What is a *binary positivity* for a locale?

Francesco Ciraulo

Department of Mathematics and Computer Science
University of Palermo

Workshop on Constructive Topology
Palermo, 6 - 7 September 2010

Prologue

1987 G. Sambin - *Intuitionistic formal spaces*

a formal topology = predicative version of a(n overt) locale
= a presentation of a locale "by generators and relations"
= a locale with "base"

"today" G. Sambin - *The Basic Picture*

a binary positivity predicate is added.

- What is a binary positivity predicate (a.k.a. "fish")?
- What is its meaning for a locale (i.e. with no reference to the base)?

Covers

A COVER relation over a set S is $\lhd \subseteq S \times \mathcal{P}(S)$, s. t.:

$$\frac{a \in U}{a \lhd U} \qquad \frac{a \lhd U \quad \forall u(u \in U \Rightarrow u \lhd V)}{a \lhd V} \qquad \frac{a \lhd U \quad a \lhd V}{a \lhd {\downarrow}U \cap {\downarrow}V}$$

where ${\downarrow}U = \{x \mid (\exists u \in U)(\underbrace{x \leq u}_{x \lhd \{u\}})\}$

A cover is essentially a Grothendieck topology on a pre-ordered set.

A *formal open subset* is $\{a \in S \mid a \lhd U\}$ (for any $U \subseteq S$).

The collection $\{$*formal open subsets*$\}$ is a frame.

Note: we can assume $S \subseteq \{$*formal open subsets*$\}$ without loss of generality (identify $a \in S$ with $\{x \in S \mid x \lhd \{a\}\}$).

Covers = locales with bases

$\mathcal{L}$ (locale)

A base for $\mathcal{L}$ is a subset $S \subseteq \mathcal{L}$ s.t.
$x = \bigvee\{a \in S \mid a \leq x\}$
(for any x in $\mathcal{L}$)

$$a \lhd U \quad \overset{def.}{\Longleftrightarrow} \quad a \leq \bigvee U$$

(for $a \in S$ and $U \subseteq S$)

$\lhd$ is a cover relation on the set S

$\mathcal{L} \cong \{$*formal open subsets*$\}$

Formal topologies with binary positivity

$$(\; S \; , \; \lhd \; , \; \ltimes \;)$$
$$(set, \; cover, \; positivity)$$

$$\lhd \subseteq S \times \mathcal{P}(S)$$

$$\frac{a \in U}{a \lhd U}$$

$$\frac{a \lhd U \quad \forall b(b \in U \Rightarrow b \lhd V)}{a \lhd V}$$

$$\frac{a \lhd U \quad a \lhd V}{a \lhd {\downarrow} U \cap {\downarrow} V}$$

$$\ltimes \subseteq S \times \mathcal{P}(S)$$

$$\frac{a \ltimes U}{a \in U}$$

$$\frac{a \ltimes U \quad \forall b(b \ltimes U \Rightarrow b \in V)}{a \ltimes V}$$

$$(Nothing : why?)$$

$$\frac{a \lhd U \quad a \ltimes V}{(\exists u \in U)(u \ltimes V)}$$

Formal closed subsets

$\{x \in S \mid x \lhd U\}$	$\{x \in S \mid x \ltimes U\}$
formal open subset	*formal closed subset*
(is an element of the locale presented by the cover)	?? ?

Questions:

1. What is the meaning of $\ltimes$?
2. What is a formal closed subset?

Note: 1 and 2 are essentially the same question:

$$F \text{ formal closed} \iff \{a \mid a \ltimes F\} = F$$

$$a \ltimes U \iff (\exists F \text{ formal closed })(a \in F \subseteq U)$$

Subsets which *split* the cover

F *splits the cover* if

$$\frac{a \lhd U \quad a \in F}{\underbrace{U \between F}_{(\exists u \in U)(u \in F)}} \; (\textit{for any } U)$$

Examples:

- $\{a \in S \mid \mathrm{Pos}(a)\}$ if the locale is overt and Pos is its (unary) positivity predicate;
- $S \cap \alpha^{-1}(\top)$ where $\alpha : \mathcal{L} \rightarrow \Omega$ is a point;
- every formal closed subset (recall that: $\frac{a \lhd U \quad a \ltimes V}{(\exists u \in U)(u \ltimes V)}$).

Problem:

characterize splitting subsets.

Splitting subsets as morphisms of sup-lattices

$$\begin{array}{cccccc} & & \{\text{formal open subsets}\} & & \{\text{propositions}\} \\ & & \parallel & & \parallel \\ \varphi_F & : & \mathcal{L} & \longrightarrow & \Omega \\ & & U & \longmapsto & U \between F \end{array}$$

F splits $\lhd \iff \varphi_F \in \mathbf{SupLat}(\mathcal{L}, \Omega)$

$$\varphi_{(\_)} \ : \ \{\textit{subsets splitting } \lhd\} \longrightarrow \mathbf{SupLat}(\mathcal{L}, \Omega)$$

is a bijection with inverse $\varphi \mapsto S \cap \varphi^{-1}(\top)$.

$$\{\textit{subsets of } S \textit{ splitting } \lhd\} \leftrightsquigarrow \mathbf{SupLat}(\mathcal{L}, \Omega)$$

$\ltimes$ as a <u>sub</u>-suplattice of $\mathbf{SupLat}(\mathcal{L}, \Omega)$

Fact: an arbitrary union of formal closed subsets is formal closed.

