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Constructive satisfiability

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Introduction

Main aim of the present thesis is a *constructive* investigation of some logical notions, namely non-deducibility and satisfiability, which are usually studied by means of non-effective methods. We use (pointfree) topology to characterize these notions semantically and apply these studies to first order intuitionistic logic as formalized in Gentzen's LJ sequent calculus.

When adopting a classical metatheory, a soundness and completeness theorem is all one needs to study a specific logical system. Indeed non-deducibility becomes equivalent to validity in *not all* interpretations and hence, under a classical reading, to the existence of a counter-model. Constructively, a negative assertion cannot generally be given a positive value instead. So non-deducibility needs an independent treatment and a direct semantics specific for it. Similar motivations arise when dealing with satisfiability, which is classically defined as either a negative (non-deducibility of the negation) or an impredicative (existence of a model) notion.

In chapter 1, after having recalled the LJ system, we give a short introduction to pointfree topology, following Sambin's approach (see [23]), together with a few new concepts needed in the following.

Chapter 2 presents a slight modification of a constructive soundness and completeness theorem for LJ firstly appeared in [22]; here is given in a form more similar to that of [9] (also see [7]).

The problem of doing a constructive analysis of non-deducibility is solved in chapter 3, where we follows some suggestions from Mostowski's works (see [15]).

Chapter 4 is devoted to a constructive and positive semantic definition of satisfiability which is independent from deducibility, in the sense that satisfiability of a formula is not defined as non deducibility of its negation. Then we propose a syntactic counterpart for satisfiability, that is a "co-inductive" calculus which is dual to that for deducibility in several aspects. For instance the notion of refutation of an hypothesis of satisfiability corresponds to that of proof of the sequent expressing that the set of formulae under consideration is inconsistent.

Finally, chapter 5 gives an application to modal logic: a logical system for tense logic is studied which seems minimal among the constructive ones.

Of course, the choice for a constructive setting needs some motivations too, as well as some explanations on the meaning we attributes to the adjective "constructive". Here "constructive" is a synonym for "that can be formalized in Maietti-Sambin's Minimal Type Theory" (see [13]). This foundational theory is characterized as the common core of (all?) other known ones. In other words, all the proofs we give hold whatever the foundational standpoint from which one looks at them is. On the contrary, not every statement one could expect, e.g. from a classical point of view, actually has got a constructive proof; thus some deductions which are standard in other foundations are never performed here. Of course, if one wants, one can start from our results and managing all transformations which are possible in some particular stronger foundation. Compatibility with other foundations is not the only advantage of being constructive, of course; another one is the possibility to read every proof as an algorithm (at least in principle). Compatibility with other foundations implies that in order to be able to read the present thesis, it is not needed at all to know Maietti-Sambin's Minimal Type Theory in its details. In fact, in order to understand what we mean by "constructive" it is enough to keep in mind some simple facts:

- we use the word "set" only when we are able to give inductive rules for constructing its elements; otherwise we use the word "collection"; for instance, natural numbers form a set whose power*set* is only a collection; similarly, a quotient of a set trough an equivalence relation is generally no longer a set itself, but only a collection;

- quantification over a set (or over a subset) is allowed; on the contrary, impredicative quantifications over collection (like the totality of the subsets of a set) are never used;

- the underlying logic is intuitionistic; so no proof by contradiction or by cases is present.

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Chapter 1

Basic notions and definitions

1.1 Gentzen's LJ sequent calculus

In order to investigate metatheoretical properties of a logic, a formal system for that logic is needed. The one we use for first order intuitionistic logic is Gentzen's LJ sequent calculus. Because of our constructive standpoint, languages we will consider will be countable (effectively enumerable in fact); thus we will use the word "language" for a list

$$\&, \vee, \rightarrow, \perp, \forall, \exists; (,); x_1, \ldots, x_n, \ldots; f_1, \ldots, f_m; R_1, \ldots, R_t$$

where the x_i's are variables, the f_i's are function symbols (possibly of arity 0, i.e. constants) and the R_i's are relation symbols. Of course, & is the conjunction, $\vee$ the disjunction, $\rightarrow$ the implication, $\perp$ the false proposition, $\forall$ and $\exists$ the universal and existential quantifiers, respectively. Negation is not primitive: $\neg\varphi$ is defined as $\varphi \rightarrow \perp$, for each formula φ. The set of all terms, written Trm, and the set of formulae, written Frm, are constructed in the usual inductive way. We will use φ, ψ and γ for formulae, Γ and Γ' for (possibly empty) finite lists of formulae. An object like $\Gamma \vdash \varphi$ (read "Γ yields φ") is called a "sequent" and its intended meaning is: "if all the formulae in Γ are true then so is φ". Here is the list of rules of LJ.

$$\frac{}{\varphi \vdash \varphi}$$

$$\frac{\Gamma \vdash \varphi \quad \Gamma \vdash \psi}{\Gamma \vdash \varphi \,\&\, \psi} \qquad \frac{\Gamma, \varphi \vdash \gamma}{\Gamma, \varphi \,\&\, \psi \vdash \gamma} \quad \frac{\Gamma, \psi \vdash \gamma}{\Gamma, \varphi \,\&\, \psi \vdash \gamma}$$

$$\frac{\Gamma, \varphi \vdash \gamma \quad \Gamma, \psi \vdash \gamma}{\Gamma, \varphi \vee \psi \vdash \gamma} \qquad \frac{\Gamma \vdash \varphi}{\Gamma \vdash \varphi \vee \psi} \quad \frac{\Gamma \vdash \psi}{\Gamma \vdash \varphi \vee \psi}$$

$$\frac{\Gamma, \varphi \vdash \psi}{\Gamma \vdash \varphi \to \psi} \qquad \frac{\Gamma \vdash \varphi \quad \psi, \Gamma' \vdash \gamma}{\Gamma, \varphi \to \psi, \Gamma' \vdash \gamma}$$

$$\frac{\Gamma \vdash \varphi(x)}{\Gamma \vdash \forall x \varphi(x)}(x \text{ not free in } \Gamma) \qquad \frac{\Gamma, \varphi(t) \vdash \psi}{\Gamma, \forall x \varphi(x) \vdash \psi}(t \in \mathrm{Trm})$$

$$\frac{\Gamma, \varphi(x) \vdash \psi}{\Gamma, \exists x \varphi(x) \vdash \psi}(x \text{ not free in } \Gamma, \psi) \qquad \frac{\Gamma \vdash \varphi(t)}{\Gamma \vdash \exists x \varphi(x)}(t \in \mathrm{Trm})$$

$$\frac{\Gamma \vdash \bot}{\Gamma \vdash \varphi}$$

$$\frac{\Gamma \vdash \varphi}{\Gamma, \psi \vdash \varphi} \qquad \frac{\Gamma, \varphi, \varphi \vdash \psi}{\Gamma, \varphi \vdash \psi} \qquad \frac{\Gamma, \varphi, \psi, \Gamma' \vdash \gamma}{\Gamma, \psi, \varphi, \Gamma' \vdash \gamma}$$

$$\frac{\Gamma \vdash \varphi \quad \varphi, \Gamma' \vdash \psi}{\Gamma, \Gamma' \vdash \psi} \text{ cut}$$

As usual, the set of provable sequents is the *smallest* one that is closed under applications of the rules. In other words, that set is defined by induction. On the contrary, we will see that a notion of a *co-inductive calculus* is possible and, in fact, necessary to deal with satisfiability.

Here below we state some fundamental proof-theoretical results we need in the following chapters. A constructive soundness and completeness theorem for LJ is proved in the next chapter.

Theorem 1.1.1 (cut-elimination) *If a sequent $\Gamma \vdash \varphi$ is provable in LJ then it can be proved without any use of the cut rule.*

We will not prove neither this theorem nor the following corollaries whose proofs can be found in [27]; we just want to stress that their proofs actually yield effective methods.

