# What is a *binary positivity* for a locale?

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### Prologue

1987 G. Sambin - Intuitionistic formal spaces

- a formal topology = predicative version of a(n overt) locale
  - a presentation of a locale
     "by generators and relations"
  - = a locale with "base"

"today" G. Sambin - *The Basic Picture* a binary positivity predicate is added.

- What is a binary positivity predicate (a.k.a. "fish")?
- What is its meaning for a locale (i.e. with no reference to the base)?

### Covers

A COVER relation over a set S is  $\triangleleft \subseteq S \times \mathcal{P}(S)$ , s. t.:

$$\frac{a \in U}{a \lhd U} \qquad \frac{a \lhd U \quad \forall u(u \in U \Rightarrow u \lhd V)}{a \lhd V} \qquad \frac{a \lhd U \quad a \lhd V}{a \lhd \downarrow U \cap \downarrow V}$$
where  $\downarrow U = \{x \mid (\exists u \in U)(\underbrace{x \leq u}_{x \lhd \{u\}})\}$ 
were is essentially a Grothendieck topology on a pre-ordered set

A <u>cover</u> is essentially Grothendieck topology on a pre-ordered set

A formal open subset is  $\{a \in S \mid a \triangleleft U\}$  (for any  $U \subseteq S$ ).

The collection { *formal open subsets*} is a frame.

Note: we can assume  $S \subseteq \{ formal open subsets \}$  without loss of generality (identify  $a \in S$  with  $\{x \in S \mid x \triangleleft \{a\}\}$ ).

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# Covers = locales with bases

 $\mathcal{L}$  (locale)

A base for 
$$\mathcal{L}$$
 is a subset  $S \subseteq \mathcal{L}$  s.t.  
 $x = \bigvee \{a \in S \mid a \leq x\}$   
(for any x in  $\mathcal{L}$ )

$$a \triangleleft U \quad \stackrel{def.}{\Longleftrightarrow} \quad a \leq \bigvee U$$
 (for  $a \in S$  and  $U \subseteq S$ )

 $\lhd$  is a cover relation on the set S

 $\mathcal{L} \cong \{ \text{formal open subsets} \}$ 

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Formal topologies with binary positivity

## Formal closed subsets

 $\{x \in S \mid x \triangleleft U\} \qquad \{x \in S \mid x \ltimes U\}$ formal open subset
(is an element of the locale ??

presented by the cover)

### Questions:

What is the meaning of ×?
What is a formal closed subset?

Note: 1 and 2 are essentially the same question:

 $F \text{ formal closed } \iff \{a \mid a \ltimes F\} = F$   $a \ltimes U \iff (\exists F \text{ formal closed })(a \in F \subseteq U)$   $(\exists F \text{ formal closed })(a \in F \subseteq U)$ 

?

# Subsets which *split* the cover

F splits the cover if

$$\underbrace{\substack{a \lhd U \quad a \in F}}_{(\exists u \in U)(u \in F)} \text{ (for any } U\text{)}$$

Examples:

■ {a ∈ S | Pos(a)} if the locale is overt and Pos is its (unary) positivity predicate;

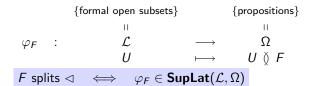
• 
$$S \cap \alpha^{-1}(\top)$$
 where  $\alpha : \mathcal{L} \to \Omega$  is a point;

every formal closed subset (recall that:  $\frac{a \triangleleft U \quad a \ltimes V}{(\exists u \in U)(u \ltimes V)}$ ).

### Problem:

### characterize splitting subsets.

# Splitting subsets as morphisms of sup-lattices



$$\varphi_{(-)} : \{ subsets \ splitting \lhd \} \longrightarrow \mathbf{SupLat}(\mathcal{L}, \Omega)$$
  
is a bijection with inverse  $\varphi \mapsto S \cap \varphi^{-1}(\top).$ 

{subsets of S splitting  $\triangleleft$ }  $\iff$  SupLat( $\mathcal{L}, \Omega$ )

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# $\ltimes$ as a <u>sub</u>-suplattice of **SupLat**( $\mathcal{L}, \Omega$ )

Fact: an arbitrary union of formal closed subsets is formal closed.

