Binary positivity in the language of locales

Francesco Ciraulo

Department of Mathematics
University of Padua

4th Workshop on Formal Topology
June 15-20 2012, Ljubljana
About this talk

1987  G. Sambin & P. Martin-Löf

formal topology  =  predicative version of a locale
                 =  “locale with base”

today  G. Sambin

POSITIVE topology  =  formal topology +
                   a BINARY POSITIVITY PREDICATE

What does a binary positivity predicate correspond to in the languages of locales?

a formal topology IS TO a locale
AS
a POSITIVE topology IS TO a locale + . . .?!?

Answer:

a suitable system of overt, weakly closed sublocales
Locales with bases

A base for a locale $\mathcal{L}$ is a subset $S \subseteq \mathcal{L}$ s.t.
\[ x = \bigvee \{ a \in S \mid a \leq x \} \]
(for every $x$ in $\mathcal{L}$)

Put
\[ a \triangleleft U \overset{\text{def.}}{\iff} a \leq \bigvee U \]
(for $a \in S$ and $U \subseteq S$)

so that
\[ \{ a \in S \mid a \triangleleft U \} \cong \bigvee U \]
(formal open subset)
(any element of $\mathcal{L}$)

Formal topology corresponding to $\mathcal{L}$
\[ \sim \]

Formal topology = axiomatization of $(S, \triangleleft)$

Overt locale = formal topology + unary positivity predicate
= $(S, \triangleleft, \text{Pos})$
\[ \uparrow \text{Pos}(a) \iff \exists \mathcal{L}(a) = 1 \]
Positive topologies \(\text{formal topologies} + \text{binary positivity}\)

\(\text{locale} (S, \triangleleft, \times)\)

\(\triangleleft \text{ binary positivity}\)

\(\times \subseteq S \times \text{Pow}(S)\)

\[
\begin{align*}
    a \times U & \quad + \quad a \times U \quad U \subseteq V \\
    a \in U & \quad + \quad a \times V & \quad + \quad a \times \{ b \in S \mid b \times U \}
\end{align*}
\]

\(\exists u \in U)(u \times V)\) (compatibility)

\(\{a \in S \mid a \times U\}\) ← “formal closed” subset

\(\text{FormalClosed}(\times) \overset{\text{def}}{=} \{\text{formal closed subsets w.r.t. } \times\}\)
Two notions of closure

Intuitionistically, **TWO** different ways of defining closure for a subspace $Y$ of a topological space:

$$-\text{int}(-Y) = \text{the complement of the interior of the complement of } X$$

$$\text{cl}(Y) = \text{the set of adherent points of } X$$

$$\text{cl}(Y) \subseteq -\text{int}(-Y) \quad \text{and so} \quad Y = -\text{int}(-Y) \quad \Rightarrow \quad Y = \text{cl}(Y)$$

but **NOT** the other way round (counterexample: discrete topology).
Example: positivities on a topological space

$(X, \tau)$ topological space

$S \subseteq \tau$ base

For $x$ a point, $\Diamond x \overset{\text{def}}{=} \{ a \in S \mid x \in a \}$ = basic open neighbourhoods of $x$.

$$a \uplus X \ U \overset{\text{def}}{\iff} (\exists x \in a)(\Diamond x \subseteq U) \iff (\exists x \in X)(a \in \Diamond x \subseteq U)$$

$$\text{FormalClosed}(\uplus) \cong \{ D \subseteq X \mid D = cl(D) \}$$

$$U \mapsto \{ x \in X \mid \Diamond x \subseteq U \}$$

$$\{ a \in S \mid a \nmid D \} \mapsto D$$

↑ Sambin’s notation for inhabited intersection

More generally:

For every subset $Y \subseteq X$, the relation

$$a \uplus Y \ U \overset{\text{def}}{\iff} (\exists y \in Y)(a \in \Diamond y \subseteq U)$$

is a positivity and

$$\text{FormalClosed}(\uplus Y) \cong \{ \text{closed sets in the subspace topology on } Y \}.$$
On the lattice of positivities

\( \text{Posty}(\mathcal{L}) \overset{\text{def}}{=} \text{all positivities on } \mathcal{L} \text{ (w.r.t. } S). \)