Corollary 1.1.2 (disjunction and existence properties) *If $\vdash \varphi \vee \psi$ is provable then either $\vdash \varphi$ is provable or $\vdash \psi$ is provable.*

If $\vdash \exists x\ \varphi(x)$ is provable then there exists a term t such that $\vdash \varphi(t)$ is provable.

Corollary 1.1.3 (consistency) *The sequent $\vdash \bot$ is* not *provable.*

1.2 A constructive theory for topology

In this section we present a constructive approach to topology; most definitions we adopt have recently been proposed by Giovanni Sambin as an outcome of a wider theory (namely the Basic Picture; see [23]). Our notions surely belong to the field of Formal Topology; very informally, Formal Topology is point-free topology seen through intuitionistic and predicative glasses. With respect to the usual notion of formal topology, the greatest novelty in the theory developed by Sambin is the introduction of a binary positivity relation, written $\ltimes$, needed for a primitive treatment of the topological notion of closure (see [23] for explanations and philosophical justifications). This relation plays the main part in the present thesis; our main discovery is that $\ltimes$ can be used to give a constructive semantics for both non-deducibility and satisfiability.

1.2.1 Subset theory

It is a fact that several foundational approaches towards mathematics exist. The most common one is based upon classical logic and Zermelo-Fraenkel set theory; in particular, the following principles are used: the principle of excluded middle, the power-set axiom and the axiom of choice. We refer to this foundational viewpoint as the classical (and impredicative) one. Besides this, there exist a lot of weaker foundational theories which refute classical logic and accept intuitionistic one instead. Maybe the most used ones are: Topos Theory, which allows the power-set axiom (thus it is impredicative) and Martin-Löf Type Theory, which refutes that axiom but accepts the axiom of choice. It is shown in [13] that Topos Theory and Martin-Löf Type Theory are constructively incompatible in the sense that accepting the principles of both theories leads to classical logic.

We want our work to be valid whatever the foundational standpoint of the reader is; hence we need our mathematics to be formalizable in a “minimalistic” theory which is contained in the intersection of all the other ones. Such a theory exists and has recently been proposed by Sambin and Maietti in [13] where it is called “Minimal Type Theory” (mTT in what follows). This theory can be described as Martin-Löf's theory deprived of the axiom of choice. Hence all definitions and proofs in the present thesis are predicative and make no use of the axiom of choice (and, of course, are based on intuitistically valid arguments).

In this paragraph we give some basic definitions and results about subsets which will be useful for the rest of the thesis. Moreover this paragraph is a concrete example which shows our way of working. The reader should check that all the facts listed below are true with respect to his (her) own foundation. Also we suggest very informally how the formalization in mTT can be done.

Let S be a collection of objects. We say that S is a set when we are able to give rules to generate its elements. A subset $U \subseteq S$ is represented formally by a propositional function, say $U(x)$, with (at most) one free variable ranging over S. Provided $U \subseteq S$ and $a \in S$, we say that a belongs to U, written $a \epsilon U$, if $U(a)$ is true. As usual, we use the standard notation

$$\{a \in S \ : \ U(a)\}$$

for the subset which formally is represented by $U(x)$. We define inclusion between two subsets, say U and V, of the same set S as the validity of the formula:

$$(\forall x \in S)(U(x) \to V(x)).$$

Equality is defined by two-sided inclusion (extensional equality) and each set-theoretical operation is obtained by reflecting the corresponding logical connective. As an example, $U \cap V$ simply is the propositional function $U(x)$ & $V(x)$; moreover $\emptyset$ is just $\bot$. In particular, we put

$$U \to V = \{a \in S \ : \ U(a) \to V(a)\}$$

and write $-U$ instead of $U \to \emptyset$. Finally, following Sambin, we write

$$U \between V$$

to express the existence of an element in $U \cap V$. Note that, from a classical point of view, $U \between V$ is just the same as $U \cap V \neq \emptyset$ but intuitionistically the former is stronger than the latter.

It should be clear that the collection of all subsets of a set S, written $\mathcal{P}S$, is accepted as a set only by impredicative foundations; hence we will never use quantification over it.

Obviously, the ones described above are not the only admissible operations on subsets. As an example, provided I is a set and $\{V_i \ : \ i \in I\}$ is a set-indexed family of subsets of a set S, one can put

$$\bigcup_{i \in I} V_i = \{a \in S \ : \ (\exists i \in I)(a \epsilon V_i)\} \quad \text{and} \quad \bigcap_{i \in I} V_i = \{a \in S \ : \ (\forall i \in I)(a \epsilon V_i)\} \ .$$

Of course, from the point of view of a reader who accepts the power-set axiom, the equations above define just arbitrary unions and intersection, all families being trivially set-based.

All the properties we need about subsets are summarized in the following statement each reader can verify on his own mind.

Proposition 1.2.1 *For any set S, the structure*

$$(\mathcal{P}S, \subseteq, \cap, \bigcup, \to, \emptyset, S)$$

is a complete (distributive) Heyting algebra.

Moreover, the relation $\between$ *satisfies the following items:*

1. $U \between V \Longleftrightarrow V \between U$

2. $U \between V \Longrightarrow (U \cap V) \between V$

3. $(\bigcup_{i \in I} W_i) \between V \Longleftrightarrow$ *there exists* $i \in I$ *such that* $W_i \between V$

for any $U, V, W_i \subseteq S$ *and any set* I.

Here distributivity is thought in the form:

$$\bigcup_{i \in I}(W \cap V_i) = W \cap (\bigcup_{i \in I} V_i)$$

for any $W \subseteq S$ and V_i as before. On the contrary, we cannot accept distributivity of union with respect to infinite intersections, because it is not intuitionistically valid as we will show in corollary 3.3.3 (see also [10] p. 204).

1.2.2 Basic topologies and convergent basic topologies

What here is called "convergent basic topology" is just Sambin's new definition of formal topology. On the contrary, a "basic topology" is a natural and very elegant generalization which is susceptible of several, also non topological, interpretations.

Definition 1.2.2 (basic topology) *A* basic topology *is a triple* $(S, \lhd, \ltimes)$ *where* S *is a set and* $\lhd$ *and* $\ltimes$ *are two relations between elements and subsets of* S *such that the following rules hold:*

$$\frac{a \,\epsilon\, U}{a \lhd U} \text{ reflexivity} \qquad \frac{a \lhd U \quad U \lhd V}{a \lhd V} \text{ transitivity}$$

where $U \lhd V$ *stands for* $(\forall a \,\epsilon\, U)(a \lhd V)$*;*

$$\frac{a \ltimes U}{a \,\epsilon\, U} \text{ co} - \text{reflexivity} \qquad \frac{a \ltimes U \quad \begin{array}{c}[b \ltimes U] \\ | \\ b \,\epsilon\, V\end{array}}{a \ltimes V} \text{ co} - \text{transitivity}$$

$$\frac{a \lhd U \quad a \ltimes V}{U \ltimes V} \text{ compatibility}$$

where $U \ltimes V$ *is* $(\exists b \,\epsilon\, U)(b \ltimes V)$.

A basic topology has got an intended topological interpretation: the set S is thought as a collection of *names* for particular open subsets of a topology (typically, S is a basis for the topology); $a \lhd U$ is read as "each point belonging to the open named a actually lies in some open whose name is in U"; lastly, $a \ltimes U$ means that there exists a point "in" a whose basic neighbourhoods are all in U. Note that in some important cases, as that of the real line, the topology can be presented by a (constructively acceptable) set, while the collection of points cannot.

In a quite natural way, it is possible to define two operators on $\mathcal{P}S$ by $\mathcal{A}U = \{a \in S : a \lhd U\}$ and $\mathcal{J}U = \{a \in S : a \ltimes U\}$. These are a *saturation* (i.e. closure) operator and a *reduction* (i.e. interior) operator, respectively, according to the following.

Definition 1.2.3 *$\mathcal{A}$ is a* saturation *operator on $\mathcal{P}S$ if*

$$\frac{U \subseteq V}{\mathcal{A}U \subseteq \mathcal{A}V} \qquad U \subseteq \mathcal{A}U \qquad \mathcal{A}\mathcal{A}U \subseteq \mathcal{A}U$$

for any $U, V \subseteq S$.

