What is a *binary positivity* for a locale?

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1987  G. Sambin - *Intuitionistic formal spaces*

- a **formal topology**  =  **predicative** version of a(n overt) locale
- =  a presentation of a locale
-  “by generators and relations”
-  =  a locale with “base”

“today”  G. Sambin - *The Basic Picture*

a **binary positivity predicate** is added.

- What is a **binary positivity predicate** (a.k.a. “fish”)?
- What is its meaning for a locale (i.e. with no reference to the base)?
Covers

A COVER relation over a set $S$ is $\triangleleft \subseteq S \times \mathcal{P}(S)$, s. t.:

$$
\begin{align*}
& a \in U \\
\hline
& a \triangleleft U \\
& \forall u (u \in U \Rightarrow u \triangleleft V)
\end{align*}
$$

$$
\hline
& a \triangleleft U \\
& a \triangleleft V
\hline
$$

where $\downarrow U = \{x \mid (\exists u \in U)(x \leq u)\}$

A cover is essentially a Grothendieck topology on a pre-ordered set.

A formal open subset is $\{a \in S \mid a \triangleleft U\}$ (for any $U \subseteq S$).

The collection $\{\text{formal open subsets}\}$ is a frame.

Note: we can assume $S \subseteq \{\text{formal open subsets}\}$ without loss of generality

(identify $a \in S$ with $\{x \in S \mid x \triangleleft \{a\}\}$).
Covers = locales with bases

\[ \mathcal{L} \text{ (locale)} \]

A \textbf{base} for \( \mathcal{L} \) is a subset \( S \subseteq \mathcal{L} \) s.t.
\[ x = \bigvee \{a \in S \mid a \leq x\} \]
(for any \( x \) in \( \mathcal{L} \))

\( a \triangleleft U \iff a \leq \bigvee U \)

(for \( a \in S \) and \( U \subseteq S \))

\( \triangleleft \) is a cover relation on the set \( S \)

\[ \mathcal{L} \cong \{\text{formal open subsets}\} \]
Formal topologies with binary positivity

\((S, \triangleleft, \times)\)

(set, cover, positivity)

\[\triangleleft \subseteq S \times \mathcal{P}(S)\]

\[a \in U \quad \frac{a \triangleleft U}{a \triangleleft V}\]

\[\forall b (b \in U \Rightarrow b \triangleleft V)\]

\[a \triangleleft U \quad a \triangleleft V \quad \frac{a \triangleleft \downarrow U \cap \downarrow V}{a \triangleleft \downarrow U \cap \downarrow V}\]

\[a \times U \quad \forall b (b \times U \Rightarrow b \in V)\]

\[a \times U \quad \frac{a \times V}{a \times V}\]

\[a \triangleleft U \quad a \times V \quad \frac{(\exists u \in U)(u \times V)}{(\exists u \in U)(u \times V)}\]

(Nothing : why?)

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Formal closed subsets

\[ \{ x \in S \mid x \hookleftarrow U \} \quad \text{formal open subset} \quad \{ x \in S \mid x \not\hookleftarrow U \} \quad \text{formal closed subset} \]

(is an element of the locale presented by the cover)

Questions:

1. What is the meaning of \( \not\hookleftarrow \)?
2. What is a formal closed subset?

Note: 1 and 2 are essentially the same question:

\[ F \text{ formal closed} \iff \{ a \mid a \not\hookleftarrow F \} = F \]
\[ a \not\hookleftarrow U \iff (\exists F \text{ formal closed })(a \in F \subseteq U) \]
Subsets which *split* the cover

*F splits the cover* if

$$\frac{a \triangleleft U \quad a \in F}{U \uplus F} \quad (\text{for any } U)$$

$$(\exists u \in U)(u \in F)$$

Examples:

- $\{a \in S \mid \text{Pos}(a)\}$ if the locale is overt and Pos is its (unary) positivity predicate;
- $S \cap \alpha^{-1}(\top)$ where $\alpha : \mathcal{L} \to \Omega$ is a point;
- every formal closed subset (recall that: $\frac{a \triangleleft U \quad a \triangleleft V}{(\exists u \in U)(u \triangleleft V)}$).

**Problem:**

characterize splitting subsets.
Splitting subsets as morphisms of sup-lattices

\( \varphi_F : \mathcal{L} \rightarrow \Omega \) gives \( \varphi_F \in \text{SupLat}(\mathcal{L}, \Omega) \) if \( F \) splits.

\[ \varphi(\_): \{ \text{subsets splitting } \triangleleft \} \rightarrow \text{SupLat}(\mathcal{L}, \Omega) \]
is a bijection with inverse \( \varphi \mapsto S \cap \varphi^{-1}(\top) \).

\[ \{ \text{subsets of } S \text{ splitting } \triangleleft \} \leftrightarrow \text{SupLat}(\mathcal{L}, \Omega) \]
as a sub-suplattice of $\text{SupLat}(\mathcal{L}, \Omega)$

Fact: an arbitrary union of formal closed subsets is formal closed.

