of K is algebraically closed, all the slopes of Frobenius are contained in the valuation group of K and K admits an extension of Frobenius (Proposition 2.1). Indeed, the pairings are defined over F and the space $\mathrm{Sol}_\lambda(M)$ is compatible with the extension $\widehat{K}^{\mathrm{alg}}/K$ by Proposition 1.9. Let us change the Frobenius endomorphism on $K[\![x]\!]_0$ by the Frobenius endomorphism σ with $\sigma(x) = x^q$.

We have supposed that at the special point we have Frobenius slopes λ_1, λ_2 with $0 \leq \lambda_1 \leq \lambda_2$ such that the non-zero Frobenius slope at the generic point is $\lambda_1 + \lambda_2$ by the specialization theorem (Theorem 6.12).

Let e_1 and e_2 be a basis of M such that the image of e_2 is a cyclic vector in $M \otimes K$ (Theorem 2.6) and let

$$\varphi(e_1, e_2) = (e_1, e_2)\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

be the Frobenius of M. Then $b \,(\mathrm{mod}\, x) \neq 0$ because M is free of rank 2 and the image of e_2 at the special fiber is a cyclic vector. Since e_2 satisfies the relation

$$b\varphi^2(e_2) - (ab^\sigma + bd^\sigma)\varphi(e_2) + b^\sigma(ad - bc)e_2 = 0,$$

the assumption of slopes at the generic fiber implies $|ab^\sigma + bd^\sigma|_0 = |b|_0$ and $|b^\sigma(ad-bc)|_0 = q^{-\lambda_1}q^{-\lambda_2}|b|_0$. We may then take a full set of horizontal sections $(e_1, e_2)\begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix}$ of $M \otimes \mathcal{A}_K(0,1)$ such that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} y_{11}^\sigma & y_{12}^\sigma \\ y_{21}^\sigma & y_{22}^\sigma \end{pmatrix} = \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix}\begin{pmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \end{pmatrix}$$

for $\gamma_1, \gamma_2 \in K$ with $|\gamma_1| = q^{-\lambda_1}$ and $|\gamma_2| = q^{-\lambda_2}$ by Theorem 2.6. Then we have a relation

$$b^\sigma y_{1i} - \frac{ab^\sigma + bd^\sigma}{\gamma_i} y_{1i}^\sigma + \frac{b(ad-bc)^\sigma}{\gamma_i \gamma_i^\sigma} y_{1i}^{\sigma^2} = 0 \qquad (\dagger)$$

for $i = 1, 2$.

Now we apply the section 7.2 to determine the log-growth, where $F = K$. Let us consider four cases (1°) - (4°).

(1°) $\lambda_1 = \lambda_2 = 0$.

In this case, by the specialization theorem (Theorem 6.12), all the slopes of $M \otimes \mathcal{E}$ are 0. Hence, we can apply Proposition 6.3 to $(M \otimes \mathcal{E})^\tau$: there is a full set of solutions in $\mathcal{E}[\![X - t]\!]_0$. Therefore, it is not our case.

(2°) $0 = \lambda_1 < \lambda_2$.

Since $\left|\frac{ab^\sigma + bd^\sigma}{\gamma_1}\right|_0 = |b^\sigma|_0 > \left|\frac{b(ad-bc)^\sigma}{\gamma_1\gamma_1^\sigma}\right|_0$ in the equality (†), y_{11}, y_{21} are bounded by Proposition 7.7. Since $(M \otimes \mathcal{E})^0 \neq 0$, y_{12} is not bounded by Proposition 7.1. Since $\left|\frac{ab^\sigma + bd^\sigma}{\gamma_2}\right|_0 / |b^\sigma|_0 = q^{\lambda_2}$ and $\left|\frac{b(ad-bc)^\sigma}{\gamma_2\gamma_2^\sigma}\right|_0 / |b|_0 = q^{\lambda_2 - \lambda_1}$ in the equality (†), y_{12} is exactly log-growth λ_2 and y_{22} is log-growth $\leq \lambda_2$ by Proposition 7.8. In this case there is a φ-∇-submodule N of M over $K[\![x]\!]_0$ of rank 1 such that N is unit-root both at the generic and special fibers by Theorem 6.8.

(3°) $0 < \lambda_1 < \lambda_2$.

Let (i, j) be $(1, 2)$ or $(2, 1)$. Since $\left|\frac{ab^\sigma + bd^\sigma}{\gamma_i}\right|_0 / |b^\sigma|_0 = q^{\lambda_i}$ and $\left|\frac{b(ad-bc)^\sigma}{\gamma_i\gamma_i^\sigma}\right|_0 / |b|_0 = q^{\lambda_i - \lambda_j}$ in the equality (†), y_{1i} is exactly log-growth λ_i and y_{2i} is log-growth $\leq \lambda_i$ by Proposition 7.8.