$\mathcal{J}$ is a reduction *operator on $\mathcal{P}S$ if*

$$\frac{U \subseteq V}{\mathcal{J}U \subseteq \mathcal{J}V} \qquad \mathcal{J}U \subseteq U \qquad \mathcal{J}U \subseteq \mathcal{J}\mathcal{J}U$$

for any $U, V \subseteq S$.

Fixed points for $\mathcal{A}$ and $\mathcal{J}$ are called *formal open* and *formal closed* subsets, respectively; their collections are written $Sat(\mathcal{A})$ and $Red(\mathcal{J})$.

Obviously the definition of basic topology can be reformulated in terms of operators. As an example, compatibility becomes

$$U \between \mathcal{J}V \Longleftrightarrow \mathcal{A}U \between \mathcal{J}V.$$

A basic topology does not satisfy any property corresponding to the usual fact that the intersection of two opens has to be open too. Instead, this is achieved in a *convergent* basic topology whose definition is given below.

In every basic topology a preorder can be defined by restricting the cover on singletons; that is:

$$a \leq b \equiv a \lhd \{b\}.$$

As usual we write $\uparrow a$ for $\{b \in S : a \leq b\}$ and put:

$$\downarrow U = \{a \in S : \uparrow a \between U\} \qquad \text{and} \qquad U \downarrow V = (\downarrow U) \cap (\downarrow V)$$

(note that $U{\downarrow}V$ can be seen as the union of all $a{\downarrow}b$ for $a \,\epsilon\, U$ and $b \,\epsilon\, V$, where $a{\downarrow}b$ is a shorthand for $\{a\}{\downarrow}\{b\}$). If one requests S to be a basis for the intended topology then one quite naturally reaches the following definition.

Definition 1.2.4 (topology) *A* convergent basic topology *("topology" in what follows) is a basic topology which satisfies the following additional rule:*

$$\frac{a \lhd U \quad a \lhd V}{a \lhd U{\downarrow}V} \downarrow$$

or, in an equivalent way,

$$(\mathcal{A}U) \cap (\mathcal{A}V) = \mathcal{A}(U{\downarrow}V).$$

If one uses an impredicative foundation then it is possible to reconstruct points through completely prime filters; this idea is followed in the definition of formal point given below (for a clearer explanation of all the definitions given in this section see paragraph 1.2.4).

Definition 1.2.5 (formal point) *A subset $\alpha \subseteq S$ is a* formal point *if it is (formal) closed, inhabited and satisfies:*

$$\frac{a \,\epsilon\, \alpha \quad b \,\epsilon\, \alpha}{a{\downarrow}b \between \alpha} \text{ convergency}$$

for any $a, b \in S$.

When one is only interested in the cover relation, as it happens in the case of the completeness theorem for LJ, it is quite natural to look at $\mathcal{P}S$ modulus $\mathcal{A}$ in the following sense. For any $U, V \subseteq S$, write $U =_{\mathcal{A}} V$ for $\mathcal{A}U = \mathcal{A}V$. It is easily seen that $=_{\mathcal{A}}$ is an equivalence relation; moreover, both the relation $\lhd$ and the operations $\downarrow$ and $\bigcup$ respect $=_{\mathcal{A}}$. The collection $\mathcal{P}S/=_{\mathcal{A}}$ is what we have just called $Sat(\mathcal{A})$; if we endow it with the preorder $\lhd$ and the operations $\downarrow$ and $\bigcup$ what we get is a complete distributive lattice.[1] In an impredicative foundation, it is easily proved that a complete distributive lattice is in fact a Heyting algebra,

[1] Note that, because of our foundational standpoint, the only way to give an infinite family of subsets is to present it as indexed by a set. Thus, completeness of $Sat(\mathcal{A})$ means that the join of a set-indexed family always exists. In particular, implication cannot be defined in the usual impredicative way.

that is, it can be endowed with an implication. Constructively speaking, it is not so easy. However, in the case of $Sat(\mathcal{A})$ one can put:

$$U \to_{\mathcal{A}} V \equiv \{a \in S : \ a{\downarrow}U \lhd V\}$$

and then prove the expected adjunction between meet (i.e. $\downarrow$) and implication. Summing up we get the following.

Proposition 1.2.6 *If $\mathcal{A}$ is the saturation operator of a topology then $Sat(\mathcal{A})$ is a complete Heyting algebra. In other words, the following hold:*

1. $U \lhd U$;
2. $(U \lhd V)$ & $(V \lhd W) \longrightarrow (U \lhd W)$;
3. $(U \lhd V)$ & $(V \lhd U) \longrightarrow (U =_{\mathcal{A}} V)$;
4. $(U \lhd V)$ & $(U \lhd W) \longleftrightarrow (U \lhd V{\downarrow}W)$;
5. $(\bigcup_{i\in I} V_i) \lhd W \longleftrightarrow (\forall i \in I)(V_i \lhd W)$;
6. $U{\downarrow}(\bigcup_{i\in I} V_i) =_{\mathcal{A}} \bigcup_{i\in I}(U{\downarrow}V_i)$;
7. $(U{\downarrow}V \lhd W) \longleftrightarrow (U \lhd V \to_{\mathcal{A}} W)$.

Lastly, it is easily checked that the infinite meet of a set-indexed family $\{V_i\}_{i\in I}$ is given by

$$\{a \in S : \ (\forall i \in I)(a \lhd V_i)\} = \bigcap_{i\in I} AV_i.$$

1.2.3 Generated basic topologies

There exists a general procedure for "generating" basic topologies, as described by Martin-Löf and Sambin in [23]. Let S be a set, $I(a)$ be a set for each $a \in S$ and $C(a,i) \subseteq S$ for each $a \in S$ and $i \in I(a)$. Think of $C(a,i)$ as a subset covering a. We want to generate the smallest cover relation satisfying $a \lhd C(a,i)$ and, at the same time, we want to characterize the largest positivity relation compatible with that cover. All this is possible by means of the following inductive generation of $\lhd$ and co-inductive generation of $\ltimes$.

$$
\frac{a \,\epsilon\, U}{a \lhd U} \qquad \frac{i \in I(a) \quad C(a,i) \lhd U}{a \lhd U} \qquad \frac{a \lhd U \quad \begin{array}{c}[b \,\epsilon\, U]\\ \vdots \\ b \,\epsilon\, P\end{array} \quad \begin{array}{c}[i \in I(b),\ C(b,i) \subseteq P]\\ \vdots \\ b \,\epsilon\, P\end{array}}{a \,\epsilon\, P}
$$

$$
\frac{a \ltimes U}{a \,\epsilon\, U} \qquad \frac{i \in I(a) \quad a \ltimes U}{C(a,i) \ltimes U} \qquad \frac{a \,\epsilon\, P \quad \begin{array}{c}[b \,\epsilon\, P]\\ \vdots \\ b \,\epsilon\, U\end{array} \quad \begin{array}{c}[i \in I(b),\ b \,\epsilon\, P]\\ \vdots \\ C(b,i) \between P\end{array}}{a \ltimes U}
$$

A very important example of generated topology will be given in the next chapter and will be used both in the completeness theorem and in the semantical treatments of non-deducibility and satisfiability: it is Coquand's topology on the set of formulae, actually a canonical model for intuitionistic logic.

