The role of the **overlap** relation in constructive mathematics

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Overview

1. **Overlap algebras & Locales**
   - Overlap Algebras and Overt Locales
   - Atoms and Points
   - Regular opens and Dense sublocales

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   - Overlap Algebras and Overt Locales
   - Atoms and Points
   - Regular opens and Dense sublocales
2. **Overlap relation & Dedekind-MacNeille completion**
   - The overlap relation for non-complete posets
   - A variation on the Dedekind-MacNeille completion

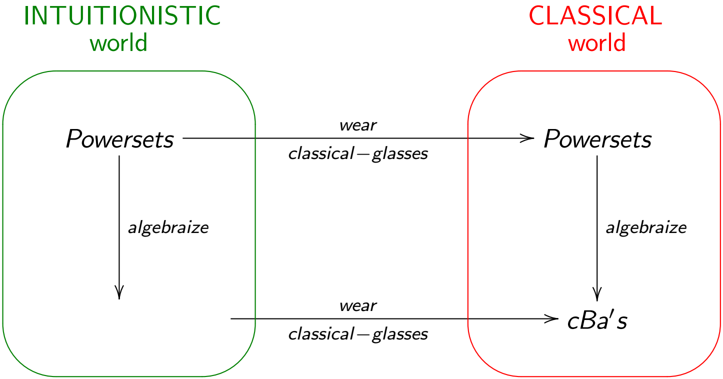
Part I

Overlap Algebras & Locales

In which the definition of overlap algebra is recalled
as well as some well and less well known facts
connecting overlap algebras, overt locales and regular open sets.

Problem:

fill in the blank with a suitable ALGEBRAIC STRUCTURE
in such a way that THE DIAGRAM COMMUTES.



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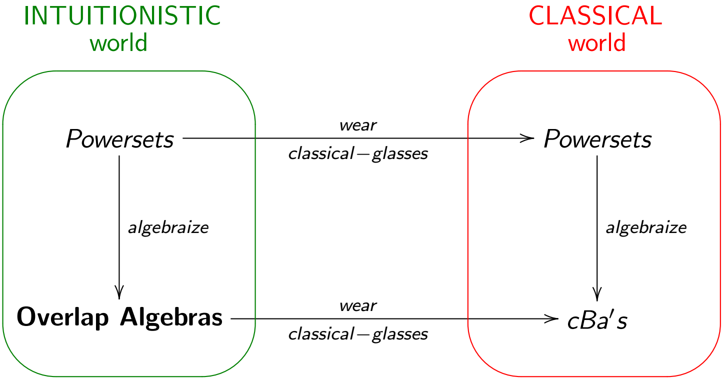
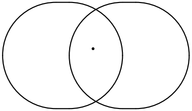


Overlap Algebras =



complete lattice $+$ $\between$ binary relation s.t.:

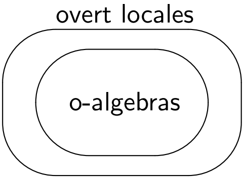
$$\frac{x \between y}{y \between x} \quad \textit{symmetry} \qquad\qquad \frac{x \between y \quad y \leq z}{x \between z} \quad \textit{monotonicity}$$

$$\frac{x \between y}{x \between (x \wedge y)} \quad \textit{refinement} \qquad\qquad \frac{x \between (\bigvee_{i \in I} y_i)}{(\exists i \in I)(x \between y_i)} \quad \textit{splitting}$$

$$\frac{\begin{matrix}[z \between x] \\ \vdots \\ z \between y\end{matrix}}{x \leq y} \quad \text{“density”}$$

Overlap Algebras as Overt Locales

A characterization



Idea: read $x \between y$ as $\mathrm{Pos}(x \wedge y)$.[1]

Overlap algebras

$=$

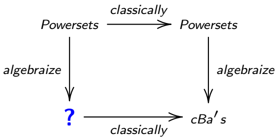
overt locales $+$

$$\cfrac{\begin{matrix}[\mathrm{Pos}(z \wedge x)]\\ \vdots \\ \mathrm{Pos}(z \wedge y)\end{matrix}}{x \leq y}\ \textit{density}^{2}$$

[1] Vice versa, $Pos(x) \equiv (x \between x)$.

[2] Not to be confused with the notion of *a dense sublocale*.

Overlap Algebras as a solution to the starting problem:



1. O-algebras are an algebraic version of powersets:
   - $\mathcal{P}(S)$ is an o-algebra (the motivating example).
   - $\mathcal{P}(S)$ is $\underbrace{\textit{atomic}}_{\text{(see below)}}$.
   - Every atomic o-algebra is a powerset (Sambin).

