Localic Groups in Formal Topology and the Closed Subgroup Theorem

Francesco Ciraulo

University of Padua

7WFTop, Venice, April 14, 2026

Pointfree Topology

- Pointfree topology replaces spaces with <u>frames</u> (of "opens")
- It works nicely constructively

  (much better than topological spaces)
- **Frames** = finite meets ($\top$ and $\wedge$), arbitrary $\bigvee$,

$$x \wedge \bigvee_i y_i = \bigvee_i (x \wedge y_i)$$

  + functions preserving $\top$, $\wedge$ and $\bigvee$
- **Locales** = "opposite" of frames
- **Formal topologies** = (overt) locales <u>with bases</u>

  = natural setting for constructive <u>and</u> predicative topology

Localic Groups

- Group objects in **Loc**
- Analogue of topological groups
- But: behaviour is very different...

The Closed Subgroup Theorem...

Isbell

Every subgroup of a localic group is closed.

Strong contrast with classical topology

e.g. $\mathbb{Q}$ is <u>not a localic subgroup</u> of $(\mathbb{R}, +, 0)$

(although it is a topological subgroup, of course)

. . . and its Constructive (impredicative) version

[Johnstone, 1989]

Every overt localic subgroup of an overt localic group is weakly closed.

Uses:

- overtness
- weak closure
- (and strong density)

Goal of the Talk

- Translate the CS thm into Formal Topology
- Clarify some predicative isssues
- Underline the role of (binary) positivity relation [Sambin, 2025]

From locales to bases and covers

- L a locale
- corresponding frame: ΩL, aka $\mathcal{O}(L)$
- base S (closed under finite meets, if needed...)
  $x = \bigvee\{a \in S \mid a \leq x\}$
- cover $(S, \lhd)$
  intuition: $a \lhd U \equiv a \leq \bigvee U$
- ΩL is recovered as the "saturated" ($\lhd$-closed) subsets
  or as a (suplattice) quotient of $\mathrm{Pow}(S)$ up to "cover"
- (essentially, a cover is Grothendieck topology on a pre-ordered set)

Predicative pointfree Topology

(in a nutshell)

Leitmotif

Reduce constructive locale theory at the level of bases...

Maps

- $f : L \to M$
- $f^- : \Omega M \to \Omega L$

  Warning: another common notation for f^- is f^*, but the latter is used in [Sambin, 2025] with a different meaning!

- f corresponds to a Formal Map...

Formal Maps

A formal map $r : (S_1, \lhd_1) \to (S_2, \lhd_2)$ is a relation between the two bases (up to a certain equivalence relation) such that

$$\frac{a \lhd_2 V}{r^- \{a\} \lhd_1 r^- V}$$

and r^- preserve finite meets (up to cover).

Here $r^- V := \{x \in S_1 \mid \exists v \in V . r(x, v)\}$ is the usual (set-theoretic) inverse image.

Points

- $p : 1 \to L$
  $\{a \in L \mid p^{-}(a) \ \mathit{true}\}$ = completely-prime filter in ΩL

- $\alpha : (\{*\}, \in) \to (S, \lhd)$
  simply a special kind of subset $\alpha \subseteq S$

  (sometimes we write $\alpha \Vdash a$ instead of $a \in \alpha$)

Open maps

- f^- always has a right adjoint $\forall_f$ (alternative notation: f_*)

$$\boxed{f^- \dashv \forall_f}$$

- f is **open** $\equiv$ and it maps open to open

  $\equiv$ f^- has left adjoint $\boxed{\exists_f \dashv f^-}$

  that satisfies Frobenius reciprocity

$$\begin{tikzcd} a \arrow[r, hook] \arrow[dr] & L \arrow[r, "f"] & M \\ & \exists_f(a) \arrow[ur, hook] & \end{tikzcd}$$

$$\exists_f(x \wedge f^- y) = \exists_f(x) \wedge y$$

  $\equiv$ f^- preserves arbitrary meets and implication

  (f^- is a cHa homomorphism)

Open maps in Formal Topology

- The category of Suplattices (= complete join-semilattices) with bases can be presented as "basic" covers with continuous relations.
- A formal map $r : (S_1, \triangleleft_1) \to (S_2, \triangleleft_2)$ is open if there exists a continuous relation $s : (S_2, \triangleleft_2) \to (S_1, \triangleleft_1)$ such that...