Given a locale 
$$\mathcal{L}$$
 and a base  $S$  (and the corresponding  $\lhd$ )  
a  $\ltimes$  (compatible with  $\lhd$ )  
is the same thing as  
a sub-suplattice of **SupLat**( $\mathcal{L}, \Omega$ ).

 $\ltimes \nleftrightarrow \{\textit{formal closed subsets}\} \hookrightarrow \{\textit{splitting subsets}\} \nleftrightarrow \textsf{SupLat}(\mathcal{L}, \Omega)$ 

$$\underbrace{\{\varphi_i\}_{i\in I}}_{\bigcap} \quad \iff \quad a \ltimes V \iff (\exists i \in I) \left(a \in \underbrace{S \cap \varphi^{-1}(\top)}_{(splitting)} \subseteq V\right)$$
$$\operatorname{SupLat}(\mathcal{L}, \Omega)$$

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# The greatest positivity relation

For any cover  $\lhd$ , there exists (impredicatively) the greatest positivity compatible with  $\lhd$ 

(it is the one corresponding to the whole of  $SupLat(\mathcal{L}, \Omega)$ ):

$$a \ltimes_{great} U \qquad \Longleftrightarrow \qquad \left( \exists \varphi \in \mathbf{SupLat}(\mathcal{L}, \Omega) \right) \left( a \in S \cap \varphi^{-1}(\top) \subseteq U \right)$$

Note: if  $\lhd$  is inductively generated (site, coverage, ...), then  $\ltimes_{great}$  is defined by <u>co-induction</u> on the same axioms.

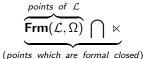
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# Positivities as inclusions in Loc (1)

Given  $(S, \lhd, \ltimes)$ , we know that:

- I there exist a unique (up to iso.) locale L such that S is a base of L and a ⊲ U iff a ≤ V U;
- **2**  $\ltimes$  can be identified with a sub-suplattice of **SupLat**( $\mathcal{L}, \Omega$ ).

Now consider:



This set of points can be used to construct a spatial locale  $\mathcal{L}'$  s.t.  $\mathcal{L}' \hookrightarrow \mathcal{L}$ .

# Positivities as inclusions in Loc (2)

Vice versa, given  $\mathcal{L}' \hookrightarrow \mathcal{L}$  ( $\mathcal{L}'$  not necessarily spatial), note that:  $\mathbf{Frm}(\mathcal{L}', \Omega)$  generates a sub-suplattice of  $\mathbf{SupLat}(\mathcal{L}, \Omega)$ 

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hence a \ltimes on \mathcal{L}.
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(Positivities obtained in this way are called reduced.)

# Summing up:given $\mathcal{L}$ (locale), S (base) and $\lhd$ (corresponding cover relation): $(S, \lhd, \ltimes)$ $(S, \lhd, \ltimes)$ $(S, \lhd, \ltimes)$ $(with \mathcal{L}' spatial)$ $(S, \lhd, \ltimes)$ $(with \ltimes reduced)$ reduced positivity $\Leftrightarrow$ spatial sublocale

### Examples

### Spatialization

$$(S, \lhd, \ltimes_{great}) \iff \underbrace{\Omega(Pt(\mathcal{L}))}_{(spatialization)} \hookrightarrow \mathcal{L}$$

### Spatial locales

Recall that  $\mathcal{L} \hookrightarrow Low(S)$  (lower subsets).

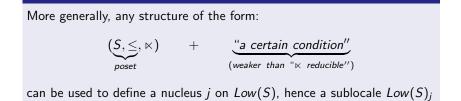
Low(S) can be presented by the cover:  $a \leq U \Leftrightarrow (\exists u \in U)(a \leq u)$ .



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### Application: locales presented by a positivity

Recall from the previous slide: any spatial locale can be presented by a structure of the form  $(S, \leq, \ltimes)$  with  $\ltimes$  reduced.



**1**  $Low(S)_j$  is not necessarily spatial;

2 classically, any locale can be presented in this way;

**3** intuitionistically, this is not the case (by a counterexample of Coquand).

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# Thank you!

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