2. Classically:
   - Overlap algebras = complete Boolean algebras (Vickers) (where: $x \between y \iff x \wedge y \neq 0$).

Atoms and points (I)

What is an ATOM?

→ (1) A minimal NON-ZERO element.

→ (2) A minimal "POSITIVE" element.

(1) is too weak: one cannot even prove that a singleton is an atom in a powerset!

(2) works well (but one needs Pos or $\between$ to define "positive" for elements).

Atoms and points (II)

TFAE

(in any overt locale)

1. a is a minimal positive element

   i.e.: $\quad \mathrm{Pos}(a) \quad$ and $\quad \mathrm{Pos}(x) \;\&\; (x \leq a) \implies (x = a)$

2. $$\underbrace{a \between x}_{\mathrm{Pos}(a \wedge x)} \iff a \leq x \qquad \text{(for any } x\text{);}$$

Atoms and points (III)

TFAE

(in any o-algebra)

1. a is an atom;
2. $\{x \mid \underbrace{a \between x}_{\mathrm{Pos}(a \wedge x)} \}$ is a completely prime filter

   (in that case $\{x \mid a \between x\} = \{x \mid a \leq x\}$).

In other words: the mapping $x \longmapsto \mathrm{Pos}(a \wedge x)$ is a frame homomorphism if and only if a is an atom.

Atoms and points (IV)

$$\{atoms\ of\ an\ o-algebra\} \quad \subseteq \quad \{points\ of\ an\ o-algebra\}$$

$$\{atomic\ o-algebras\} \quad \subseteq \quad \{spatial\ o-algebras\}$$

Open questions:

- What is a spatial o-algebra like?
- What is the relationship between o-algebras and discrete locales[3]?

[3] Discrete locale = overt + the diagonal map is open.

Regular elements (I)

Even if Sambin's "density" does not hold for an overt locale, in general, nevertheless:

$$\frac{\begin{array}{c}[\mathrm{Pos}(z \wedge x)] \\ \vdots \\ \mathrm{Pos}(z \wedge y)\end{array}}{x \leq y}\ \textit{density}$$

(in any overt locale)

there exists a unique nucleus r s.t.:

$$\frac{\begin{array}{c}[\mathrm{Pos}(z \wedge x)] \\ \vdots \\ \mathrm{Pos}(z \wedge y)\end{array}}{x \leq r(y)}$$

Regular elements (II)

For $\mathcal{L}$ overt, let: $\mathcal{L}_r \overset{def}{=} \{x \in \mathcal{L} \mid x = r(x)\}$

1. $\mathcal{L}_r$ is a o-algebra
   (hence an overt locale (w.r.t. the same Pos of $\mathcal{L}$))

2. $\mathcal{L}_{--} = \{x \in \mathcal{L} \mid x = --x\} \subseteq \mathcal{L}_r$
   (the least dense sublocale of $\mathcal{L}$) (classically $\mathcal{L}_{--} = \mathcal{L}_r$)

3. $\mathcal{L}_r$ is the least $\underbrace{\textit{positively dense}}_{\mathrm{Pos}(r(x)) \Rightarrow \mathrm{Pos}(x)}$ sublocale of $\mathcal{L}$
   (this notion has been studied by Bas Spitters)

4. $\mathcal{L}_r = \mathcal{L}$ iff $\mathcal{L}$ is an o-algebra
   (in particular, every o-algebra is of the kind $\mathcal{L}_r$)

Regular elements (III)

REGULAR open sets = the interior of its closure.

Topological closure:

$\nearrow$ (1) complement of the interior of the complement.

$\searrow$ (2) set of adherent points.

Regular elements:

$\nearrow$ (1) $x = --x$

$\searrow$ (2) $x = r(x)$

Part II

Overlap Algebras & Dedekind-MacNeille completion

In which a new approach to
the Dedekind-MacNeille completion
is presented (which works for posets with overlap).

Posets with overlap

Idea: modify the definition of an overlap relation in such a way that it would make sense for arbitrary posets.

$$\frac{x \between y}{x \between (x \wedge y)}\ \textit{refinement} \quad \rightsquigarrow \quad \frac{x \between y}{\exists z (x \between z \ \ \& \ \ z \leq x \ \ \& \ \ z \leq y)}$$

$$\frac{x \between (\bigvee_{i \in I} y_i)}{(\exists i \in I)(x \between y_i)}\ \textit{splitting} \quad \rightsquigarrow \quad \frac{\bigvee_{i \in I} y_i \text{ exists} \qquad x \between (\bigvee_{i \in I} y_i)}{(\exists i \in I)(x \between y_i)}$$

N.B.:
usually, the addition of an overlap relation greatly enrich the underlying structure.

