Binary positivity in the language of locales

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About this talk

1987 G. Sambin & P. Martin-Löf

formal topology = predicative version of a locale
= "locale with base"

today G. Sambin

POSITIVE topology = formal topology +
a BINARY POSITIVITY PREDICATE

What does a binary positivity predicate correspond to in the languages of locales?

a formal topology IS TO a locale
AS
a POSITIVE topology IS TO a locale + ...?!?

Answer:
a suitable system of overt, weakly closed sublocales

Locales with bases

A base for a locale $\mathcal{L}$ is a subset $S \subseteq \mathcal{L}$ s.t. $x = \bigvee\{a \in S \mid a \leq x\}$ (for every x in $\mathcal{L}$)

Put $a \lhd U \stackrel{def}{\iff} a \leq \bigvee U$ (for $a \in S$ and $U \subseteq S$)

so that $\{a \in S \mid a \lhd U\} \cong \bigvee U$

(*formal open subset*) (any element of $\mathcal{L}$)

Formal topology corresponding to $\mathcal{L}$ $\rightsquigarrow$ $(S, \lhd)$

Formal topology $=$ axiomatization of $(S, \lhd)$

$$\begin{aligned} \text{Overt locale} &= \text{formal topology} + \text{unary positivity predicate} \\ &= (S, \lhd, Pos) \\ &\quad \llcorner\ Pos(a) \iff \text{“}\exists_{\mathcal{L}}(a) = 1\text{”} \end{aligned}$$

Positive topologies $(\overbrace{\textit{formal topologies}}^{\textit{locales}} + \text{binary positivity})$

$$(\overbrace{S, \lhd}^{\textit{locale}}, \ltimes)$$

$\ltimes$ ↰ *binary positivity*

$$\ltimes \subseteq S \times Pow(S)$$

$$\frac{a \ltimes U}{a \in U} + \frac{a \ltimes U \quad U \subseteq V}{a \ltimes V} + \frac{a \ltimes U}{a \ltimes \{b \in S \mid b \ltimes U\}} +$$

$$\frac{\overbrace{a \lhd U}^{a \leq \bigvee U} \quad a \ltimes V}{(\exists u \in U)(u \ltimes V)} \ (\textit{compatibility})$$

$\boxed{\{a \in S \mid a \ltimes U\}}$ $\leftarrow$ "*formal closed*" *subset*

$$\textit{FormalClosed}(\ltimes) \stackrel{def}{=} \{\textit{formal closed subsets w.r.t.} \ltimes\}$$

Two notions of closure

Intuitionistically, TWO different ways of defining *closure* for a subspace Y of a topological space:

$$-int(-Y) \quad = \quad \underline{\text{the complement of the interior of the complement of } X}$$

$$cl(Y) \quad = \quad \underline{\text{the set of adherent points of } X}$$

$cl(Y) \subseteq -int(-Y)$ and so $Y = -int(-Y) \implies Y = cl(Y)$

but NOT the other way round (counterexample: discrete topology).

Example: positivities on a topological space

(X, τ) topological space
$S \subseteq \tau$ base

For x a point, $\Diamond x \overset{def}{=} \{a \in S \mid x \in a\}$ = basic open neighbourhoods of x.

$$\boxed{a \ltimes_X U \overset{def}{\iff} (\exists x \in a)(\Diamond x \subseteq U) \iff (\exists x \in X)(a \in \Diamond x \subseteq U)}$$

$$\begin{array}{rcl} FormalClosed(\ltimes_X) & \cong & \{D \subseteq X \mid D = cl(D)\} \\ U & \longmapsto & \{x \in X \mid \Diamond x \subseteq U\} \\ \{a \in S \mid a \between D\} & \longleftarrow & D \end{array}$$

$\hookleftarrow$ *Sambin's notation for inhabited intersection*

More generally:

For every subset $Y \subseteq X$, the relation

$$a \ltimes_Y U \overset{def}{\iff} (\exists\, y \in Y)(a \in \Diamond y \subseteq U)$$

is a *positivity* and

$$FormalClosed(\ltimes_Y) \cong \{closed\ sets\ in\ the\ subspace\ topology\ on\ Y\}.$$

On the lattice of positivities

$\mathcal{L}$ locale
S base

$Posty(\mathcal{L}) \overset{def}{=}$ all positivities on $\mathcal{L}$ (w.r.t. S).