- \( \text{Posty}(\mathcal{L}) \) is ordered by INCLUSION:

\[ \kappa_1 \leq \kappa_2 \overset{\text{def}}{=} (\forall a \in S, \forall U \subseteq S)(a \kappa_1 U \Rightarrow a \kappa_2 U) \]

- \( \text{Posty}(\mathcal{L}) \) is a SUPLATTICE with:

\[ a(\bigvee_i \kappa_i)U \iff \exists i(a \kappa_i U) \]

So \( \text{Posty}(\mathcal{L}) \) has:

**MINIMUM** \[ a \kappa_{\min} U \iff \text{falsum} \]

**MAXIMUM** \[ a \kappa_{\max} U \iff a \kappa U \text{ for some positivity } \kappa \]

\[ \uparrow \text{ (a more explicit characterization below)} \]
Splitting subsets & suplattice morphisms
(completely-prime up-sets & sup-preserving maps)

$Z \subseteq \mathcal{L}$ is splitting if

$$\frac{x \leq \bigvee Y \ x \in Z}{Y \not\sqsubseteq Z}$$

that is, $(\bigvee Y) \in Z \iff Y \not\sqsubseteq Z$

for every $\{x\} \cup Y \subseteq \mathcal{L}$. Let us put

$$Split(\mathcal{L}) \overset{\text{def}}{=} \{\text{splitting subsets of } \mathcal{L}\}.$$

Facts:

1. $(Split(\mathcal{L}), \bigcup)$ is a suplattice;
2. there exists an isomorphism of suplattices

$$Split(\mathcal{L}) \cong SupLat(\mathcal{L}, \Omega)$$

$$Z \mapsto [x \mapsto \{\ast \mid x \in Z\}]$$

$$\varphi^{-1}(\{\ast\}) \iff \varphi$$

where $\Omega = \text{Pow}(\{\ast\})$ is the frame of truth values.

Examples: points, $\text{Pos}$.
Thanks to *compatibility*.

\[ \text{FormalClosed}(\times) \leftrightarrow \text{Split}(\mathcal{L}) \cong \text{SupLat}(\mathcal{L}, \Omega) \]

\[ U \mapsto \uparrow U = \{ x \in \mathcal{L} \mid u \leq x \text{ for some } u \in U \} \]
\[ \left\{ a \in S \mid a \times U \right\} \mapsto \left\{ x \in \mathcal{L} \mid (\exists a \in S)(a \leq x \text{ and } a \times U) \right\} \]

\( \uparrow \) is a sub-suplattice of \( \text{Split}(\mathcal{L}) \) (w.r.t. union)

\[ \text{FormalClosed}(\times) \leftrightarrow \text{SupLat}(\mathcal{L}, \Omega) \]
\[ \left\{ a \in S \mid a \times U \right\} \mapsto \left[ x \mapsto \left\{ \ast \mid (\exists a \in S)(a \leq x \text{ and } a \times U) \right\} \right] \]

\( \uparrow \) is a sub-suplattice of \( \text{SupLat}(\mathcal{L}, \Omega) \)
Positivities AS sub-suplattices of $\text{Split}(\mathcal{L})$

A base-independent description of positivities

For each base $S$ of $\mathcal{L}$, the following defines a BIJECTION between

$$Posty(\mathcal{L}) \text{ w.r.t. } S \quad \text{and} \quad \{\text{sub-suplattices of } \text{Split}(\mathcal{L})\}$$

\[
\begin{align*}
a \times_F U & \iff (\exists Z \in F) (a \in Z \cap S \subseteq U) \\
\times & \quad \mapsto \quad \{\uparrow U \mid U \in \text{FormalClosed}(\times)\} \\
F & \quad \mapsto \quad \{Z \cap S \mid Z \in F\} \cong F
\end{align*}
\]

Note: $\text{FormalClosed}(\times_F) = \{Z \cap S \mid Z \in F\} \cong F$

### Corollaries

- For $S_1, S_2$ bases of $\mathcal{L}$: $\text{Posty}(\mathcal{L})$ w.r.t. $S_1 \cong \text{Posty}(\mathcal{L})$ w.r.t. $S_2$.