Overtness

- L is **overt** (aka open as a locale) if $L \xrightarrow{!} 1$ is <u>open</u>

- $(S, \lhd, Pos)$ cover with **positivity predicate** ($=$ formal topology in the original 1987's sense):

$$\frac{a \in Pos \qquad a \lhd U}{Pos \between U} \qquad a \lhd \{a\} \cap Pos$$

or, equivalently, $a \lhd \{b \in Pos \mid b \lhd a\}$

where $X \between Y \equiv (\exists a \in S)(a \in X \cap Y)$.

Subobjects

$$(S, \lhd') \stackrel{i}{\hookrightarrow} (S, \lhd)$$

- Given by larger covers $\lhd \subseteq \lhd'$:

$$\frac{a \lhd U}{a \lhd' U}$$

- Correspond to nuclei:

$$a \lhd' U \equiv a \leq j\left(\bigvee U\right)$$

- Inclusion is given equality:

$$i(a, b) \text{ iff } a = b \qquad i^{-} U = U$$

See [Vickers, 2007] for more on sublocales in formal topology...

Images (and the surjection-inclusion factorization)

$$\begin{tikzcd} L \arrow[r, "f"] \arrow[dr, two heads] & M \\ & f[L] = M_j \arrow[u, hook] \end{tikzcd} \qquad \begin{tikzcd} (S_1, \lhd_1) \arrow[r, "r"] \arrow[dr, two heads] & (S_2, \lhd_2) \\ & (S_2, \lhd_2') \arrow[u, hook] \end{tikzcd}$$

$$j = \forall_f \circ f^-$$

$$a \lhd_2' U \equiv r^- a \lhd_1 r^- U$$

Fact:

the image of an overt topology is overt with $Pos'(a) \equiv Pos \between r^- a$

Intersections

$$\begin{tikzcd} (S, \overset{?}{\lhd'}) \arrow[r, hook] \arrow[d, hook] & (S, \lhd_2) \arrow[d, hook] \\ (S, \lhd_1) \arrow[r, hook] & (S, \lhd) \end{tikzcd}$$

$\lhd'$ = <u>least cover</u> such that $\lhd_1 \subseteq \lhd'$ and $\lhd_2 \subseteq \lhd'$

Intersection

Defined by induction...

provided that $\lhd_1$ and $\lhd_2$ are <u>inductively generated...</u>

“*The least cover such that* . . .

- [Coquand et al., 2003]
- Johnstone’s C-ideals on a site $(S, \wedge, C)$

- In many cases, we need our covers $\lhd$ to be inductively generated.
- E.g. when constricting intersections, products, pullbacks,. . .

Strongly dense subobjects

$$L_j \cong (S, \lhd') \hookrightarrow (S, \lhd) \cong L \xrightarrow{!} 1$$

- **dense**: $\quad j(\bot) \leq \bot \qquad \dfrac{a \lhd' \emptyset}{a \lhd \emptyset}$
- **strongly dense**: $\quad j(!^{-}p) \leq !^{-}p \qquad \dfrac{a \lhd' P}{a \lhd P}$

  for every truth value $p \in \Omega$ and $P = \{x \in S \mid p\}$

Strong density + overtness

[Johnstone, 1989, Fox, 2005]

1. If $\lhd'$ is strongly dense in $\lhd$, then

$$\lhd' \text{ } overt \;\Leftrightarrow\; \lhd \text{ } overt$$

   and, in that case, $Pos' = Pos$.

2. If $\lhd$ and $\lhd'$ are both overt, then

$$\lhd' \text{ } strongly\ dense \;\Leftrightarrow\; Pos = Pos' \;\Leftrightarrow\; Pos \subseteq Pos'$$

Strong density for a formal topologies

$(S, \lhd', Pos') \hookrightarrow (S, \lhd, Pos)$

dense: $a \lhd' \emptyset \Rightarrow a \lhd \emptyset$

strongly dense: $Pos(a) \Rightarrow Pos'(a)$

$$\begin{array}{rc} \text{CLASS:} & a \lhd' \emptyset \Rightarrow a \lhd \emptyset \\ \text{iff} & \neg Pos'(a) \Rightarrow \neg Pos'(a) \\ \text{iff} & Pos(a) \Rightarrow Pos'(a) \\ & (\text{strong density} \equiv \text{density}) \end{array}$$

Weak Closure of $(S, \lhd') \hookrightarrow (S, \lhd)$

$(S, \overline{\lhd'})$ = largest sublocale s.t. $\boxed{(S, \lhd') \hookrightarrow (S, \overline{\lhd'}) \text{ strongly dense}}$

$$(S, \lhd') \hookrightarrow (S, \overline{\lhd'}) \hookrightarrow (S, \lhd)$$

weak closure $\hookrightarrow$ closure
closed $\Rightarrow$ weakly closed

CLASS: weak closure = closure

$\overline{\lhd'}$

= least cover with $\lhd \subseteq \overline{\lhd'} \subseteq \lhd'$ s.t. ...