- $Posty(\mathcal{L})$ is ordered by INCLUSION:

$$\ltimes_1 \leq \ltimes_2 \overset{def}{\iff} (\forall a \in S, \forall U \subseteq S)(a \ltimes_1 U \Rightarrow a \ltimes_2 U)$$

- $Posty(\mathcal{L})$ is a SUPLATTICE with: $a\big(\bigvee_i \ltimes_i\big)U \iff \exists i(a \ltimes_i U)$

So $Posty(\mathcal{L})$ has:

MINIMUM $\quad a \ltimes_{min} U \iff \textit{falsum}$

MAXIMUM $\quad a \ltimes_{max} U \iff a \ltimes U$ for some positivity $\ltimes$

↳ (a more explicit characterization below)

Splitting subsets & suplattice morphisms

(completely-prime up-sets & sup-preserving maps)

$Z \subseteq \mathcal{L}$ is <u>*splitting*</u> if $\boxed{\dfrac{x \leq \bigvee Y \quad x \in Z}{Y \between Z}}$, that is, $(\bigvee Y) \in Z \iff Y \between Z$

for every $\{x\} \cup Y \subseteq \mathcal{L}$. Let us put $\boxed{Split(\mathcal{L}) \stackrel{def}{=} \{splitting\ subsets\ of\ \mathcal{L}\}}$.

Facts:

1. $\big(Split(\mathcal{L}), \bigcup\big)$ is a suplattice;
2. there exists an isomorphism of suplattices

$$\begin{array}{rcl} Split(\mathcal{L}) & \cong & \mathbf{SupLat}(\mathcal{L}, \Omega) \\ Z & \mapsto & [x \mapsto \{* \mid x \in Z\}] \\ \varphi^{-1}(\{*\}) & \leftarrow\!\shortmid & \varphi \end{array}$$

where $\Omega = Pow(\{*\})$ is the frame of truth values.

Examples: points, *Pos*.

$\mathit{FormalClosed}(\ltimes) \hookrightarrow \mathit{Split}(\mathcal{L}) \cong \mathbf{SupLat}(\mathcal{L}, \Omega)$

Thanks to *compatibility*...

$$\begin{array}{rcl} \mathit{FormalClosed}(\ltimes) & \hookrightarrow & \mathit{Split}(\mathcal{L}) \\ U & \mapsto & \uparrow U \;= \{x \in \mathcal{L} \mid u \leq x \text{ for some } u \in U\} \\ \{a \in S \mid a \ltimes U\} & \mapsto & \{x \in \mathcal{L} \mid (\exists a \in S)(a \leq x \text{ and } a \ltimes U)\} \end{array}$$

$\Lsh$ is a <u>sub-suplattice</u> of $\mathit{Split}(\mathcal{L})$ (w.r.t. union)

$$\begin{array}{rcl} \mathit{FormalClosed}(\ltimes) & \hookrightarrow & \mathbf{SupLat}(\mathcal{L}, \Omega) \\ \{a \in S \mid a \ltimes U\} & \mapsto & [x \mapsto \{* \mid (\exists a \in S)(a \leq x \text{ and } a \ltimes U)\}] \end{array}$$

$\Lsh$ is a <u>sub-suplattice</u> of $\mathbf{SupLat}(\mathcal{L}, \Omega)$

Positivities AS sub-suplattices of $\mathit{Split}(\mathcal{L})$

A base-independent description of positivities

$\mathcal{L}$ locale

For each base S of $\mathcal{L}$, the following defines a BIJECTION between

$$\begin{array}{ccc} \mathit{Posty}(\mathcal{L}) \text{ w.r.t. } S & \text{and} & \{\text{sub-suplattices of } \mathit{Split}(\mathcal{L})\} \\ \ltimes & \longmapsto & \{\uparrow U \mid U \in \mathit{FormalClosed}(\ltimes)\} \\ \boxed{a \ltimes_F U \stackrel{def}{\Longleftrightarrow} (\exists Z \in F)(a \in Z \cap S \subseteq U)} & \longleftarrow & F \end{array}$$