- For every $\times$ one has: $\times = \times_{\text{FormalClosed}(\times)}$, that is,

\[
a \times U \iff a \in Z \subseteq U \quad \text{for some } Z \in \text{FormalClosed}(\times)
\]
The greatest positivity $\succcurlyeq_{\text{max}}$ in terms of splitting subsets

In the previous isomorphism

$$Posty(\mathcal{L}) \cong \{\text{sub-suplattices of } \text{Split}(\mathcal{L})\}$$

$\succcurlyeq_{\text{max}}$ corresponds to $\text{Split}(\mathcal{L})$

so

$$a \succcurlyeq_{\text{max}} U \iff (\exists Z \in \text{Split}(\mathcal{L})) (a \in Z \cap S \subseteq U)$$

and

$$\text{FormalClosed}(\succcurlyeq_{\text{max}}) = \{Z \cap S \mid Z \in \text{Split}(\mathcal{L})\} \cong \text{Split}(\mathcal{L})$$

Constructively...

If $\mathcal{L} = (S, \triangleleft)$ is inductively generated, then $\succcurlyeq_{\text{max}}$ is generated by coinduction.

(Martin-Löf & Sambin - *Generating Positivity by Coinduction*)
Formal closed subsets AS . . .


Points of the lower powerlocale $\in\mathbb{I}$

$\in\mathbb{I}$ overt, weakly closed sublocales of $\mathcal{L}$

$\in\mathbb{I}$ suplattice morphisms $\mathcal{L} \to \Omega = \text{Pow}(\{\ast\})$

$\in\mathbb{I}$ splitting subsets of $\mathcal{L}$

$\in\mathbb{I}$ $\bigcup_{\otimes \in \text{Posty}(\mathcal{L})} \text{FormalClosed}(\otimes)$

$= \text{FormalClosed}(\otimes_{\text{max}})$

See also Spitters, *Locatedness and overt sublocales*, APAL 162 (2010) and

Weakly closed sublocales

See Vickers, *Sublocales in Formal Topology*, JSL 72 (2007) and Johnstone's *Elephant* for the more general notion of *fibrewise closed*.

A weakly closed sublocale of \((S, \triangleleft)\) is one generated as follows

1. for each \(a \in S\) fix a (possibly empty) set \(I(a)\) of PROPOSITIONS;
2. for \(a \in S\) and \(P \in I(a)\) impose the EXTRA condition \(a \triangleleft \{x \in S \mid P\}\).

CLASSICALLY: this is just a closed sublocale.

Warning

In the spatial case, \(cl(Y)\) need not be weakly closed!

- Weakly closed sublocales are closed under binary joins. (Johnstone)
- If \(\{Y \subseteq X \mid cl(Y) = Y\}\) is closed under binary unions, then LLPO. (Bridges)
Overt, weakly closed sublocales

\((S, \lhd') \leftrightarrow (S, \lhd)\) is weakly closed AND overt

IFF

there exists \(P \in \text{SupLat}(\mathcal{L}, \Omega) \cong \text{Split}(\mathcal{L})\) s.t.

\(\lhd'\) can be generated by imposing the EXTRA axioms

\[ a \lhd' \{ x \in S \mid P(a) = 1 \} \], or equivalently \[ a \lhd' \{ a \} \cap P \], for all \(a \in S\).

\(P\) is the **unary positivity predicate** of \((S, \lhd')\).

**Fact:**

point of \((S, \lhd')\) = point of \((S, \lhd)\) s.t. all its basic neighbourhoods are in \(P\)
Formal closed subsets AS overt weakly-closed sublocales

Given \((S, \triangleleft, \triangleright)\) and \(U = \{x \in S \mid x \triangleright U\}\) (formal closed subset)

\[\uparrow U\] splitting subset

\[x \mapsto \{\ast \mid x \in (\uparrow U)\}\] suplattice morphism \(\mathcal{L} \to \Omega\)

**Overt, weakly closed sublocale**

generated by imposing \(a \triangleleft \{a\} \cap U\) with UNARY positivity given by \((\_ \) \triangleright U\).