Weak Closure in the OVERT case

$$(S, \lhd', Pos') \hookrightarrow (S, \overline{\lhd'}) \hookrightarrow (S, \lhd)$$

$\overline{\lhd'}$ = least cover with $\lhd \subseteq \overline{\lhd'} \subseteq \lhd'$ s.t. $\boxed{(S, \overline{\lhd'}) \text{ is overt w.r.t. } Pos'}$.

$\overline{\lhd'}$ = least cover s.t.

$$\lhd \subseteq \overline{\lhd'} \qquad a \overline{\lhd'} (\{a\} \cap Pos')$$

(provided that $\lhd$ is inductively generated)

Overt, Weakly Closed Sublocales

- $H \subseteq S$ is splitting if:

$$\frac{a \in H \quad a \lhd U}{H \between U}$$

($H \cong$ c.p. semi-filter $\cong$ suplattice homomorphism $\Omega L \to \Omega$)

- Let $\lhd^H$ be the least cover s.t.

$$\lhd \subseteq \lhd^H \qquad a \lhd^H (\{a\} \cap H)$$

Overt, weakly closed =

$$(S, \lhd^H, H)$$

for some splitting H.

Products = tensor products of frames

$$(S_1, \lhd_1) \times (S_2, \lhd_2) \;=\; (S_1 \otimes S_2, \lhd_\otimes)$$

- $S_1 \otimes S_2 = \{a \otimes b \mid a \in S_1, b \in S_2\}$
- Intuition: $a \otimes b$ is an "open rectangle"
- $(a_1 \otimes b_1) \wedge (a_2 \otimes b_2) = (a_1 \wedge a_2) \otimes (b_1 \wedge b_2)$
- $\lhd_\otimes$ is the cover generated by:

$$\frac{a \lhd_1 U \quad b \lhd_2 V}{a \otimes b \;\lhd_\otimes\; U \otimes V}$$

where $U \otimes V = \{u \otimes v \mid u \in U, v \in V\}$

[Coquand et al., 2003, Maietti and Valentini, 2004]

Products

$$
\begin{tikzcd}
 & L_1 \\
M \arrow[ur, "f"] \arrow[r, dotted, "{(f,g)}"] \arrow[dr, "g"'] & L_1 \times L_2 \arrow[u, "\pi_1"'] \arrow[d, "\pi_2"] \\
 & L_2
\end{tikzcd}
$$

$$\pi_1^-(a) = a \otimes \top$$

$$(f, g)^-(a \otimes b) = f^-(a) \wedge g^-(b)$$

$$\pi_2^-(b) = \top \otimes b$$

Overtness of Product (and open projections)

$(S_1 \otimes S_2, \lhd_\otimes)$ is overt iff both $(S_1, \lhd_1)$ and $(S_2, \lhd_2)$ are overt

and in that case

- $Pos(a \otimes b) \Leftrightarrow Pos(a) \;\&\; Pos(b)$
- the projections π_1 and π_2 are open maps
  $\exists_{\pi_1}(a \otimes b) = \{x \mid \exists y \in Pos,\ x \otimes y \lhd a \otimes b\}$

Localic Group Structure

A *localic group* is a group object in **Loc**:

$$L \times L \xrightarrow{m} L \qquad \textit{multiplication}$$

$$1 \xrightarrow{e} L \qquad \textit{unit}$$

$$L \xrightarrow{n} L \qquad \textit{inverse}$$

satisfying the usual group laws

- Associativity
- Unit laws
- Inverse laws

Associativity

Intuition: $(x \cdot y) \cdot z = x \cdot (y \cdot z)$

$$\begin{tikzcd} L \times L \times L \arrow[r, "m \times id"] \arrow[d, "id \times m"'] & L \times L \arrow[d, "m"] \\ L \times L \arrow[r, "m"'] & L \end{tikzcd}$$

$$m(m(x, y), z) = m(x, m(y, z))$$

$(x, y, z : X \to L)$

As usual, I'm identifying $L \times (L \times L)$ and $(L \times L) \times L$.

