The role of the overlap relation in constructive mathematics

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Overview

1 Overlap algebras & Locales
   - Overlap Algebras and Overt Locales
   - Atoms and Points
   - Regular opens and Dense sublocales
Overview

1. **Overlap algebras & Locales**
   - Overlap Algebras and Overt Locales
   - Atoms and Points
   - Regular opens and Dense sublocales

2. **Overlap relation & Dedekind-MacNeille completion**
   - The overlap relation for non-complete posets
   - A variation on the Dedekind-MacNeille completion
Part I

Overlap Algebras & Locales

In which the definition of overlap algebra is recalled as well as some well and less well known facts connecting overlap algebras, overt locales and regular open sets.
Problem:

fill in the blank with a suitable ALGEBRAIC STRUCTURE in such a way that THE DIAGRAM COMMUTES.
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\[ \begin{array}{c}
\text{INTUITIONISTIC} \\
\text{world} \\
\text{Powersets} \quad \xrightarrow{\text{algebraize}} \quad \text{Overlap Algebras} \\
\downarrow \\
\text{CLASSICAL} \\
\text{world} \\
\text{Powersets} \quad \xrightarrow{\text{algebraize}} \quad cBa’s
\end{array} \]
Overlap Algebras =

complete lattice + binary relation s.t.:

\[
\begin{align*}
\frac{x \equiv y}{y \equiv x} & \quad \text{symmetry} \\
\frac{x \equiv y \\ y \leq z}{x \equiv z} & \quad \text{monotonicity} \\
\frac{x \equiv y}{x \equiv (x \land y)} & \quad \text{refinement} \\
\frac{x \equiv (\bigvee_{i \in I} y_i)}{(\exists i \in I)(x \equiv y_i)} & \quad \text{splitting} \\
\end{align*}
\]

[z \equiv x] \\
\vdots \\
z \equiv y \\
\frac{x \leq y}{} \quad \text{“density”}

Francesco Ciraulo - Palermo (IT) - The overlap relation in constructive mathematics - (CMPC 2010)
Overlap Algebras as Overt Locales
A characterization

Idea: read $x \equiv y$ as $\text{Pos}(x \land y)$.  \(^1\)

Overlap algebras

\[ = \]

$[\text{Pos}(z \land x)]$  

$\vdots$  

\[ \text{Pos}(z \land y) \quad \text{density}^{2} \]

\[ x \leq y \]

\(^1\)Vice versa, $\text{Pos}(x) \equiv (x \equiv x)$.

\(^2\)Not to be confused with the notion of a dense sublocale.
Overlap Algebras as a solution to the starting problem:

1. O-algebras are an algebraic version of powersets:
   - \( P(S) \) is an o-algebra (the motivating example).
   - \( P(S) \) is \textit{atomic}.
     (see below)
   - Every atomic o-algebra is a powerset (Sambin).

2. Classically:
   - Overlap algebras = complete Boolean algebras (Vickers)
     (where: \( x \equiv y \iff x \land y \neq 0 \)).
Atoms and points (I)

What is an ATOM?

(1) A minimal **NON-ZERO** element.

(2) A minimal **“POSITIVE”** element.

(1) is too weak: one cannot even prove that a singleton is an atom in a powerset!

(2) works well (but one needs Pos or \(\geq\) to define “positive” for elements).
Atoms and points (II)

TFAE
(in any overt locale)

1. $a$ is a minimal positive element
   i.e.: $\text{Pos}(a)$ and $\text{Pos}(x) \land (x \leq a) \implies (x = a)$

2. $a \equiv x \iff a \leq x$
   (for any $x$),
   $\text{Pos}(a \land x)$
Atoms and points (III)

TFAE

(in any o-algebra)

1. \( a \) is an atom;

2. \( \{ x \mid a \wedge x \} \) is a completely prime filter
   
   \( \text{Pos}(a \wedge x) \)

   (in that case \( \{ x \mid a \wedge x \} = \{ x \mid a \leq x \} \)).

In other words: the mapping \( x \mapsto \text{Pos}(a \wedge x) \) is a frame homomorphism if and only if \( a \) is an atom.
Atoms and points (IV)

\{\textit{atoms of an o–algebra}\} \subseteq \{\textit{points of an o–algebra}\}

\{\textit{atomic o–algebras}\} \subseteq \{\textit{spatial o–algebras}\}

Open questions:

- What is a spatial o-algebra like?
- What is the relationship between o-algebras and discrete locales\(^3\)?

\(^3\)Discrete locale = overt + the diagonal map is open.
Regular elements (I)

Even if Sambin’s “density” does not hold for an overt locale, in general, nevertheless:

\[
\begin{align*}
[\text{Pos}(z \land x)] \\
\vdots \\
\text{Pos}(z \land y) \\
x \leq y \quad \text{density}
\end{align*}
\]

(in any overt locale)

there exists a unique nucleus \( r \) s.t.:

\[
\begin{align*}
[\text{Pos}(z \land x)] \\
\vdots \\
\text{Pos}(z \land y) \\
x \leq r(y)
\end{align*}
\]
Regular elements (II)

For $\mathcal{L}$ overt, let:

$$\mathcal{L}_r \overset{\text{def}}{=} \{ x \in \mathcal{L} \mid x = r(x) \}$$

1. $\mathcal{L}_r$ is a o-algebra
   (hence an overt locale (w.r.t. the same Pos of $\mathcal{L}$))

2. $\mathcal{L}_{\neg\neg} = \{ x \in \mathcal{L} \mid x = \neg\neg x \} \subseteq \mathcal{L}_r$
   (the least dense sublocale of $\mathcal{L}$) (classically $\mathcal{L}_{\neg\neg} = \mathcal{L}_r$)

3. $\mathcal{L}_r$ is the least positively dense sublocale of $\mathcal{L}$
   \[ \text{Pos}(r(x)) \Rightarrow \text{Pos}(x) \]
   (this notion has been studied by Bas Spitters)

4. $\mathcal{L}_r = \mathcal{L}$ iff $\mathcal{L}$ is an o-algebra
   (in particular, every o-algebra is of the kind $\mathcal{L}_r$)
Regular elements (III)

REGULAR open sets = the interior of its closure.