Given a locale $\mathcal{L}$ and a base S (and the corresponding $\lhd$)

a $\ltimes$ (compatible with $\lhd$)

is the same thing as

a <u>sub</u>-suplattice of $\mathbf{SupLat}(\mathcal{L}, \Omega)$.

$$\ltimes \leftrightsquigarrow \{\textit{formal closed subsets}\} \hookrightarrow \{\textit{splitting subsets}\} \leftrightsquigarrow \mathbf{SupLat}(\mathcal{L}, \Omega)$$

$$\underbrace{\{\varphi_i\}_{i \in I}}_{\substack{\bigcap \\ \mathbf{SupLat}(\mathcal{L}, \Omega)}} \quad \leftrightsquigarrow \quad a \ltimes V \overset{def}{\iff} (\exists i \in I)\big(a \in \underbrace{S \cap \varphi^{-1}(\top)}_{(splitting)} \subseteq V\big)$$

The greatest positivity relation

For any cover $\lhd$, there exists (impredicatively)
the greatest positivity compatible with $\lhd$

(it is the one corresponding to the whole of $\mathbf{SupLat}(\mathcal{L}, \Omega)$):

$$a \ltimes_{great} U \qquad \Longleftrightarrow \qquad \big(\exists \varphi \in \mathbf{SupLat}(\mathcal{L}, \Omega)\big)\big(a \in S \cap \varphi^{-1}(\top) \subseteq U\big)$$

Note: if $\lhd$ is inductively generated (site, coverage, ...),
then $\ltimes_{great}$ is defined by co-induction on the same axioms.

Positivities as inclusions in **Loc** (1)

Given $(S, \lhd, \ltimes)$, we know that:

1. there exist a unique (up to iso.) locale $\mathcal{L}$ such that S is a base of $\mathcal{L}$ and $a \lhd U$ iff $a \leq \bigvee U$;
2. $\ltimes$ can be identified with a sub-suplattice of $\mathbf{SupLat}(\mathcal{L}, \Omega)$.

Now consider:

$$\underbrace{\overbrace{\mathbf{Frm}(\mathcal{L}, \Omega)}^{points\ \ of\ \ \mathcal{L}} \bigcap \ltimes}_{(points\ \ which\ \ are\ \ formal\ \ closed)}$$

This set of points can be used to construct a <u>spatial</u> locale $\mathcal{L}'$ s.t. $\mathcal{L}' \hookrightarrow \mathcal{L}$.

Positivities as inclusions in **Loc** (2)

Vice versa, given $\mathcal{L}' \hookrightarrow \mathcal{L}$ ($\mathcal{L}'$ not necessarily spatial), note that:

$$\mathbf{Frm}(\mathcal{L}', \Omega) \text{ generates a sub-suplattice of } \mathbf{SupLat}(\mathcal{L}, \Omega)$$

hence a $\ltimes$ on $\mathcal{L}$.

(Positivities obtained in this way are called *reduced*.)

Summing up:

given $\mathcal{L}$ (locale), S (base) and $\lhd$ (corresponding cover relation):

$(S, \lhd, \ltimes)$ $\dashrightarrow$ $\mathcal{L}' \hookrightarrow \mathcal{L}$ (with $\mathcal{L}'$ spatial)

$(S, \lhd, \ltimes)$ (with $\ltimes$ reduced) $\dashleftarrow$ $\mathcal{L}' \hookrightarrow \mathcal{L}$

reduced positivity $\leftrightsquigarrow$ spatial sublocale

Examples

Spatialization

$$(S, \lhd, \ltimes_{great}) \quad \leftrightsquigarrow \quad \underbrace{\Omega(Pt(\mathcal{L}))}_{(spatialization)} \hookrightarrow \mathcal{L}$$

Spatial locales

Recall that $\mathcal{L} \hookrightarrow Low(S)$ (lower subsets).

$Low(S)$ can be presented by the cover: $a \leq U \Leftrightarrow (\exists u \in U)(a \leq u)$.

$$(S, \leq, \underbrace{\ltimes}_{reduced}) \quad \leftrightsquigarrow \quad \underbrace{\mathcal{L}}_{(spatial)} \hookrightarrow Low(S)$$

Application:
locales presented by a positivity

Recall from the previous slide: any spatial locale can be presented by a structure of the form $(S, \leq, \ltimes)$ with $\ltimes$ reduced.

More generally, any structure of the form:

$$\underbrace{(S, \leq}_{poset}, \ltimes) \qquad + \qquad \underbrace{\text{“}a\ certain\ condition\text{”}}_{(weaker\ than\ \text{“}\ltimes\ reducible\text{”})}$$

can be used to define a nucleus j on $Low(S)$, hence a sublocale $Low(S)_j$

1. $Low(S)_j$ is not necessarily spatial;
2. classically, <u>any</u> locale can be presented in this way;
3. intuitionistically, this is not the case (by a counterexample of Coquand).

Thank you!