Note: $\mathit{FormalClosed}(\ltimes_F) = \{Z \cap S \mid Z \in F\} \cong F$

Corollaries

- For S_1, S_2 bases of $\mathcal{L}$: $\quad \mathit{Posty}(\mathcal{L}) \text{ w.r.t. } S_1 \cong \mathit{Posty}(\mathcal{L}) \text{ w.r.t. } S_2$.
- For every $\ltimes$ one has: $\boxed{\ltimes = \ltimes_{\mathit{FormalClosed}(\ltimes)}}$, that is,

$$a \ltimes U \iff a \in Z \subseteq U \quad \text{for some } Z \in \mathit{FormalClosed}(\ltimes)$$

The greatest positivity $\ltimes_{max}$ in terms of splitting subsets

In the previous isomorphism

$$Posty(\mathcal{L}) \qquad \cong \qquad \{\text{sub-suplattices of } Split(\mathcal{L})\}$$

$$\ltimes_{max} \quad \textit{corresponds to} \quad Split(\mathcal{L})$$

so $\boxed{a \ltimes_{max} U \quad \Longleftrightarrow \quad \big(\exists Z \in Split(\mathcal{L})\big)\big(a \in Z \cap S \subseteq U\big)}$

and $\boxed{FormalClosed(\ltimes_{max}) \quad = \quad \{Z \cap S \mid Z \in Split(\mathcal{L})\} \quad \cong \quad Split(\mathcal{L})}$

Constructively...

If $\mathcal{L} = (S, \lhd)$ is **inductively generated**, then $\ltimes_{max}$ is generated by **coinduction**.

(Martin-Löf & Sambin - *Generating Positivity by Coinduction*)

Formal closed subsets AS . . .

Bunge & Funk, *Constructive Theory of the Lower Power Locale*, **MSCS** 6 (1996)

Points of the lower powerlocale $\cong$

$\cong$ overt, weakly closed sublocales of $\mathcal{L}$

$\cong$ suplattice morphisms $\mathcal{L} \to \Omega = Pow(\{*\})$

$\cong$ splitting subsets of $\mathcal{L}$

$$\cong \bigcup_{\ltimes \in Posty(\mathcal{L})} FormalClosed(\ltimes)$$

$$= FormalClosed(\ltimes_{max})$$

See also Spitters, *Locatedness and overt sublocales*, **APAL** 162 (2010) and
Vickers, *Constructive points of powerlocales*, **Math. Proc. Cambridge Philos. Soc.** 122 (1997)

Weakly closed sublocales

See Vickers, *Sublocales in Formal Topology*, **JSL** 72 (2007)
and Johnstone's *Elephant* for the more general notion of *fibrewise closed*.

A weakly closed sublocale of $(S, \lhd)$ is one generated as follows

1. for each $a \in S$ fix a (possibly empty) set $I(a)$ of PROPOSITIONS;
2. for $a \in S$ and $P \in I(a)$ impose the EXTRA condition $\boxed{a \lhd \{x \in S \mid P\}}$.

CLASSICALLY: this is just a closed sublocale.

Warning

In the spatial case, $cl(Y)$ need not be weakly closed!

- Weakly closed sublocales are closed under binary joins. (Johnstone)
- If $\{Y \subseteq X \mid cl(Y) = Y\}$ is closed under binary unions, then LLPO. (Bridges)

Overt, weakly closed sublocales

$(S, \lhd') \hookrightarrow (S, \lhd)$ is weakly closed AND overt

IFF

there exists $P \in \mathbf{SupLat}(\mathcal{L}, \Omega) \cong \mathit{Split}(\mathcal{L})$ s.t.

$\lhd'$ can be generated by imposing the EXTRA axioms

$a \lhd' \{x \in S \mid P(a) = 1\}$, or equivalently $a \lhd' \{a\} \cap P$, for all $a \in S$.

P is the unary positivity predicate of $(S, \lhd')$.

Fact:

point of $(S, \lhd')$ $=$ point of $(S, \lhd)$ s.t. all its basic neighbourhoods are in P

Formal closed subsets AS overt weakly-closed sublocales

Given $(S, \lhd, \ltimes)$ and $U = \{x \in S \mid x \ltimes U\}$	positive topology (formal closed subset)
$\uparrow U$	splitting subset
$x \mapsto \{* \mid x \in (\uparrow U)\}$	suplattice morphism $\mathcal{L} \to \Omega$

overt, weakly closed sublocale

generated by imposing $a \lhd \{a\} \cap U$

with UNARY positivity given by $(\_) \ltimes U$.