(4°) $0 < \lambda_1 = \lambda_2$.

Let $z_1 e_1 + z_2 e_2$ be a nonzero horizontal section of M in $\mathcal{A}_K(0, 1)$ of log-growth $\lambda < \lambda_1$. Then the K-vector space $H^0_\lambda(M)$ is of dimension one which is stable under the Frobenius by Proposition 4.8. This Frobenius structure is given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} z_1^\sigma \\ z_2^\sigma \end{pmatrix} = \gamma \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

for $\gamma \in K$ with $|\gamma| = q^{-\lambda_1}$. By applying Propositions 7.7 and 7.8 to the Frobenius equation on z_1 similar to the equality (†), z_1 is exactly of log-growth λ_1. This contradicts our hypothesis. Hence, $H_{\lambda_1}(M) = V(M)$ and $H_\lambda(M) = 0$ for $\lambda < \lambda_1$.

In the case where $\lambda_1 < \lambda_2$ we have

$$S_\lambda(V(M)) = \begin{cases} 0 & \text{if } \lambda < \lambda_1 \\ K\langle y_{11}e_1 + y_{21}e_2 \rangle & \text{if } \lambda_1 \leq \lambda < \lambda_2 \\ V(M) & \text{if } \lambda \geq \lambda_2, \end{cases}$$

and in the case where $\lambda_1 = \lambda_2$ we have

$$S_\lambda(V(M)) = \begin{cases} 0 & \text{if } \lambda < \lambda_1 \\ V(M) & \text{if } \lambda \geq \lambda_1, \end{cases}$$

Let $e_1^\vee, e_2^\vee$ be the dual basis of the dual φ-∇-module $M^\vee$ with respect to e_1, e_2. Then a full set of horizontal sections of $M^\vee$ is given by $(e_1^\vee, e_2^\vee) \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix}^{-1}$. Let us put

$y = \det \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix} \in K[\![x]\!]_0$ ($ye_1 \wedge e_2$ is an invertible horizontal section of $\det M$). The Frobenius slope filtration of the space of the horizontal sections of $V(M^\vee)$ is given by

$$(S_\lambda(V(M)))^\perp = \begin{cases} 0 & \text{if } \lambda \geq \lambda_2 \\ K\langle \frac{1}{y}(-y_{21}e_1^\vee + y_{11}e_2^\vee)\rangle & \text{if } \lambda_1 \leq \lambda < \lambda_2 \\ V(M^\vee) & \text{if } \lambda < \lambda_1 \end{cases}$$

for $\lambda_1 < \lambda_2$ and

$$(S_\lambda(V(M)))^\perp = \begin{cases} 0 & \text{if } \lambda \geq \lambda_1 \\ V(M^\vee) & \text{if } \lambda < \lambda_1 \end{cases}$$

for $\lambda_1 = \lambda_2$.

On the other hand, the increasing filtrations of solutions of M in $\mathcal{A}_K(0,1)$ is given by

$$\mathrm{Sol}_\lambda(M^\vee) = \begin{cases} 0 & \text{if } \lambda < \lambda_1 \\ K\langle (y_{11}, y_{21})\rangle & \text{if } \lambda_1 \leq \lambda < \lambda_2 \\ \mathrm{Sol}(M^\vee) & \text{if } \lambda \geq \lambda_2. \end{cases}$$

for $\lambda_1 < \lambda_2$ and

$$\mathrm{Sol}_\lambda(M^\vee) = \begin{cases} 0 & \text{if } \lambda < \lambda_1 \\ \mathrm{Sol}(M^\vee) & \text{if } \lambda \geq \lambda_1. \end{cases}$$

for $\lambda_1 = \lambda_2$ using the natural pairing (Proposition 1.2). Hence the log-growth filtration $\{(M^\vee)^\lambda\}$ of $M^\vee$ at the special point coincides with $\{(S_\lambda(V(M)))^\perp\}$.

This completes the proof of Theorem 7.2 in the case of the special point. □

7.5 The Gaussian hypergeometric case

Let us apply our theorem 7.2 to the log-growth of the solutions of the Gaussian hypergeometric case. Recall that in [14, 8] it was only noticed that at the supersingular point such solutions are unbounded without any measure of the size of unboundness.