Surely, the most important topological space is the real line. Of course, the collection of all the classical real numbers does not form a (constructive) set. On the contrary, its natural topology can be presented via a small basis, namely the set of all open intervals with rational endpoints. Thus, let $S = \mathbb{Q} \times \mathbb{Q}$ and consider the axiom set

$$
I((a,b)) = \{((q_1,q_2),(q_3,q_4)) \in S \times S : \ q_1 < a < q_3 < q_2 < b < q_4\} \cup \{*\}
$$

$$
C((a,b),((q_1,q_2),(q_3,q_4))) = \{(q_1,q_2),(q_3,q_4)\}
$$
$$
C((a,b),*) = \{(q,p) \in S : \ a < q < p < b\}.
$$

The resulting basic topology (actually a convergent one), written $\mathcal{R}$, is the constructive version of the real line, in the sense that its formal points correspond to the real numbers (see [7]).[2]

1.2.4 Representable basic topologies

Let X and S be two sets and r a binary relation between them. We call (X,r,S) a *basic pair*. Put:

$$
rx = \{a \in S : \ xra\} \quad \textit{and} \quad r^{-}a = \{x \in X : \ xra\}.
$$

Next define four operators on subsets by:

$$
a \,\epsilon\, rD \equiv r^{-}a \between D \qquad x \,\epsilon\, r^{-}U \equiv rx \between U
$$

[2]The ideas underlying this inductive generation of the real line are due to A. Joyal.

$$x \,\epsilon\, r^*U \equiv rx \subseteq U \qquad a \,\epsilon\, r^{-*}D \equiv r^-a \subseteq D$$

for any $D \subseteq X$ and $U \subseteq S$. The following characterizing properties can be proved:

- r is left adjoint of r^* (r^* is right adjoint of r), written $r \dashv r^*$, that is:

  $$rD \subseteq U \equiv D \subseteq r^*U$$

  for any $D \subseteq X$ and $U \subseteq S$;

- $r^- \dashv r^{-*}$;

- r and r^- are *symmetric*, written $r \cdot\!|\!\cdot r^-$, that is:

  $$rD \between U \equiv D \between r^-U$$

  for any D and U as before.

As a consequence of the above items, several simple facts hold. Here we list some of the more interesting ones:

- r, r^-, r^* and r^{-*} are monotonic;
- $rr^*r = r \quad , \quad r^*rr^* = r^* \quad , \quad r^-r^{-*}r^- = r^- \quad , \quad r^{-*}r^-r^{-*} = r^{-*} \quad ;$
- r and r^- distribute over (arbitrary) unions;
- r^* and r^{-*} distribute over (arbitrary) intersections;
- $(X,\ r^*r,\ r^-r^{-*})$ and $(S,\ r^{-*}r^-,\ rr^*)$ are basic topologies.

In order to see the link with topology, think of X as the set of points of a topological space and S as a set of names for (basic) open subsets. Read xra as "the point x lies in the basic open whose name is a"; thus r^-a is just the open subsets whose name is a. It is easy to see that r^-r^{-*} and r^*r read just like the standard topological interior and closure operators, written *int* and *cl*, respectively. By symmetry, $\mathcal{A} = r^{-*}r^-$ and $\mathcal{J} = rr^*$ also are a saturation and a reduction operator respectively and they are linked by compatibility. It is quite surprising to find out that there is a lattice isomorphism between $Red(int)$ (that is the lattice of open subsets) and $Sat(\mathcal{A})$; this is why the elements of $Sat(\mathcal{A})$ are called "formal open subsets". Similarly, for *cl* and $\mathcal{J}$. Moreover, the functors providing the two isomorphisms (and their inverses) are just the four operators r^{-*}, r^-, r and r^* defined above. The basic topology $(S, \mathcal{A}, \mathcal{J})$ is said to be *represented* by the basic pair (X, r, S).

What we have briefly described just above is the beginning of a wider theory by Giovanni Sambin called “The Basic Picture”. For a complete treatment such as for mathematical and philosophical implications of this theory see [23]. Representable basic topologies are taken as a paradigm every time one wants to give a definition on the formal side. As an example, the best way for describing a point, say x, from the viewpoint of S is to look at the collection of its basic neighbourhoods, i.e. rx. Thus, the properties defining a formal points are simply those characterizing a subset of the form rx. As another example, we want to describe the genesis of the definition of convergent basic topology. Let $\{r^-a : a \in S\}$ be a basis; thus, for any $a, b \in S$, $r^-a \cap r^-b$ is an open subset. It is an easy exercise in topology to verify that every open subset is the union of all basic open subsets contained in it. Thus

$$r^-a \cap r^-b = \bigcup_{r^-c \subseteq r^-a \cap r^-b} r^-c.$$

Now, $r^-c \subseteq r^-a \cap r^-b$ is the same as $(r^-c \subseteq r^-a)$ & $(r^-c \subseteq r^-b)$, i.e. $(c \lhd a)$ & $(c \lhd b)$; but this means exactly that c belongs to $a{\downarrow}b$. So

$$r^-a \cap r^-b = r^-(a{\downarrow}b)$$

(remember that r^- distributes over unions). As a consequence, the intersection of two open subsets is open too:

$$\begin{array}{rcccl} r^-U \cap r^-V & = & (r^- \bigcup_{a\epsilon U}\{a\}) \cap (r^- \bigcup_{b\epsilon V}\{b\}) & = & \\ & = & (\bigcup_{a\epsilon U} r^-a) \cap (\bigcup_{b\epsilon V} r^-b) & = & \\ & = & \bigcup_{a\epsilon U,\ b\epsilon V}(r^-a \cap r^-b) & = & \\ & = & \bigcup_{a\epsilon U,\ b\epsilon V} r^-(a{\downarrow}b) & = & \\ & = & r^- \bigcup_{a\epsilon U,\ b\epsilon V}(a{\downarrow}b) & = & r^-(U{\downarrow}V). \end{array}$$

Finally, we can apply the isomorphism between $Red(int)$ and $Sat(\mathcal{A})$, namely the operator r^{-*}, to obtain

$$\mathcal{A}U \cap \mathcal{A}V = \mathcal{A}(U{\downarrow}V).$$

An important example of representable topology, which we will use in several cases in what follows, is given by the rational line. Let $\mathcal{Q}$ be the basic pair where $X = \mathbb{Q}$ is the set of rationals, S is $\mathbb{Q} \times \mathbb{Q}$ and

$$q\ r\ (a, b) \equiv a < q < b$$

for any $q, a, b \in \mathbb{Q}$ ($<$ is the standard order on the rationals). Write $\mathcal{Q}$ for the basic topology induced on S. It is possible to prove that $\mathcal{Q}$ is convergent and, moreover, its formal points are exactly all the subsets of the kind rq, $q \in \mathbb{Q}$. Thus

$\mathcal{Q}$ is exactly the topology of the rational numbers. Note that $\lhd_{\mathcal{Q}}$ strictly contains $\lhd_{\mathcal{R}}$, in the sense that

$$(a,b) \lhd_{\mathcal{R}} U \Longrightarrow (a,b) \lhd_{\mathcal{Q}} U$$

but the contrary generally fails.[3]

1.3 Additional topological topics

1.3.1 Bi-convergent basic topologies

Much of the novelty in Sambin's approach stands in the primitive treatment of the topological notion of closure; closed subsets are defined in terms of the positivity relation and not as complement of the open ones. We said that the $\downarrow$-rule arises when one wants the intersection of two open subsets to be open too; similarly, one could want $Sat(cl)$ to be closed under binary union (note that closure under arbitrary intersection always holds). Here we show how to express this property without mentioning points. In order to find the right definition, let us consider a topological space (X, r, S) and use classical reasonings. It is easy to prove that every (concrete) closed subset is the image under r^* of some subset of S; moreover, r^* is classically equivalent to $-r^-{-}$. A bit of calculation is needed:

$$\begin{array}{rcccl} r^*U \cup r^*V & = & (-r^- - U) \cup (-r^- - V) & = & \\ & = & -((r^- - U) \cap (r^- - V)) & = & \\ & = & -r^-((-U){\downarrow}(-V)) & = & r^* - ((-U){\downarrow}(-V)). \end{array}$$

Classically speaking, $-((-U){\downarrow}(-V))$ is the same as $U \Downarrow V$ if one puts:

$$U \Downarrow V = (\Downarrow U) \cup (\Downarrow V) \qquad \text{and} \qquad \Downarrow U = \{a \in S : \uparrow a \subseteq U\}.$$

Now we have only to *transport* the above equation on the formal side. In other words, we can apply the isomorphism between $Sat(cl)$ and $Red(\mathcal{J})$, namely r, to obtain

$$(\mathcal{J}U) \cup (\mathcal{J}V) = \mathcal{J}(U \Downarrow V)$$

which can be rewritten as

$$\frac{a \ltimes U \Downarrow V}{(a \ltimes U) \vee (a \ltimes V)} \Downarrow$$

(the other direction always holds). Finally, we can give the following.