This is the smallest sublocale for which $(\_) \ltimes U$ is a unary positivity predicate.
Its points are the points of $(S, \lhd)$ which are "contained" in U.

Positivities AS ...

A formal closed subset of $\mathcal{L} = (S, \lhd)$ is...

... $\{a \in S \mid a \ltimes U\}$ for some $U \subseteq S$ and some $\ltimes \in Posty(\mathcal{L})$

... $\{a \in S \mid a \ltimes_{max} U\}$ for some $U \subseteq S$

... $S \cap Z$ for some $Z \in Split(\mathcal{L})$

... $S \cap \varphi^{-1}(1)$ for some $\varphi \in \mathbf{SupLat}(\mathcal{L}, \Omega)$

... $\{a \in S \mid Pos(a)\}$ for some $\underbrace{(S, \lhd', Pos) \hookrightarrow (S, \lhd)}_{overt\ \ and\ \ weakly\ \ closed}$

A positivity on a locale $\mathcal{L}$ is...

$$\ldots \text{a suplattice of} \begin{cases} \textit{splitting subsets of } \mathcal{L} \\ \textit{suplattice morphisms from } \mathcal{L} \textit{ to } \Omega \\ \textit{overt, weakly closed sublocales of } \mathcal{L} \end{cases}$$

and $\boxed{\ltimes_{max} \cong Split(\mathcal{L}) \cong \mathbf{SupLat}(\mathcal{L}, \Omega) \cong \{\text{overt, weakly closed sublocales of } \mathcal{L}\}}$

Morphisms between positive topologies

A morphism $f : (\overbrace{S_1, \lhd_1}^{\mathcal{L}_1}, \ltimes_1) \longrightarrow (\overbrace{S_2, \lhd_2}^{\mathcal{L}_2}, \ltimes_2)$ is

- a morphism $f : \mathcal{L}_1 \longrightarrow \mathcal{L}_2$ of **locales**
  (with Ωf the corresponding morphism of frames in the opposite direction)
- such that:

$$\left.\begin{array}{r} a \leq \Omega f(b) \\ a \ltimes_1 U \end{array}\right\} \quad \Longrightarrow \quad b \ltimes_2 \{y \in S_2 \mid \underbrace{\exists u \in U . u \leq \Omega f(y)}_{\Omega f(y) \in \uparrow U}\}$$

for all $a \in U \subseteq S_1$ and $b \in S_2$.

?!?

Positive topology $= (\mathcal{L}, \Phi)$, with $\boxed{\mathcal{L} \text{ a locale}}$ and $\boxed{\Phi \hookrightarrow \mathbf{SupLat}(\mathcal{L}, \Omega)}$

Given $(\mathcal{L}_1, \Phi_1)$, $(\mathcal{L}_2, \Phi_2)$ and $f : \mathcal{L}_1 \longrightarrow \mathcal{L}_2$ in **Loc**

TFAE

1. f is a morphism of positive topologies;
2. the map $U \mapsto S_2 \cap (\Omega f)^{-1}(\uparrow U)$
   maps formal closed of $(\mathcal{L}_2, \Phi_2)$ to formal closed of $(\mathcal{L}_1, \Phi_1)$;
3. $(\Omega f)^{-1}$ maps elements of $Split(\mathcal{L}_1)$ corresponding to Φ_1
   to elements of $Split(\mathcal{L}_2)$ corresponding to Φ_2;
4. $(\forall \varphi \in \Phi_1)(\varphi \circ \Omega f \in \Phi_2)$, that is, $\Phi_1 \circ \Omega f \subseteq \Phi_2$.

$$\frac{[\Omega \xleftarrow{\varphi} \mathcal{L}_1] \in \Phi_1}{[\Omega \xleftarrow{\varphi} \mathcal{L}_1 \xleftarrow{\Omega f} \mathcal{L}_2] \in \Phi_2}$$

The category **PTop** of Positive Topologies

Objects	$(\mathcal{L}, F)$	$(\mathcal{L}, \Phi)$
	$\mathcal{L}$ locale $\quad +$	
binary positivity	$F \hookrightarrow \mathit{Split}(\mathcal{L})$	$\Phi \hookrightarrow \mathbf{SupLat}(\mathcal{L}, \Omega)$