This is the smallest sublocale for which \((\_ \) \triangleright U\) is a unary positivity predicate. Its points are the points of \((S, \triangleleft)\) which are “contained” in \(U\).
Positivities AS . . .

A formal closed subset of \( \mathcal{L} = (S, \prec) \) is . . .

\[
\{ a \in S \mid a \ltimes U \} \text{ for some } U \subseteq S \text{ and some } \ltimes \in \text{Posty}(\mathcal{L})
\]

\[
\{ a \in S \mid a \ltimes_{\text{max}} U \} \text{ for some } U \subseteq S
\]

\[
S \cap Z \text{ for some } Z \in \text{Split}(\mathcal{L})
\]

\[
S \cap \varphi^{-1}(1) \text{ for some } \varphi \in \text{SupLat}(\mathcal{L}, \Omega)
\]

\[
\{ a \in S \mid \text{Pos}(a) \} \text{ for some } (S, \prec', \text{Pos}) \hookrightarrow (S, \prec)
\]

\( \text{overt and weakly closed} \)

A positivity on a locale \( \mathcal{L} \) is . . .

\[
\text{a suplattice of } \begin{cases} 
\text{splitting subsets of } \mathcal{L} \\
\text{suplattice morphisms from } \mathcal{L} \text{ to } \Omega \\
\text{overt, weakly closed sublocales of } \mathcal{L}
\end{cases}
\]

and \( \ltimes_{\text{max}} \cong \text{Split}(\mathcal{L}) \cong \text{SupLat}(\mathcal{L}, \Omega) \cong \{ \text{overt, weakly closed sublocales of } \mathcal{L} \} \)
Morphisms between positive topologies

A morphism $f : (S_1, \sqcap_1, \times_1) \rightarrow (S_2, \sqcap_2, \times_2)$ is

- a morphism $f : \mathcal{L}_1 \rightarrow \mathcal{L}_2$ of locales
  (with $\Omega f$ the corresponding morphism of frames in the opposite direction)

- such that:

  $a \leq \Omega f(b) \quad \text{and} \quad a \times_1 U \quad \Rightarrow \quad b \times_2 \{ y \in S_2 \mid \exists u \in U. u \leq \Omega f(y) \} \quad \Omega f(y) \in \uparrow U$

for all $a \in U \subseteq S_1$ and $b \in S_2$. 
Positive topology \( = (\mathcal{L}, \Phi) \), with \( \mathcal{L} \) a locale and \( \Phi \hookrightarrow \text{SupLat}(\mathcal{L}, \Omega) \)

Given \( (\mathcal{L}_1, \Phi_1), (\mathcal{L}_2, \Phi_2) \) and \( f : \mathcal{L}_1 \longrightarrow \mathcal{L}_2 \) in \( \text{Loc} \)

TFAE

1. \( f \) is a morphism of positive topologies;

2. the map \( U \mapsto S_2 \cap (\Omega f)^{-1}(\uparrow U) \)
maps formal closed of \( (\mathcal{L}_2, \Phi_2) \) to formal closed of \( (\mathcal{L}_1, \Phi_1) \);

3. \( (\Omega f)^{-1} \) maps elements of \( \text{Split}(\mathcal{L}_1) \) corresponding to \( \Phi_1 \)
to elements of \( \text{Split}(\mathcal{L}_2) \) corresponding to \( \Phi_2 \);

4. \( (\forall \varphi \in \Phi_1)(\varphi \circ \Omega f \in \Phi_2) \), that is, \( \Phi_1 \circ \Omega f \subseteq \Phi_2 \).

\[
\begin{align*}
[\Omega \leftarrow \varphi \mathcal{L}_1] & \in \Phi_1 \\
[\Omega \leftarrow \varphi \mathcal{L}_1 \xleftarrow{\text{of}} \mathcal{L}_2] & \in \Phi_2
\end{align*}
\]
The category $\textbf{PTop}$ of Positive Topologies

<table>
<thead>
<tr>
<th>Objects</th>
<th>$(\mathcal{L}, F)$</th>
<th>$(\mathcal{L}, \Phi)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathcal{L}$ locale +</td>
<td></td>
<td></td>
</tr>
<tr>
<td>binary positivity</td>
<td>$F \hookrightarrow \text{Split}(\mathcal{L})$</td>
<td>$\Phi \hookrightarrow \text{SupLat}(\mathcal{L}, \Omega)$</td>
</tr>
</tbody>
</table>