[3]Let i be a fixed irrational number in (a,b), r_q a rational number s.t. $0 < r_q < |i - q|$ for any $q \in (a,b) \cap \mathbb{Q}$ and $I_q = (q - r_q, q + r_q)$. Thus, $(a,b) \cap \mathbb{Q} \subseteq \bigcup I_q$ so $(a,b) \lhd_{\mathcal{Q}} \{I_q : q \in (a,b) \cap \mathbb{Q}\}$, but the same does not hold for $\lhd_{\mathcal{R}}$ because i does not belong to anyone of the I_q's.

Definition 1.3.1 *A convergent basic topology is called a* bi-convergent basic topology *if it satisfies the* $\Downarrow$*-rule.*

An example of bi-convergent basic topology is given by $\mathcal{Q}$, the topology of the rational numbers. Because $\mathcal{Q}$ is representable, it is enough to prove that

$$rq \subseteq (\Downarrow U \cup \Downarrow V) \Longrightarrow (rq \subseteq U) \vee (rq \subseteq V)$$

(to obtain bi-convergency use the adjunction $r \dashv r^*$ and then apply the operator r).

Firstly, we prove that any couple of basic neighbourhoods of q is contained either in U or in V. In fact, let (a, b) and (c, d) be two elements belonging to rq and consider $(\max\{a, c\}, \min\{b, d\})$. This surely is one of the neighbourhoods of q so it belongs to either $\Downarrow U$ or $\Downarrow V$; in other words either any element covering it is in U or in V. Thus either both (a, b) and (c, d) are in U or both of them are in V.

Secondly, we want to prove that either all basic neighbourhoods of q belong to U or all of them belong to V. Let $I_0, I_1, \ldots, I_n, \ldots$ be an enumeration of all elements in rq and suppose, by inductive hypothesis, that $\{I_0, \ldots, I_{n-1}\}$ is contained in either U or V. For instance let $\{I_0, \ldots, I_{n-1}\} \subseteq U$. Now consider I_n and use the previous step: if $\{I_n, I_0\} \subseteq U$ then $\{I_0, \ldots, I_{n-1}, I_n\} \subseteq U$ and we are done; instead, if $\{I_n, I_0\} \subseteq V$ then consider $\{I_n, I_1\}$; if this is contained in U then we are done, otherwise repeat the same argument with I_2 and so on. Eventually, if $\{I_n, I_j\} \subseteq V$ for any $0 \leq j < n$ then $\{I_0, \ldots, I_{n-1}, I_n\} \subseteq V$, of course.

1.3.2 About further topological operators

We have already seen how to define interior and closure (and their formal counterparts) as suitable compositions of two of the four operators r, r^-, r^* and r^{-*} arising from a relation (X, r, S). Of course, other four objects can be obtained in this way:

$$r^- r \qquad r^* r^{-*} \qquad r r^- \qquad r^{-*} r^*$$

the first two operators being on $\mathcal{P}X$ the others on $\mathcal{P}S$. Our claim is that they are definable in terms of *int*, *cl*, $\mathcal{A}$ and $\mathcal{J}$, provided (X, cl, int) and $(S, \mathcal{A}, \mathcal{J})$ are *convergent*. As we are more interested in the formal side, we only give the explicit definitions in that context only. Let us put

$$\mathcal{E}U = \{a \in S : \ a{\downarrow}U \ltimes S\} \qquad \text{and} \qquad \mathcal{U}U = \{a \in S : \ \mathcal{E}a \subseteq \mathcal{J}U\}$$

($\mathcal{E}a$ stands for $\mathcal{E}\{a\}$, as usual). It is possible to prove that $\mathcal{E} = rr^-$ and $\mathcal{U} = r^{-*}r^*$ provided the topology is definable. For instance:

$$a{\downarrow}U \ltimes S \equiv r^-(a{\downarrow}U) \between r^*S \equiv (r^-a) \cap (r^-U) \between X \equiv r^-a \between r^-U \equiv a \,\epsilon\, rr^-U.$$

We will see that the operator $\mathcal{E}$ is just needed to study satisfiability (see chapter 4). For this reason we anticipate a brief list of properties about it.

1. $U \between \mathcal{E}V \equiv U{\downarrow}V \ltimes S$;
2. $U \between \mathcal{E}V \equiv \mathcal{E}U \between V$;
3. $U \subseteq V \Longrightarrow \mathcal{E}U \subseteq \mathcal{E}V$;
4. $\mathcal{E}\bigcup_{i\in I} V_i = \bigcup_{i\in I}\mathcal{E}V_i$;
5. $\mathcal{E}\mathcal{A} = \mathcal{E}$, hence $\mathcal{E}$ respects $=_{\mathcal{A}}$;
6. $U \between \mathcal{E}V \equiv \mathcal{A}U \between \mathcal{E}V$.

1.4 Overlap algebras

In this section we give an algebraic description of the collection of all subsets of a set. The idea we follow is essentially the same that brought to the definition of Boolean algebra. The difference is that we want to do things constructively. One could say that the intuitionistic versions of Boolean algebras already exist and they are just the Heyting algebras. However, we want to axiomatize also the overlap relation, written $\between$. Note that it cannot be defined in terms of order and complement unless one uses classical logic. The definitions we give can be found in [23] where a complete justification is provided.

Definition 1.4.1 (see [19], [20] and [23]) *An* overlap algebra*, or* o-algebra*, is a structure* $(P, =, \leq, \gtrless, \wedge, \bigvee, \rightarrow, 0, 1)$ *where:*

- $(P, =)$ *is a collection with "equality" (that is, an equivalence relation on* P*);*
- $(P, =, \leq, \wedge, \bigvee, \rightarrow, 0, 1)$ *is a complete (and distributive*[4]*) Heying algebra;*
- $\gtrless$ *is a binary relation on* P*, called* overlap*, satisfying:*
  1. *if* $p \gtrless q$ *then* $q \gtrless p$*;*

[4]A complete Heyting algebra always satisfies distributivity of binary meet with respect to infinite join, i.e. $p \wedge (\bigvee_{i\in I} g_i) = \bigvee_{i\in I}(p \wedge q_i)$.

2. *if* $p \between q$ *then* $p \wedge q \between q$*;*

3. $(\bigvee_{i \in I} p_i) \between q$ *iff there exists* $i \in I$ *such that* $p_i \between q$*.*

To be precise, the original definition by Sambin requires a complete lattice instead of a complete Heyting algebra. Moreover, a link between $\leq$ and $\between$ is present, namely the following rule (called "density")

$$\frac{\begin{array}{c}[p \between r] \\ \vdots \\ q \between r\end{array}}{p \leq q}$$

which is not convenient for our purpose.

Of course, if X is a set then the structure $\mathcal{P}X$ (with the obvious relations, operations and constants) is an o-algebra. Conversely, an "atomic" (in the standard sense) o-algebra turns out to be exactly the powerset of some set; the proof of this fact is in [23], chapter 10 (also see [20] and [19]). Here we only want to note that the language of o-algebras allows a very elegant definition of "atom"; namely, $a \in P$ is an atom if

$$a \leq p \Longleftrightarrow a \between p$$

for all $p \in P$. Note that atomicity of an o-algebra implies "density"; indeed, the premise of the density rule yields that the family of atoms below p is contained in that one corresponding to q; so $p \leq q$ because the algebra is atomic. Hence Sadocco's and Sambin's representation result is true for our modified definition of o-algebra too. The following are some basic but useful facts about $\between$.