Morphisms	$f : \mathcal{L}_1 \to \mathcal{L}_2$ of locales s.t.	
	$(\Omega f)^{-1}[F_1] \subseteq F_2$	$\Phi_1 \circ \Omega f \subseteq \Phi_2$

Embedding **Loc** into **PTop**

$$\mathbf{PTop}\big((\mathcal{L}_1, \Phi_1), (\mathcal{L}_2, \Phi_2)\big) = \{f \in \mathbf{Loc}(\mathcal{L}_1, \mathcal{L}_2) \mid \Phi_1 \circ \Omega f \subseteq \Phi_2\}$$

$\boxed{\mathbf{Loc}(\mathcal{L}, \mathcal{L}') \quad \cong \quad \mathbf{PTop}\big((\mathcal{L}, \Phi), (\mathcal{L}', \Phi_{max})\big)}$ for every $\Phi \hookrightarrow \mathbf{SupLat}(\mathcal{L}, \Omega)$

$\leftharpoonup \mathbf{SupLat}(\mathcal{L}', \Omega)$

By identifying $\boxed{\mathcal{L}}$ with $\boxed{\big(\mathcal{L}, \overbrace{\mathbf{SupLat}(\mathcal{L}, \Omega)}^{\Phi_{max}}\big)}$, we get: $\boxed{\mathbf{Loc} \hookrightarrow \mathbf{PTop}}$.

Fact: **Loc** is a reflective subcategory of **PTop**.

Points of a positive topologies

Def: a point of $(S, \lhd, \ltimes)$ is...

...a point of $(S, \lhd)$ which "belongs" to $\mathit{FormalClosed}(\ltimes)$

i.e. a point whose set of basic neighbourhoods belongs to $\mathit{FormalClosed}(\ltimes)$.

$$\boxed{\begin{array}{rcl} \mathit{Points}(\mathcal{L},\Phi) & \cong & \mathit{Points}(\mathcal{L}) \cap \Phi \\ & \cong & \mathbf{Frame}(\mathcal{L},\Omega) \cap \Phi \\ & \cong & \{\varphi \in \Phi \mid \varphi \text{ preserves finite meets}\} \\ & & \\ & \cong & \mathbf{PTop}\big(\mathbf{1}, (\mathcal{L},\Phi)\big) \end{array}}$$

where $\mathbf{1}$ = terminal object of **PTop** = terminal locale + Φ_{max}

Idea: a positivity is *a way for selecting points*.

Sambin & Trentinaglia, *On the meaning of positivity relations...*, **J.UCS** 11 (2005)

Note that: $\boxed{\mathit{Points}(\mathcal{L},\Phi_{max}) = \mathit{Points}\big(\mathcal{L},\mathbf{SupLat}(\mathcal{L},\Omega)\big) = \mathit{Points}(\mathcal{L})}$.

One formal closed subset, three positive topologies

Let $(S, \lhd, \ltimes)$ be a positive topology.

For any $H \in \mathit{FormalClosed}(\ltimes)$ one can construct:

1. an overt, weakly closed sublocale $(S, \lhd_H)$
   by adding the extra axiom schema $a \lhd \{a\} \cap H$
   (H acts as the unary positivity predicate)

2. another positivity $\ltimes_H$ defined by: $a \ltimes_H U \overset{def}{\iff} a \ltimes H \cap U$
   (j.w.w. G. Sambin and M. Maietti)

And one can show that:

$$
\begin{aligned}
Pt(S, \lhd_H) &= Pt(S, \lhd_H, \ltimes_H) = \{\alpha \in Pt(S, \lhd) \mid \alpha^{-1}(\top) \cap S \subseteq H\} \\
&= Pt\,(S, \lhd, \ltimes_H)
\end{aligned}
$$

↖ this is PREDICATIVE (even when $\lhd$ is not generated)

Two adjunctions between **Top** and **PTop**

1. Extending the usual adjunction between **Top** and **Loc**:

$$\begin{array}{ccccc} \textbf{Top} & & \textbf{Loc} & & \textbf{PTop} \\ X & \mapsto & \Omega X & \mapsto & (\Omega X, \boxed{\ltimes_{max}}) \\ \boxed{Pt(\mathcal{L})} & \leftarrow\!\shortmid & \mathcal{L} & \leftarrow\!\shortmid & (\mathcal{L}, \Phi) \end{array}$$

2. A new adjunction:

$$\begin{array}{ccc} \textbf{Top} & & \textbf{PTop} \\ X & \mapsto & (\Omega X, \boxed{\ltimes_X}) \\ \boxed{Pt(\mathcal{L}, \Phi)} & \leftarrow\!\shortmid & (\mathcal{L}, \Phi) \end{array}$$

Recall that $a \ltimes_X U \iff (\exists x \in X)(a \in \Diamond x \subseteq U)$ where $\Diamond x$ = basic neighbourhoods of x.