Suppose that the residue field k of K is algebraically closed with characteristic $p > 2$. Let $f : Y \to X$ be the Legendre's family of elliptic curves over $\mathrm{Spec}\, k$, that is, Y is defined by the homogeneous equation $zy^2 = x(x-z)(x-uz)$ in the projective space $\mathbb{P}^2_X$ over $X = \mathrm{Spec}\, k[u, \frac{1}{u(u-1)}]$. Let $\overline{a} \in k, \overline{a} \neq 0, 1$, let us take a lift a of $\overline{a}$ in $\mathcal{V}$ and denote by $Y_{\overline{a}}$ the fiber $f^{-1}(\overline{a})$ of the Legendre's family.

The first relative rigid cohomology

$$\mathcal{M} = \underline{H}^1_{\mathrm{rig}}(Y/X)$$

is an overconvergent F-isocrystal on X/K of rank 2 since the Legendre's family has a lift over $\mathrm{Spec}\, \mathcal{V}$ [3, Théorème 5]. By the Poincare duality we have $\mathcal{M}^\vee \cong \mathcal{M}(1)$, where (1) means the Tate twist (the Frobebnius is changed by $q\varphi$ and the connection is unchanged). We denote by M (resp. M_a) the associated solvable φ-∇-module over E (resp. $K[\![x-a]\!]_0$) (Remarks 3.10). Then M is isomorphic to the first rigid cohomology of the generic fiber of the Legendre's family as φ-modules and $V(M_a)$ is isomorphic to the first rigid cohomology of $Y_{\overline{a}}/K$ by the base change theorem [30, Theorem 4.1.1].

By the knowledge of the Frobenius slope filtration for such an F-isocrystal, we conclude:

(1) Suppose that $Y_{\overline{a}}$ is ordinary. Then the pairs of the breaks of Frobenius slope filtration and their multiplicities of M_a at the special point are $(0,1)$ and $(1,1)$. Hence, the pairs of the breaks of log-growth filtration and their multiplicities of M_a at the special point are $(0,1)$ and $(1,1)$.

(2) Suppose that $Y_{\overline{a}}$ is supersingular. Then the pair of the break of Frobenius slope filtration and its multiplicity of M_a at the special point is $(\frac{1}{2},2)$. Hence, the pair of the break of log-growth filtration and its multiplicity of M_a at the special point is $(\frac{1}{2},2)$.

(3) Since M is isomophic to the first rigid cohomology of the generic fiber of the Legendre's family as φ-modules, the breaks of Frobenius slope filtration and their multiplicities of M (resp. at the generic point of M_a, i.e., of $M_a \otimes \mathcal{E}_a \cong M \otimes \mathcal{E}_a$ (see Remark 3.10)) are $(0,1)$ and $(1,1)$ by the specialization thereom 6.12 (an ordinary $\overline{a}$ always exists!). Hence, the pairs of the breaks of log-growth filtration and their multiplicities of M (resp. of $M_a \otimes \mathcal{E}_a$) are $(0,1)$ and $(1,1)$.

7.6 An example of higher rank case

In the unipotent case we can generalize our main result to the cases of arbitrary rank.

Since the ∇-module M over E in Example 5.1 comes from a convergent F-isocrystal on $\mathbb{P}^1_k \setminus \{0,\infty\}$, M has a Frobenius structure which is represented by the matrix [5, section 5], [25, 4.3] (for the general Frobenius in [29, 4.2.3]):

$$\begin{pmatrix} u_0 & u_1 & u_2 & \cdots & u_{r-1} \\ & qu_0 & qu_1 & \cdots & qu_{r-2} \\ & & \ddots & \ddots & \vdots \\ & & & q^{r-2}u_0 & q^{r-2}u_1 \\ & & & & q^{r-1}u_0 \end{pmatrix}$$

for $u_0, \cdots, u_{r-1} \in K, u_0 \neq 0$, where our Frobenius σ on $\mathcal{E}$ is given by $\sigma(x) = x^q$. Let λ_1 be the first slope, i.e., $|u_0| = q^{-\lambda_1}$. Since the dual of M is isomorphic to M as ∇-modules, we have

$$(S_\lambda M)^\perp = (M^\vee)^{\lambda-\lambda_1}$$

for any λ.

We also have the coincidence of the special log-growth filtration and the Frobenius slope filtration at $x = \overline{a} \in k\,(\overline{a} \neq 0)$ for the unipotent cases. □

References

[1] Ax, J. "Zeros of polynomials over local fields, the Galois action", J. Algebra **15** (1970), 412-428.