Given a locale $\mathcal{L}$ and a base $S$ (and the corresponding $\triangleleft$)
a $\ltimes$ (compatible with $\triangleleft$)
is the same thing as

a sub-suplattice of $\text{SupLat}(\mathcal{L}, \Omega)$.

$\ltimes \iff \{\text{formal closed subsets}\} \hookrightarrow \{\text{splitting subsets}\} \iff \text{SupLat}(\mathcal{L}, \Omega)$

$\{\varphi_i\}_{i \in I} \iff a \ltimes V \overset{\text{def}}{\iff} (\exists i \in I)(a \in S \cap \varphi^{-1}(\top) \subseteq V)$

(splitting)
The greatest positivity relation

For any cover $\sqsubset$, there exists (impredicatively)
the greatest positivity compatible with $\sqsubset$

(it is the one corresponding to the whole of $\text{SupLat}(\mathcal{L}, \Omega)$):

$$a \nRightarrow_{\text{great}} U \iff (\exists \varphi \in \text{SupLat}(\mathcal{L}, \Omega))(a \in S \cap \varphi^{-1}(\top) \subseteq U)$$

Note: if $\sqsubset$ is inductively generated (site, coverage, ...),
then $\nRightarrow_{\text{great}}$ is defined by co-induction on the same axioms.
Positivities as inclusions in \textbf{Loc} (1)

Given \((S, \triangleleft, \heartsuit)\), we know that:

1. there exist a unique (up to iso.) locale \(L\) such that \(S\) is a base of \(L\) and \(a \triangleleft U\) iff \(a \leq \bigvee U\);
2. \(\heartsuit\) can be identified with a sub-suplattice of \(\text{SupLat}(L, \Omega)\).

Now consider:

\[
\begin{array}{c}
\text{points of } L \\
\text{Frm}(L, \Omega) \bigcap \heartsuit \\
\text{(points which are formal closed)}
\end{array}
\]

This set of points can be used to construct a \underline{spatial} locale \(L'\) s.t. \(L' \hookrightarrow L\).
Positivities as inclusions in \textbf{Loc} (2)

Vice versa, given $\mathcal{L}' \hookrightarrow \mathcal{L}$ ($\mathcal{L}'$ not necessarily spatial), note that:

$\text{Frm}(\mathcal{L}', \Omega)$ generates a sub-suplattice of $\text{SupLat}(\mathcal{L}, \Omega)$

hence a $\ltimes$ on $\mathcal{L}$.

(Positivities obtained in this way are called \textit{reduced}.)

\begin{itemize}
  \item Summing up:
  \begin{itemize}
    \item given $\mathcal{L}$ (locale), $S$ (base) and $\ltimes$ (corresponding cover relation):
      \begin{align*}
        (S, \ltimes, \ltimes) & \quad \longrightarrow \quad \mathcal{L}' \hookrightarrow \mathcal{L} \\
        & \quad \text{(with } \mathcal{L}' \text{ spatial)}
      \end{align*}
    \item \begin{align*}
        (S, \ltimes, \ltimes) & \quad \longleftarrow \quad \mathcal{L}' \hookrightarrow \mathcal{L} \\
        & \quad \text{(with } \ltimes \text{ reduced)}
      \end{align*}
  \end{itemize}

  \begin{itemize}
    \item \textit{reduced positivity} $\leftrightarrow$ \textit{spatial sublocale}
  \end{itemize}
\end{itemize}
Examples

Spatialization

\[(S, \triangleleft, \ltimes_{\text{great}}) \leftrightarrow \Omega(Pt(\mathcal{L})) \hookrightarrow \mathcal{L}\]

(spatialization)

Spatial locales

Recall that \(\mathcal{L} \hookrightarrow Low(S)\) (lower subsets).

\(Low(S)\) can be presented by the cover: \(a \leq U \iff (\exists u \in U)(a \leq u)\).

\[(S, \leq, \ltimes) \leftrightarrow \mathcal{L} \hookrightarrow Low(S)\]

(reduced) (spatial)
Application:
locales presented by a positivity

Recall from the previous slide: any spatial locale can be presented by a structure of the form \((S, \leq, \Join)\) with \(\Join\) reduced.

More generally, any structure of the form:

\[
(S, \leq, \Join) \quad + \quad \text{“a certain condition”''}
\]

\[\text{poset} \quad \text{(weaker than “\(\Join\) reducible’’)}\]

can be used to define a nucleus \(j\) on \(\text{Low}(S)\), hence a sublocale \(\text{Low}(S)_j\).

1. \(\text{Low}(S)_j\) is not necessarily spatial;
2. classically, any locale can be presented in this way;
3. intuitionistically, this is not the case (by a counterexample of Coquand).
Thank you!