**Morphisms**

$f : \mathcal{L}_1 \to \mathcal{L}_2$ of locales s.t.

$$(\Omega f)^{-1}[F_1] \subseteq F_2$$

$\Phi_1 \circ \Omega f \subseteq \Phi_2$$
Embedding \textbf{Loc} into \textbf{PTop}

\[
\text{PTop}((\mathcal{L}_1, \Phi_1), (\mathcal{L}_2, \Phi_2)) = \{ f \in \text{Loc}(\mathcal{L}_1, \mathcal{L}_2) \mid \Phi_1 \circ \Omega f \subseteq \Phi_2 \}
\]

\[
\text{Loc}(\mathcal{L}, \mathcal{L}') \cong \text{PTop}((\mathcal{L}, \Phi), (\mathcal{L}', \Phi_{\text{max}})) \quad \text{for every} \quad \Phi \hookrightarrow \text{SupLat}(\mathcal{L}, \Omega)
\]

By identifying \( \mathcal{L} \) with \( (\mathcal{L}, \text{SupLat}(\mathcal{L}, \Omega)) \), we get: \( \text{Loc} \hookrightarrow \text{PTop} \).

Fact: \textbf{Loc} is a \textbf{reflective} subcategory of \textbf{PTop}.
Points of a positive topologies

Def: a point of \((S, \sqsubset, \otimes)\) is... 
...a point of \((S, \sqsubset)\) which “belongs” to \(\text{FormalClosed}(\otimes)\)  
i.e. a point whose set of basic neighbourhoods belongs to \(\text{FormalClosed}(\otimes)\).

\[
\begin{align*}
\text{Points}(\mathcal{L}, \Phi) & \equiv \text{Points}(\mathcal{L}) \cap \Phi \\
\text{Frame}(\mathcal{L}, \Omega) & \cap \Phi \\
\{ \varphi \in \Phi \mid \varphi \text{ preserves finite meets} \} & \\
\text{PTop}(1, (\mathcal{L}, \Phi)) &
\end{align*}
\]

where 1 = terminal object of \(\text{PTop} = \text{terminal locale} + \Phi_{\text{max}}\)

Idea: a positivity is a way for selecting points.

Note that: \[\text{Points}(\mathcal{L}, \Phi_{\text{max}}) = \text{Points}(\mathcal{L}, \text{SupLat}(\mathcal{L}, \Omega)) = \text{Points}(\mathcal{L}).\]
One formal closed subset, three positive topologies

Let \((S, \triangleleft, \triangleright)\) be a positive topology.

For any \(H \in \text{FormalClosed}(\triangleright)\) one can construct:

1. an overt, weakly closed sublocale \((S, \triangleleft_H)\) by adding the extra axiom schema \(a \triangleleft \{a\} \cap H\) (\(H\) acts as the unary positivity predicate)

2. another positivity \(\trianglerighteq_H\) defined by:

\[
a \trianglerighteq_H U \iff a \trianglerighteq H \cap U
\]

(j.w.w. G. Sambin and M. Maietti)

And one can show that:

\[
Pt(S, \triangleleft_H) = Pt(S, \triangleleft_H, \trianglerighteq_H) = \{\alpha \in Pt(S, \triangleleft) | \alpha^{-1}(\top) \cap S \subseteq H\} = Pt(S, \triangleleft, \trianglerighteq_H)
\]

\(\uparrow\) this is PREDICATIVE (even when \(\triangleleft\) is not generated)
Two adjunctions between \textbf{Top} and \textbf{PTop}

1. Extending the usual adjunction between \textbf{Top} and \textbf{Loc}:

\[
\begin{array}{ccc}
\text{Top} & \xrightarrow{\sim} & \text{Loc} \\
X & \mapsto & \Omega X \\
Pt(L) & \leftarrow & L \\
\end{array} \qquad \begin{array}{ccc}
\text{PTop} & \xrightarrow{\sim} & (\Omega X, \vartriangleleft_{\text{max}}) \\
(\Omega X, \vartriangleleft_{\text{max}}) & \leftarrow & (L, \Phi) \\
\end{array}
\]