[2] Baldassarri, F., Chiarellotto, B. "Algebraic versus Rigid Cohomology with Lagarithmic Coefficients", 11-50, *"Barsotti Symposium in Algebraic Geometry"* (Messing W., Cristante V. Eds), Perspectives in Math.vol. **15**, Acad. Press, 1994.

[3] Berthelot, P. " Géomètrie rigide et cohomologie des variétès algébriques de caractéristique p", Bull. Soc. Math. France, Mémoire **23** (1986), 7-32.

[4] Chiarellotto, B. "Espaces de Berkovich et équatiions différentielles p-adiques: une note". Rend Sem Mat. Padova **103** (2000), 193-209.

[5] Chiarellotto, B., Le Stum, B. "F-isocristaux unipotents" Compositio Math. **116** (1999), 81-110.

[6] Christol, G. "*Modules différentiels et équations différentielles p-adiques*" . Queen's papers in Pure and Applied Math. , **66**, Queen's Univ., Kingston ON, 1983.

[7] Christol, G. Mebkhout, Z. " Sur le théorème de l'indice des équations différentielles p-adiques III" Annals of Math. **151** (2000) 385-457.

[8] Crew, R. "Finiteness theorems for the cohomology of an overconvergent isocrystal on a curve", Ann. Scient. E.N.S. **31** (1998), 717-764.

[9] Crew, R. "Arithmetic $\mathcal{D}$-modules on the unit disk", preprint.

[10] de Jong, A.J. "Homomorphisms of Barsotti-Tate groups and crystals in positive characteristics" Invent. Math. **134** (1998), 301-333. Erratum **138** (1999) 225.

[11] Deligne, P. "*Equations différentielles à points singuliers réguliers*", LNM **163**, Springer, 1970.

[12] Demazure, M. "*Lectures on p-divisible groups*", LNM **302**, Springer, 1972.

[13] Dwork, B.M. "On the Zeta function of a hypersurface II". Annals of Math. **80** (1964), 227-299.

[14] Dwork, B.M. "p-adic cycles ", Publ.I.H.E.S. **37** (1969) 27-116.

[15] Dwork, B.M. "On p-adic differential equations III " , Invent. Math. **20** (1973), 35-45.

[16] Dwork, B.M. "On p-adic differential equations II " , Annals of Math. **98** (1973), 366-376.

[17] Dwork, B.M. "*Lectures on p-adic differential equations*". GL **253** (1982), Springer Verlag.

[18] Dwork, B.M., Robba Ph. "On ordinary p-adic linear differential equations", T.A.M.S. **231** (1977), 1-46.

[19] Fontaine, J.-M. "Représentation p-adiques des corps locaux I", *The Grothendieck Festschrift, vol.II*, 249-309, Progr. Math. **87**, Birkhauser, Boston MA, 1990.

[20] Grothendieck, A. "*Groupes de Barsotti-Tate et cristaux de Dieudonné*", Université de Montréal, Montréal, 1974.

[21] Katz, N. "Slope filtration of F-crystals" , Asterisque **63** (1979), 113-164.

[22] Kedlaya, K. "A p-adic monodromy theorem", Annals of Math. **160** (2004), 93-184

[23] Kedlaya, K. "Slope filtrations revisited", Doc. Math. **10** (2005), 447-525.

[24] Lang, S. "*Algebra*", Revised third edition, GTM 211, Springer, 2002.

[25] Matsuda, S. "Katz's correspondance for quasi-unipotent overconvergent isocrystals" Compositio Math. **134** (2002), 1-34.

[26] Robba, Ph., Christol, G. "*Équation Différentielles p-adiques. Application aux sommes exponentielles*" Actualité Math. Hermann Paris, 1994.

[27] Robba, Ph. "On the Index of p-adic differential operators I" . Ann. of Math. **101** (1975), 280-316.

[28] Robba, Ph. "Sur les équations différentielles lineaires p-adiques (II)", Pacific J. of Math. **98** (1982), 393-418.

[29] Tsuzuki, N. "Slope filtration of quasi-unipotent overconvergent F-isocrystals" Annales de l'institut Fourier, Grenoble **48** (1998), 379-412.

[30] Tsuzuki, N. "On base change theorem and coherence in rigid cohomology", Documenta Math., Extra Volume, Kazuya Kato's fiftieth birthday (2003), 891-918.

Bruno Chiarellotto

Dipartimento di Matematica
Università degli Studi di Padova
Via Belzoni 7
35100 Padova, Italy

Nobuo Tsuzuki

Department of Mathematics
Graduate School of Science
Hiroshima University
Higashi-Hiroshima 739-8526, Japan