Proposition 1.4.2 *In every o-algebra all the following hold:*

1. $p \vee q \between r$ *if and only if* $p \between r$ *or* $q \between r$*;*

2. $\neg(0 \between p)$*;*

3. *if* $p \between q$ *and* $p \leq r$ *then* $r \between q$*;*

4. $(p \wedge q = 0) \rightarrow \neg(p \between q)$*;*

5. $p \between q \rightarrow p \wedge q \neq 0$*.*

PROOF: 1. is a special case of the 3rd rule about $\lessgtr$; the same for 2. (0 is the join of the empty family). Item 3. can be proved as follows: from $p \leq r$ one has $p \vee r = r$; on the other hand, $p \lessgtr q$ yields $p \vee r \lessgtr q$ by 1.; thus $r \lessgtr q$. Finally, 4. (and then 5., which is logically equivalent to it): from $p \lessgtr q$ one gets $p \wedge q \lessgtr q$ and then $0 \lessgtr q$ by hypothesis, contradicting item 2.. q.e.d.

Note that assuming "density" allows the derivation of the converse of item 4, thus: $p \wedge q \lessgtr r$ implies $p \lessgtr q$; this together with $\neg(p \lessgtr q)$ yields a contradiction and hence anything follows; in particular $0 \lessgtr r$ and $p \wedge q \leq 0$ by density. If in addition one uses classical logic then $\lessgtr$ becomes definable. Besides, the premises of the density rule are classically equivalent to $-q \leq -p$; so density, together with classical logic, implies $p = --p$. Summing up, an o-algebra which satisfies density is exactly a boolean algebra, provided the reader's metalanguage is classical. On the contrary, in an o-algebra as defined above, $\lessgtr$ does not need to be definable even if one adopts classical logic. For instance, the set $\{0, a, 1\}$ linearly ordered by $0 < a < 1$ is a complete Heyting algebra and $x \wedge y \neq 0$ defines an "overlap" (which in fact is the only possible one if density is assumed). However we can put $x \lessgtr y$ true if $x = 1 = y$ and false otherwise and obtain another overlap relation (not satisfying density).

Now we want to simulate a relation; actually, what we really need are the four operators on subsets induced by r, namely: r, r^-, r^* and r^{-*}. Again in [23], chapter 10, it is proved that a relation is exactly the same as a symmetric pair of adjunctions, in the sense of the following definition.

Definition 1.4.3 *For X and S sets, a quadruple of operators $F, G' : \mathcal{P}X \to \mathcal{P}S$ and $F', G : \mathcal{P}S \to \mathcal{P}X$ is said to form a* symmetric pair of adjunctions *if:*

- *$F \dashv G$ (F is left adjoint to G) i.e. $F(D) \subseteq U \equiv D \subseteq G(U)$;*
- *$F' \dashv G'$ (F' is left adjoint to G') i.e. $F'(U) \subseteq D \equiv U \subseteq G'(D)$;*
- *$F \cdot\!|\!\cdot F'$ (F is symmetric to F') i.e. $F(D) \between U \equiv D \between F'(U)$;*

(for every $D \subseteq X$ and $U \subseteq S$).

Proposition 1.4.4 *For every two sets X, S and every quadruple of operators $F, G' : \mathcal{P}X \to \mathcal{P}S$ and $F', G : \mathcal{P}S \to \mathcal{P}X$, the following are equivalent:*

- *F, G, F', G' form a symmetric pair of adjunctions;*

- *$F = r$, $F' = r^-$, $G = r^*$ and $G' = r^{-*}$ for some relation r between X and S.*

PROOF: One direction is by paragraph 1.2.4. As for the other one, define

$$x\, r\, y \;\equiv\; y \,\epsilon\, F(x)$$

(see [23], theorem 10.1.13 for details). q.e.d.

Note that for every symmetric pair of adjunctions all the properties listed in paragraph 1.2.4 are true. Of course, the notion of a symmetric pair of adjunctions can be reformulated in the language of o-algebras in a natural way. Thus we give the following.

Definition 1.4.5 *An* o-relation *from an o-algebra $\mathcal{P}$ into the o-algebra $\mathcal{Q}$ is a quadruple of functions (F, F', G, G'), where $F, G' : \mathcal{P} \to \mathcal{Q}$ and $F', G : \mathcal{Q} \to \mathcal{P}$, such that:*

1. *$F \dashv G$ that is: $F(p) \leq q \equiv p \leq G(q)$;*
2. *$F' \dashv G'$ that is: $F'(q) \leq p \equiv q \leq G'(p)$;*
3. *$F \cdot|\cdot F'$ that is: $F(p) \between q \equiv p \between F'(q)$.*

1.4.1 Convergent topologies vs o-algebras

Let $\mathcal{S} = (S, \lhd, \ltimes, \downarrow)$ be a convergent topology. Consider the binary relation on $\mathcal{P}S$ defined by

$$U \between V \equiv U{\downarrow}V \ltimes S \equiv U \between \mathcal{E}V$$

(for the definition of $\mathcal{E}$ see paragraph 1.3.2). We want to prove that the structure

$$\mathcal{PS} = (\mathcal{P}S, =_{\mathcal{A}}, \lhd, \between, \downarrow, \bigcup, \to_{\mathcal{A}}, \emptyset, S)$$

(the complete Heyting algebra $Sat(\mathcal{A})$ enriched with the relation $\between$) is an o-algebra. In other words, we have to check that $\between$ is an overlap relation according to definition 1.4.1. By the very definition, $U \between V$ is the same as $(\exists a \,\epsilon\, U)(\exists b \,\epsilon\, V)(a{\downarrow}b \ltimes S)$ or also, by an easy proof, $(\exists a \lhd U)(\exists b \lhd V)(a{\downarrow}b \ltimes S)$. Since $a{\downarrow}b = b{\downarrow}a$ item 1 is satisfied. Item 2 follows from $a{\downarrow}b =_{\mathcal{A}} (a{\downarrow}b){\downarrow}b$ which is easily proved. Lastly, item 3 is true by purely logical arguments.

We say that $\mathcal{PS}$ is the o-algebra represented by the topology $\mathcal{S}$.

Chapter 2

A constructive completeness theorem for intuitionistic logic

In this chapter we present a slight modification of a soundness and completeness theorem for LJ given by Sambin in [22]. We also use an idea from Coquand and Smith (see [9]), namely the inductive generation of the canonical cover. Our aim is to show that the original proof by Sambin can be adapted to the definition of topology given above. As usual, the cover relation is enough to give a semantics for LJ-deducibility, thus the binary positivity relation is not used in the present chapter, apart from paragraph 2.2.1 where it is invoked in order to give some intuitive explanations. On the contrary, the positivity relation will play a fundamental role in the rest of the thesis.

2.1 Interpreting formulae in a topology

To interpret terms a set D is needed together with functions corresponding to the function symbols in the language. The interpretation is carried on as usual starting from an assignment to the variables. In this way, each term is interpreted as an element in D.

Let $\mathcal{S} = (S, \lhd, \ltimes)$ be a topology. A valuation in the topology $\mathcal{S}$ (corresponding to a given interpretation of terms) is a function

$$V \ : \ \mathrm{Frm} \longrightarrow \mathcal{P}S$$

which is recursively defined in the following way. Firstly fix $V(p(d_1, \ldots, d_n))$, for each instance of an atomic formula p; then put:

- $V(\bot) = \emptyset$;
- $V(\varphi \,\&\, \psi) = V(\varphi){\downarrow}V(\psi)$;
- $V(\varphi \vee \psi) = V(\varphi) \cup V(\psi)$;
- $V(\varphi \to \psi) = V(\varphi) \to_{\mathcal{A}} V(\psi)$;
- $V(\forall x \varphi(x)) = \{a \in S : (\forall d \in D)(a \lhd V(\varphi(d)))\}$;
- $V(\exists x \varphi(x)) = \bigcup_{d \in D} V(\varphi(d))$.

Finally, if $\Gamma = \gamma_1, \ldots, \gamma_n$ is a list of formulae, put

$$V(\Gamma) = V(\gamma_1){\downarrow} \cdots {\downarrow}V(\gamma_n)$$

if $n > 0$, while $V(\Gamma) = S$ if Γ is the empty list.