2. A new adjunction:

\[
\begin{array}{ccc}
\text{Top} & \xrightarrow{\sim} & \text{PTop} \\
X & \mapsto & (\Omega X, \vartriangleleft_{X}) \\
Pt(L, \Phi) & \leftarrow & (L, \Phi) \\
\end{array}
\]

Recall that \(a \vartriangleleft_X U \iff (\exists x \in X)(a \in \diamond x \subseteq U)\) where \(\diamond x = \) basic neighbourhoods of \(x\).

\textbf{CLASSICALLY} : \(\vartriangleleft_X = \vartriangleleft_{\text{max}}\) and so \(Pt(\Omega X, \vartriangleleft_X) = Pt(\Omega X, \vartriangleleft_{\text{max}}) \cong Pt(\Omega X)\).
Two notions of sobriety

points = \( Pt(\Omega X) = Pt(\Omega X, \Join_{\max}) \)

“strong” points = \( Pt(\Omega X, \Join) \)

\( \alpha \in Pt(\Omega X) \) is “strong” if for all \( a \in \alpha \), there exists \( x \in X \) s.t. \( a \in \Diamond x \subset \alpha \).

sober \( X = Pt(\Omega X, \Join_{\max}) \)
weak sober \( X = Pt(\Omega X, \Join) \)

If \( X \) is \( T2 \), then \( X \) is weakly sober.

On the contrary, if “\( T2 \Rightarrow \text{sober} \)” were true, then LPO would hold.

Fourman & Scott, *Sheaves and Logic*, in *Applications of sheaves*, LNM 753 (1979)
Spatiality for positive topologies

A positive topology \((\mathcal{L}, \ltimes)\) is \((\text{bi-})\text{spatial}\) if

\[
(\mathcal{L}, \ltimes) = (\Omega Pt(\mathcal{L}, \ltimes), \ltimes Pt(\mathcal{L}, \ltimes))
\]

that requires TWO things:

1. \(\mathcal{L} = \Omega Pt(\mathcal{L}, \ltimes)\) \hspace{1cm} (stronger than usual spatiality)
2. \(\ltimes = \ltimes Pt(\mathcal{L}, \ltimes)\) \hspace{1cm} ("reducibility")

By unfolding definitions:

1. \(x \leq y \ \text{IFF} \ (\forall \alpha \in Pt(\mathcal{L}) \cap \Phi) (\alpha(x) \leq \alpha(y))\)
2. \(a \ltimes U \ \text{IFF} \ (\exists \alpha \in Pt(\mathcal{L}, \ltimes)) (a \in \alpha^{-1}(\top) \cap S \subseteq U)\)

i.e. \(\Phi\) coincides with its sub-su-plattice spanned by \(Pt(\mathcal{L}) \cap \Phi\).

---

\(^1\)Rinaldi, Sambin and Schuster have a joint work in progress about reducibility in Ring Theory.
Positivity relations on suplattices

The notion of a binary positivity predicate makes sense also for the category $\textbf{SupLat}$ of suplattices and sup-preserving maps.

$$\text{basic topology} = \text{suplattice } \mathcal{L} + \text{ positivity } \Phi \hookrightarrow \textbf{SupLat}(\mathcal{L}, \Omega)$$


As before, every suplattice $\mathcal{L}$ can be identified with the basic topology $(\mathcal{L}, \Phi_{\text{max}})$. 
Positivity on topoi (?)
Future work (?)

By adopting the view of TOPOI as generalized spaces...

$\mathcal{L}$ locale $\rightsquigarrow$ $\mathcal{E}$ topos
$\Omega \rightsquigarrow$ Set
$\varphi \in \text{SupLat}(\mathcal{L}, \Omega) \rightsquigarrow$ functor from $\mathcal{E}$ to $\text{Set}$ that preserves colimits

... $\rightsquigarrow$ ...

**Aim:** to obtain a PREDICATIVE account of some Topos Theory
(e.g. of closed subtopoi).
References


Najlepša hvala!

Thank you very much!