For instance, <u>any lattice with overlap is automatically distributive</u>.

Moreover (<u>classically</u>):

$$\begin{array}{rc} \text{bounded lattice} & + \\ \text{pseudo-complement} & + \\ \text{overlap} & = \\ \hline \text{Boolean algebra} & \end{array}$$

Example: Heyting algebras with overlap

What happens if one adds an overlap relation to a Heyting algebra?

Classically:

Heyting algebra $+$ overlap relation $=$ Boolean algebra .

Not surprisingly:

many classical examples of Boolean algebras (which are no longer so intuitionistically) become Heyting algebras with overlap!

Example:

a suitable version of the classical Boolean algebra of

finite-cofinite subsets.

Towards a new kind of completion

The case of a Boolean algebra (classically).

The Dedekind-MacNeille completion $DMN(S)$ of a poset $(S, \leq)$ can be presented as the complete lattice of all formal open subsets associated to a (basic) cover relation on S, namely:

$$a \lhd U \quad \overset{def}{\Longleftrightarrow} \quad (\forall b \in S)\big(\ (\forall u \in U)(u \leq b) \ \Rightarrow \ a \leq b \ \big) \ .$$

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If S is a Boolean algebra AND we adopt a classical metalanguage, then:

$$\begin{aligned} a \lhd U \quad & \Leftrightarrow \quad (\forall b \in S)\big(\ (\forall u \in U)(u \leq -b) \ \Rightarrow \ a \leq -b\ \big) \\ & \Leftrightarrow \quad (\forall b \in S)\big(\ (\forall u \in U)(u \wedge b = 0) \ \Rightarrow \ a \wedge b = 0\ \big) \\ & \Leftrightarrow \quad (\forall b \in S)\big(\ \underbrace{a \wedge b \neq 0}_{a \between b} \ \Rightarrow \ (\exists u \in U)(\underbrace{u \wedge b \neq 0}_{u \between b})\ \big) \end{aligned}$$

Completion via overlap (I)

In any poset with overlap:

$$a \lhd_{DMN} U \quad \overset{def}{\Longleftrightarrow} \quad (\forall b \in S)\big(\ a \between b \ \Rightarrow\ (\exists u \in U)(u \between b)\ \big)$$

"Accidentally" (!?), this is the (basic) cover represented by the basic pair $(S, \between, S)$.

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"Accidentally" (!?), this is the (basic) cover represented by the basic pair $(S, \between, S)$.

If S is already complete:

$$a \lhd_{DMN} U \quad \Leftrightarrow \quad \underbrace{(\forall b \in S)\big(\ a \between b \ \Rightarrow \ \underbrace{(\exists u \in U)(u \between b)}_{(\bigvee U) \between b}\ \big)}_{a \leq \bigvee U}$$

Completion via overlap (II)

$DMN(S)$: usual Dedekind-MacNeille completion.

$DMN_{\between}(S)$: completion via overlap.

When S is a Boolean algebra:

1. $DMN(S)$ is a <u>complete</u> Boolean algebra;
   (actually, the "least" cBa which "contains" S)

2. $DMN_{\between}(S)$ is an <u>overlap algebra</u>;
   (actually, the "least" o-algebra which "contains" S)[4]

[4](w.r.t. a suitable notion of morphism)

Completion via overlap (III)

For any poset with overlap:

1. $DMN_{\between}(S)$ is an overlap algebra;
2. there exists an embedding $S \hookrightarrow DMN(S)$ which preserves all existing joins (and meets);
3. $DMN_{\between}(S)$ embeds in any other o-algebra satisfying 2.

Classically:

the completion via overlap of a poset with overlap is always a cBa!

Actually, $DMC(S) \subseteq DMN_{\between}(S)$.

Future work

- Spatial o-algebras & Discrete locales
- Completions via overlap
- Inductive-Coinductive generation[5]

[5] For inductively generated formal topologies, a positivity relation $\ltimes$, hence the positivity predicate Pos, can be defined co-inductively.
This suggests that the overlap relation $\between$ can probably be defined by co-induction every time $\leq$ is defined by induction.

References

- G. Sambin, "*The Basic Picture: Structures for Constructive Topology*", Oxford University Press, to appear.
- F. C., G. Sambin, "*The overlap algebra of regular opens*", **J. Pure Appl. Algebra** 214 (11), pp. 1988 -1995 (November 2010).
- F. C., "*Regular opens in Formal Topology and a representation theorem for overlap algebras*", submitted.
- F. C., M. E. Maietti, P. Toto, "*Constructive version of Boolean algebra*", submitted.

Thank you!