Definition 2.1.1 *A sequent $\Gamma \vdash \varphi$ is* valid *in an interpretation if*

$$V(\Gamma) \lhd V(\varphi)$$

holds in the corresponding topology.

A sequent is valid, written $\Gamma \models \varphi$, if it is valid in all possible interpretations.

Because of the definition of validity, the valuating function respects the equivalence relation $=_{\mathcal{A}}$, thus the valuation of a formula can be read as an element of $Sat(\mathcal{A})$. In other words, we interpret formulae in (particular) complete Heyting algebras. Also note that in case the topology is representable, the functor r^- allows to transfer all definitions to the concrete side. Unsurprisingly, what one gets this way is exactly the usual topological semantics.

Theorem 2.1.2 (soundness) *If $\Gamma \vdash \varphi$ then $\Gamma \models \varphi$.*

Proof: The proof is an easy exercise in the theory of complete Heyting algebras. We only give details for some particular cases. For instance, the rule

$$\frac{\Gamma, \varphi \vdash \gamma \quad \Gamma, \psi \vdash \gamma}{\Gamma, \varphi \vee \psi \vdash \gamma}$$

is valid if

$$\frac{Z{\downarrow}U \lhd W \quad Z{\downarrow}V \lhd W}{Z{\downarrow}(U \cup V) \lhd W}$$

holds for any $U, V, W, Z \subseteq S$, which in the abstract algebraic setting can be proved in the following way:

$$\dfrac{\dfrac{\dfrac{c \wedge a \leq d}{a \leq c \to d} \quad \dfrac{c \wedge b \leq d}{b \leq c \to d}}{a \vee b \leq c \to d}}{c \wedge (a \vee b) \leq d}$$

Similarly, the validity of

$$\dfrac{\Gamma \vdash \varphi \quad \psi, \Gamma' \vdash \gamma}{\Gamma, \varphi \to \psi, \Gamma' \vdash \gamma}$$

follows from

$$\dfrac{\dfrac{\dfrac{c \leq a \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{a \to b \leq a \to b}}{a \wedge (a \to b) \leq b} \quad \dfrac{b \wedge d \leq e}{b \leq d \to e}}{a \wedge (a \to b) \leq d \to e}}{a \leq (a \to b) \to (d \to e)}}{c \leq (a \to b) \to (d \to e)}}{c \wedge (a \to b) \leq d \to e}}{c \wedge (a \to b) \wedge d \leq e}$$

q.e.d.

2.2 Topology on the set of formulae

In order to prove the completeness theorem, we need to construct a canonical (i.e. syntactical) interpretation. Let $\mathcal{C} = (\mathrm{Frm}, \lhd, \ltimes)$ be the basic topology generated by the following recursively defined axiom-set. Here $\uparrow \varphi$ stands for the collection of all ψ such that $\varphi \vdash \psi$:

- $I(\bot) = \{*\} \cup \uparrow \bot = \{*\} \cup \mathrm{Frm}$;
- $I(\varphi \vee \psi) = \{*\} \cup \uparrow (\varphi \vee \psi)$;
- $I(\exists x\ \varphi(x)) = \{*\} \cup \uparrow (\exists x\ \varphi(x))$;
- $I(\varphi) = \uparrow \varphi$, otherwise;

where $*$ is a new symbol. Then consider the following basic covers:

- $C(\varphi, \psi) = \{\psi\}$, for $\psi \in \uparrow \varphi$;
- $C(\bot, *) = \emptyset$;

- $C(\varphi \vee \psi, *) = \{\varphi, \psi\}$;
- $C(\exists x\ \varphi(x), *) = \{\varphi(t) :\ t \in \mathrm{Trm}\}$.

The general rules for the cover given in paragraph 1.2.3 explicitly become (recall that $U \lhd V$ means that all elements in U are covered by V; on the contrary, $U \ltimes V$ states the existence of at least one $a \,\epsilon\, U$ such that $a \ltimes V$):

$$\frac{\varphi \,\epsilon\, U}{\varphi \lhd U}\ \mathrm{refl} \qquad \frac{\varphi \vdash \psi \quad \psi \lhd U}{\varphi \lhd U}\ \vdash \qquad \frac{}{\bot \lhd U}\ \bot$$

$$\frac{\varphi \lhd U \quad \psi \lhd U}{\varphi \vee \psi \lhd U}\ \vee \qquad \frac{\{\varphi(t) : t \in \mathrm{Trm}\} \lhd U}{\exists x \varphi(x) \lhd U}\ \exists$$

and, in addition, the following principle of proof by induction is fulfilled:

$$\frac{\varphi \lhd U \quad \begin{array}{c}[\psi \,\epsilon\, U]\\ \vdots \\ \psi \,\epsilon\, P\end{array} \quad \begin{array}{c}[\psi \vdash \gamma\ ,\ \gamma \,\epsilon\, P]\\ \vdots \\ \psi \,\epsilon\, P\end{array} \quad [\bot \,\epsilon\, P] \quad \begin{array}{c}[\{\psi, \gamma\} \subseteq P]\\ \vdots \\ (\psi \vee \gamma) \,\epsilon\, P\end{array} \quad \begin{array}{c}[\{\psi(t) : t \in \mathrm{Trm}\} \subseteq P]\\ \vdots \\ (\exists x\ \psi(x)) \,\epsilon\, P\end{array}}{\varphi \,\epsilon\, P}\ .$$

Besides, the relation $\ltimes$ satisfies:

$$\frac{\varphi \ltimes U}{\varphi \,\epsilon\, U}\ \mathrm{co}-\mathrm{refl} \qquad \frac{\varphi \vdash \psi \quad \varphi \ltimes U}{\psi \ltimes U}\ \mathrm{co}\!-\vdash \qquad \frac{\bot \ltimes U}{}\ \mathrm{co}\!-\!\bot$$

$$\frac{\varphi \vee \psi \ltimes U}{\{\varphi, \psi\} \ltimes U}\ \mathrm{co}\!-\!\vee \qquad \frac{(\exists x\ \varphi(x)) \ltimes U}{\{\varphi(t) : t \in \mathrm{Trm}\} \ltimes U}\ \mathrm{co}\!-\!\exists$$

$$\frac{\varphi \,\epsilon\, P \quad \begin{array}{c}[\psi \,\epsilon\, P]\\ \vdots \\ \psi \,\epsilon\, U\end{array} \quad \begin{array}{c}[\psi \vdash \gamma\ ,\ \psi \,\epsilon\, P]\\ \vdots \\ \gamma \,\epsilon\, P\end{array} \quad [\neg(\bot \,\epsilon\, P)] \quad \begin{array}{c}[(\psi \vee \gamma) \,\epsilon\, P]\\ \vdots \\ \{\psi, \gamma\} \between P\end{array} \quad \begin{array}{c}[(\exists x\ \psi(x)) \,\epsilon\, P]\\ \vdots \\ \{\psi(t) : t \in \mathrm{Trm}\} \between P\end{array}}{\varphi \ltimes U}$$

the last one being the co-induction rule.

Lemma 2.2.1 *For any $\varphi, \psi \in \mathrm{Frm}$, $\varphi \lhd \psi \iff \varphi \vdash \psi$.*

PROOF:

$\Leftarrow$) For any ψ, $\psi \lhd \psi$ holds by reflexivity; so, if $\varphi \vdash \psi$ then $\varphi \lhd \psi$ by the $\vdash$-rule.

$\Rightarrow$) The proof is an easy induction on the deduction of $\varphi \lhd \psi$. For instance, if the last rule in the derivation is $\vdash$ then there exists γ such that $\varphi \vdash \gamma$ and $\gamma \lhd \psi$; by the inductive hypothesis, $\gamma \vdash \psi$ holds and then $\varphi \vdash \psi$ by cut. As another example, let us suppose that φ is $\exists x\ \gamma(x)$ and the last rule is the

$$\frac{\{\gamma(t) : t \in \mathrm{Trm}\} \lhd \psi}{\exists x \gamma(x) \lhd \psi}\ \exists\ .$$

The inductive hypothesis yields $\gamma(t) \vdash \psi$ for all $t \in \mathrm{Trm}$; thus it is possible to find a variable which is not free in ψ and $\exists x \gamma(x) \vdash \psi$ follows by a suitable rule of LJ. q.e.d.