(1) complement of the interior of the complement.

Topological closure:

(2) set of adherent points.

Regular elements:

(1) $x = - - x$

(2) $x = r(x)$
Part II

Overlap Algebras & Dedekind-MacNeille completion

In which a new approach to the Dedekind-MacNeille completion is presented (which works for posets with overlap).
Idea: modify the definition of an overlap relation in such a way that it would make sense for arbitrary posets.

\[ x \cong y \frac{}{x \cong (x \wedge y)} \quad \text{refinement} \quad \Rightarrow \quad \exists z (x \cong z \land z \leq x \land z \leq y) \]

\[ x \cong (\bigvee_{i \in I} y_i) \frac{}{(\exists i \in I)(x \cong y_i)} \quad \text{splitting} \quad \Rightarrow \quad \bigvee_{i \in I} y_i \text{ exists} \quad x \cong (\bigvee_{i \in I} y_i) \]

\[ (\exists i \in I)(x \cong y_i) \]
N.B.: usually, the addition of an overlap relation greatly enrich the underlying structure.

For instance, any lattice with overlap is automatically distributive.

Moreover (classically):

\[
\begin{align*}
\text{bounded lattice} & \quad + \\
\text{pseudo-complement} & \quad + \\
\text{overlap} & \quad = \\
\text{Boolean algebra}
\end{align*}
\]
Example: Heyting algebras with overlap

What happens if one adds an overlap relation to a Heyting algebra?

Classically:

Heyting algebra + overlap relation = Boolean algebra.

Not surprisingly:

many classical examples of Boolean algebras (which are no longer so intuitionistically) become Heyting algebras with overlap!

Example:

a suitable version of the classical Boolean algebra of

finite-cofinite subsets.
Towards a new kind of completion
The case of a Boolean algebra (classically).

The Dedekind-MacNeille completion $DMN(S)$ of a poset $(S, \leq)$ can be presented as the complete lattice of all formal open subsets associated to a (basic) cover relation on $S$, namely:

$$a \triangleleft U \iff (\forall b \in S)((\forall u \in U)(u \leq b) \Rightarrow a \leq b).$$
Towards a new kind of completion
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$$a \triangleleft U \iff (\forall b \in S)( (\forall u \in U)(u \leq b) \implies a \leq b) .$$

If $S$ is a Boolean algebra AND we adopt a classical metalanguage, then:

$$a \triangleleft U \iff (\forall b \in S)( (\forall u \in U)(u \leq -b) \implies a \leq -b)$$

$$\iff (\forall b \in S)( (\forall u \in U)(u \land b = 0) \implies a \land b = 0)$$

$$\iff (\forall b \in S)( a \land b \neq 0 \implies (\exists u \in U)(u \land b \neq 0) )$$
Completion via overlap (I)

In any poset with overlap:

\[ a \triangleleft_{DMN} U \overset{\text{def}}{\iff} (\forall b \in S)(a \preceq b \Rightarrow (\exists u \in U)(u \succeq b)) \]

"Accidentally" (!?), this is the (basic) cover represented by the basic pair \((S, \preceq, S)\).
Completion via overlap (I)

In any poset with overlap:

\[
a \triangleleft_{DMN} U \iff (\forall b \in S)(a \cong b \Rightarrow (\exists u \in U)(u \cong b))
\]

“Accidentally” (!?), this is the (basic) cover represented by the basic pair \((S, \cong, S)\).

If \(S\) is already complete:

\[
a \triangleleft_{DMN} U \iff (\forall b \in S)(a \cong b \Rightarrow (\exists u \in U)(u \cong b))
\]

\[\forall U \cong b \quad a \leq \bigvee U\]
Completion via overlap (II)

\[ DMN(S) : \text{usual Dedekind-MacNeille completion.} \]

\[ DMN_\bowtie(S) : \text{completion via overlap.} \]

When \( S \) is a Boolean algebra:

1. \( DMN(S) \) is a complete Boolean algebra;
   (actually, the “least” cBa which “contains” \( S \))

2. \( DMN_\bowtie(S) \) is an overlap algebra;
   (actually, the “least” o-algebra which “contains” \( S \))\(^4\)

\(^4\)(w.r.t. a suitable notion of morphism)
For any poset with overlap:

1. \( DMN_\forall(S) \) is an overlap algebra;
2. there exists an embedding \( S \hookrightarrow DMN(S) \) which preserves all existing joins (and meets);
3. \( DMN_\forall(S) \) embeds in any other o-algebra satisfying 2.

Classically:

the completion via overlap of a poset with overlap is always a cBa!

Actually, \( DMC(S) \subseteq DMN_\forall(S) \).
Future work

- Spatial o-algebras & Discrete locales
- Completions via overlap
- Inductive-Coinductive generation\(^5\)

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\(^5\)For inductively generated formal topologies, a positivity relation \(\succ\), hence the positivity predicate \(\text{Pos}\), can be defined co-inductively. This suggests that the overlap relation \(\sim\) can probably be defined by co-induction every time \(\leq\) is defined by induction.
References


Thank you!