Unit Laws

Intuition: $x \cdot e = x = e \cdot x$

$$\begin{tikzcd} L \arrow[r, "(e{,}id)"] \arrow[d, "(id{,}e)"'] \arrow[dr, "id"] & L \times L \arrow[d, "m"] \\ L \times L \arrow[r, "m"'] & L \end{tikzcd}$$

$$m(x, e) = x = m(e, x)$$

Here, I'm identifying $1 \xrightarrow{e} L$ with the corresponding constant map $X \xrightarrow{!} 1 \xrightarrow{e} L$.

Remark: m is split epi (= injective map of frames)

Inverse Laws

Intuition: $x \cdot x^{-1} = e = x^{-1} \cdot x$

$$\begin{tikzcd}
L \arrow[r, "(n{,}id)"] \arrow[d, "(id{,}n)"'] \arrow[dr, "e"] & L \times L \arrow[d, "m"] \\
L \times L \arrow[r, "m"'] & L
\end{tikzcd}$$

$$m(x, n(x)) = e = m(n(x), x)$$

I'm identifying $1 \xrightarrow{e} L$ with the corresponding constant map $L \xrightarrow{!} 1 \xrightarrow{e} L$.

Inverse and multiplication of sublocales

$$\begin{tikzcd} L_j \arrow[r, hook] \arrow[drr, two heads] & L \arrow[r, "n"] & L \\ & & L_j^{-1} \arrow[u, hook] \end{tikzcd}$$

$$\begin{tikzcd} L_1 \times L_2 \arrow[r, hook] \arrow[drr, two heads] & L \times L \arrow[r, "m"] & L \\ & & L_1 \cdot L_2 \arrow[u, hook] \end{tikzcd}$$

Inverse and multiplication are open maps

- $n^2 = id$

  in particular, n is an open map

  and the inverse of an open sublocale is open

  $$a^{-1} = n^- a$$

  (the inverse of the open sublocale a is the open sublocale $n^- a$)
- m is an open map

The multiplication of two opens is open

$$\begin{tikzcd} (a \otimes b) \arrow[r, hook] \arrow[drr] & L \times L \arrow[r, "m"] & L \\ & & a \cdot b \arrow[u, hook] \end{tikzcd}$$

$$a \cdot b = \exists_m (a \otimes b)$$

Multiplication is open (proof sketch)

$\alpha := (m, n \circ \pi_2)$

$$\alpha : L \times L \to L \times L$$

$$\alpha(x, y) = (m(x, y), n(y))$$

- $\alpha^2 = id$
- α is open

- $m = \pi_1 \circ \alpha$
- m is open

$$\exists_m = \exists_{\pi_1} \circ \alpha^-$$

A new base for a localic group...

- Given a "group" $((S, \lhd), m, n, e)$
- construct a new base S' closed under $(\_)^{-1}$ and $(\_) \cdot (\_)$ (and a corresponding new cover $\lhd'$)
- and, then, write the group laws as suitable conditions on $(S', \lhd')$

$$\big(S, \lhd, Pos, \cdot, (\_)^{-1}, e\big)$$

- $(a^{-1})^{-1} =_{\lhd} a \qquad a \cdot (b \cdot c) =_{\lhd} (a \cdot b) \cdot c$
- $Pos(a^{-1}) \iff Pos(a) \qquad Pos(a \cdot b) \iff Pos(a) \ \& \ Pos(b)$
- $a \lhd U \ \& \ b \lhd V \ \Rightarrow \ a \cdot b \lhd U \cdot V$

- $Pos(a) \Rightarrow e \Vdash a \cdot a^{-1}$
- $e \Vdash b \Rightarrow a \lhd a \cdot b \qquad e \Vdash a \Rightarrow b \lhd a \cdot b$
- $c \lhd \{b \mid \exists a(e \Vdash a \ \& \ a \cdot b \lhd c)\} \qquad c \lhd \{a \mid \exists b \in E(e \Vdash b \ \& \ a \cdot b \lhd c)\}$
- $e \Vdash b \Rightarrow \top \lhd \{a \mid a \cdot a^{-1} \lhd b \ \& \ a^{-1} \cdot a \lhd b\}$

Group laws at the level of basic opens

Remark

Conditions like these

$$e \Vdash a \Rightarrow b \lhd a \cdot b \qquad c \lhd \{b \mid \exists a(e \Vdash a \;\&\; a \cdot b \lhd c)\}$$

are perhaps less transparent than

$$m(e, id) = id$$

BUT they can be given without mentioning categorical products!
(They make sense, in principle, also for non inductively generated covers.)