CLASSICALLY: $\ltimes_X = \ltimes_{max}$ and so $Pt(\Omega X, \ltimes_X) = Pt(\Omega X, \ltimes_{max}) \cong Pt(\Omega X)$.

Two notions of sobriety

$$\text{points} = Pt(\Omega X) = Pt(\Omega X, \ltimes_{max})$$
$$\text{“strong” points} = Pt(\Omega X, \ltimes_X)$$

$\alpha \in Pt(\Omega X)$ is “strong” if for all $a \in \alpha$, there exists $x \in X$ s.t. $a \in \Diamond x \subseteq \alpha$.

$$\text{sober } X = Pt(\Omega X, \ltimes_{max})$$
$$\text{weak sober } X = Pt(\Omega X, \ltimes_X)$$

If X is $T2$, then X is weakly sober.

On the contrary, if “ $T2 \Rightarrow$ *sober* ” were true, then LPO would hold.

Fourman & Scott, *Sheaves and Logic*, in *Applications of sheaves*, **LNM** 753 (1979)

Aczel & Fox, *Separation properties in constructive topology*, **OLG** 48 (2005)

Spatiality for positive topologies

A positive topology $(\mathcal{L}, \ltimes)$ is (bi-)spatial if

$$(\mathcal{L}, \ltimes) \quad = \quad \big(\Omega Pt(\mathcal{L}, \ltimes), \ltimes_{Pt(\mathcal{L},\ltimes)}\big)$$

that requires TWO things:

1. $\mathcal{L} = \Omega Pt(\mathcal{L}, \ltimes)$ (stronger than usual spatiality)
2. $\ltimes \;=\; \ltimes_{Pt(\mathcal{L},\ltimes)}$ ("*reducibility*")[1]

By unfolding definitions:

1. $x \leq y$ IFF $\big(\forall \alpha \in Pt(\mathcal{L}) \cap \Phi\big)\big(\alpha(x) \leq \alpha(y)\big)$
2. $a \ltimes U$ IFF $\big(\exists \alpha \in Pt(\mathcal{L}, \ltimes)\big)\big(a \in \alpha^{-1}(\top) \cap S \subseteq U\big)$

i.e. Φ coincides with its sub-suplattice spanned by $Pt(\mathcal{L}) \cap \Phi$.

[1] Rinaldi, Sambin and Schuster have a joint work in progress about reducibility in Ring Theory.

Positivity relations on suplattices

The notion of a *binary positivity predicate* makes sense also for the category **SupLat** of suplattices and sup-preserving maps.

$$\underline{\text{basic topology}} \quad = \quad \begin{array}{l} \text{suplattice } \mathcal{L} \, + \\ \text{positivity } \Phi \hookrightarrow \mathbf{SupLat}(\mathcal{L}, \Omega) \end{array}$$

See C. & Sambin, *A constructive Galois connection between closure and interior*, **JSL**, to appear and my talk in Kanazawa 2010.

As before, every suplattice $\mathcal{L}$ can be identified with the basic topology $(\mathcal{L}, \Phi_{max})$.

Positivity on topoi (?)

Future work (?)

By adopting the view of TOPOI as generalized spaces...

$$\begin{array}{rcl} \mathcal{L} \text{ locale} & \rightsquigarrow & \mathcal{E} \text{ topos} \\ \Omega & \rightsquigarrow & \mathit{Set} \\ \varphi \in \mathbf{SupLat}(\mathcal{L}, \Omega) & \rightsquigarrow & \text{functor from } \mathcal{E} \text{ to } \mathit{Set} \text{ that preserves colimits} \\ \ldots & \rightsquigarrow & \ldots \end{array}$$

Aim: to obtain a PREDICATIVE account of some Topos Theory (e.g. of *closed* subtopoi).

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Najlepša hvala! Thank you very much!