Overt subgroup are weakly closed

Recall that

[Johnstone, 1989]

Every overt localic subgroup of an overt localic group is weakly closed.

So every subgroup of $(S, \lhd, Pos, \cdot, (\_)^{-1}, e)$ is of the form

$$(S, \lhd^H, H)$$

with H a suitable splitting subset...

Subgroups

$(S, \lhd^H, H)$ is a subgroup if

- $e \Vdash a \Rightarrow a \in H$
- $H \cdot H := \{a \cdot b \mid a, b \in H\} \subseteq H$
- $H^{-1} := \{a^{-1} \mid a \in H\} \subseteq H$

Sambin's Positive Topologies

$$(S, \lhd, \ltimes)$$

$$\frac{a \ltimes U}{a \in U} \qquad \frac{a \ltimes U \quad (\forall u \in U)(u \ltimes V)}{a \ltimes V} \qquad \boxed{\frac{a \lhd U \qquad a \ltimes V}{(\exists u \in U)(u \ltimes V)}}$$

that is, $\boxed{\{a \mid a \ltimes V\} \text{ is splitting}}$

A positivity relation $\ltimes$ on $(S, \lhd)$

corresponds to a (suitable) collection of splitting subsets...

On positivity relations

$$(S, \lhd, \ltimes)$$

$\ltimes$ $=$ a (suitable) collection of splitting subsets

$\cong$ a collection of overt, weakly closed sublocales [C.-Vickers, 2016]

- $\ltimes_{max} \cong$ ALL overt, weakly closed sublocales of S

$\ltimes_{max}$

If $\lhd$ is inductively generated, then $\ltimes_{max}$ is generated by co-induction.

See Appendix by Martin-Löf and Sambin in [Sambin, 2025]

"Generating Positivity by Coinduction"

(Martin-Löf-Sambin)

$\{a \mid a \ltimes_{max} V\}$ is the largest splitting subset contained in V, that is,

$$\frac{a \ltimes_{max} V}{a \in V} \qquad\qquad \frac{a \lhd U \qquad a \ltimes_{max} V}{(\exists u \in U)(u \ltimes_{max} V)}$$

$$\frac{a \in P \subseteq V \qquad \forall a \forall U (a \lhd U \;\&\; a \in P \implies U \between P)}{a \ltimes_{max} V}$$

(predicative, if $\lhd$ inductively generated).

Note: $\frac{a \ltimes_{max} U \quad (\forall u \in U)(u \ltimes_{max} V)}{a \ltimes_{max} V}$ follows by coinduction.

The Positivity of Subgroups

- Given $(S, \lhd, Pos, \cdot, (\_)^{-1}, e)$
  with $\lhd$ inductively generated
- define $\ltimes$ by co-induction as

$$\frac{a \in H \subseteq V \qquad H \textit{ splitting} \qquad \{a \mid e \Vdash a\} \subseteq H = H^{-1} = H \cdot H}{a \ltimes V}$$

  that is, consider only the splitting subsets H that correspond to subgroups!

- Subgroups $\rightarrow$ Giovanni's formal closed subsets of $(S, \lhd, \ltimes)$

References

Coquand, T., Sambin, G., Smith, J., and Valentini, S. (2003).
Inductively generated formal topologies.
Annals of Pure and Applied Logic.

Fox, C. (2005).
Point-set and point-free topology in constructive set theory.
PhD thesis, University of Manchester.

Johnstone, P. T. (1989).
A constructive "closed subgroup theorem" for localic groups and groupoids.
Cahiers de Topologie et Géométrie Différentielle Catégoriques.

Maietti, M. E. and Valentini, S. (2004).
A structural investigation on formal topology: coreflection of formal covers and exponentiability.
Journal of Symbolic Logic.

Sambin, G. (2025).
Positive Topology: A New Practice in Constructive Mathematics.
Oxford University Press.

Vickers, S. (2007).
Sublocales in formal topology.
Journal of Symbolic Logic.

Grazie!