Corollary 2.2.2 *For any $\varphi, \psi \in \mathrm{Frm}$, $\varphi{\downarrow}\psi =_{\mathcal{A}} \varphi \ \&\ \psi$.*

PROOF: Let γ be a formula belonging to $\varphi{\downarrow}\psi$; thus it is covered by both φ and ψ. This means that both φ and ψ are derivable from γ (previous lemma) that is, $\gamma \vdash \varphi \ \&\ \psi$. The previous lemma again forces γ to be covered by $\varphi \ \&\ \psi$ and hence we get $\varphi{\downarrow}\psi \lhd \varphi \ \&\ \psi$ as γ was generic.

The statement $\varphi \ \&\ \psi \lhd \varphi{\downarrow}\psi$ follows by reflexivity from $\varphi \ \&\ \psi \ \epsilon\ \varphi{\downarrow}\psi$ which is easily proved by using the previous lemma once again. q.e.d.

Actually, $\mathcal{C}$ is a topology, that is, its cover satisfies the $\downarrow$ rule. To prove this fact we need two lemmas.

Lemma 2.2.3 *The rule*

$$\frac{\varphi \lhd U}{\varphi \ \&\ \psi \lhd U{\downarrow}\psi}\ \star$$

holds for any $\varphi, \psi \in \mathrm{Frm}$ and $U \subseteq \mathrm{Frm}$.

PROOF: This lemma can be seen as a theorem of "$\star$-elimination". The proof is by induction on the derivation of $\varphi \lhd U$.

$$\dfrac{\dfrac{\varphi \ \epsilon\ U}{\varphi \lhd U}\ \mathrm{refl}}{\varphi \ \&\ \psi \lhd U{\downarrow}\psi}\ \star \quad \rightsquigarrow \quad \dfrac{\dfrac{\dfrac{\dfrac{}{\varphi \ \&\ \psi \lhd \varphi} \quad \dfrac{}{\varphi \ \&\ \psi \lhd \psi}}{\varphi \ \&\ \psi \ \epsilon\ \varphi{\downarrow}\psi}\ \mathrm{def} \quad \dfrac{\varphi \ \epsilon\ U}{\varphi{\downarrow}\psi \subseteq U{\downarrow}\psi}\ \mathrm{def}}{\varphi \ \&\ \psi \ \epsilon\ U{\downarrow}\psi}}{\varphi \ \&\ \psi \lhd U{\downarrow}\psi}\ \mathrm{refl}$$

$$\dfrac{\dfrac{\varphi \vdash \gamma \quad \gamma \lhd U}{\varphi \lhd U}\,\vdash}{\varphi \,\&\, \psi \lhd U{\downarrow}\psi}\,\star \qquad\rightsquigarrow\qquad \dfrac{\dfrac{\varphi \vdash \gamma}{\varphi \,\&\, \psi \vdash \gamma \,\&\, \psi}\,\text{logic} \quad \dfrac{\gamma \lhd U}{\gamma \,\&\, \psi \lhd U{\downarrow}\psi}\,\star}{\varphi \,\&\, \psi \lhd U{\downarrow}\psi}\,\vdash$$

$$\dfrac{\dfrac{}{\bot \lhd U}\,\bot}{\bot \,\&\, \psi \lhd U{\downarrow}\psi}\,\star \qquad\rightsquigarrow\qquad \dfrac{\dfrac{}{\bot \,\&\, \psi \vdash \bot}\,\text{logic} \quad \dfrac{}{\bot \lhd U{\downarrow}\psi}\,\bot}{\bot \,\&\, \psi \lhd U{\downarrow}\psi}\,\vdash$$

$$\dfrac{\dfrac{\varphi_1 \lhd U \quad \varphi_2 \lhd U}{\varphi_1 \vee \varphi_2 \lhd U}\,\vee}{(\varphi_1 \vee \varphi_2) \,\&\, \psi \lhd U{\downarrow}\psi}\,\star \qquad\rightsquigarrow\qquad \begin{array}{c} \dfrac{\dfrac{\varphi_1 \lhd U}{\varphi_1 \,\&\, \psi \lhd U{\downarrow}\psi}\,\star \quad \dfrac{\varphi_2 \lhd U}{\varphi_2 \,\&\, \psi \lhd U{\downarrow}\psi}\,\star}{(\varphi_1 \,\&\, \psi) \vee (\varphi_2 \,\&\, \psi) \lhd U{\downarrow}\psi}\,\vee \\ \vdots \\ (\varphi_1 \vee \varphi_2) \,\&\, \psi \lhd U{\downarrow}\psi \end{array}$$

$$\dfrac{\dfrac{\varphi(t) \lhd U \quad (\forall t \in \mathrm{Trm})}{\exists x \varphi(x) \lhd U}\,\exists}{(\exists x \varphi(x)) \,\&\, \psi \lhd U{\downarrow}\psi}\,\star \qquad\rightsquigarrow\qquad \begin{array}{c} \dfrac{\dfrac{\varphi(t) \lhd U}{\varphi(t) \,\&\, \psi \lhd U{\downarrow}\psi}\,\star \quad (\forall t \in \mathrm{Trm})}{\exists x(\varphi(x) \,\&\, \psi) \lhd U{\downarrow}\psi}\,\exists \\ \vdots \\ (\exists x \varphi(x)) \,\&\, \psi \lhd U{\downarrow}\psi \end{array}$$

q.e.d.

Lemma 2.2.4 *The rule*

$$\frac{\varphi \lhd U}{\varphi{\downarrow}V \lhd U{\downarrow}V}$$

holds for any $\varphi \in \mathrm{Frm}$ *and* $U, V \subseteq \mathrm{Frm}$.

PROOF: Let γ be a generic element of $\varphi{\downarrow}V$; then there exists $\psi \,\epsilon\, V$ such that $\gamma \,\epsilon\, \varphi{\downarrow}\psi$; as a consequence, $\gamma \vdash \varphi \,\&\, \psi$. Assume $\varphi \lhd U$. By the previous lemma, we have $\varphi \,\&\, \psi \lhd U{\downarrow}\psi$. On the other hand, $U{\downarrow}\psi$ is contained in $U{\downarrow}V$. Thus $\varphi \,\&\, \psi \lhd U{\downarrow}V$ holds by reflexivity and transitivity; $\gamma \lhd U{\downarrow}V$ follows by the $\vdash$-rule. q.e.d.

Proposition 2.2.5 $\mathcal{C}$ *is convergent, that is, it satisfies the* $\downarrow$ *rule.*

PROOF:

$$\dfrac{\dfrac{\dfrac{}{\varphi \vdash \varphi \,\&\, \varphi}\,\text{logic} \qquad \dfrac{\varphi \lhd V}{\varphi \,\&\, \varphi \lhd V{\downarrow}\varphi}\,\text{lemma 2.2.3}}{\varphi \lhd V{\downarrow}\varphi}\,\vdash \qquad \dfrac{\varphi \lhd U}{\varphi{\downarrow}V \lhd U{\downarrow}V}\,\text{lemma 2.2.4}}{\varphi \lhd U{\downarrow}V}\,\text{transitivity}$$

q.e.d.

2.2.1 Formal points of the canonical topology

Recall that a formal point is a convergent and inhabited closed subset. The following proposition gives a characterization of formal points in the canonical topology.

Proposition 2.2.6 *Let $\mathcal{C}$ be the canonical topology; then $\alpha \subseteq \mathrm{Frm}$ is a formal point if and only if it satisfies the following rules:*

$$\dfrac{}{\top \,\epsilon\, \alpha} \qquad \dfrac{\bot \,\epsilon\, \alpha}{} \qquad \dfrac{a \vdash b \quad a \,\epsilon\, \alpha}{b \,\epsilon\, \alpha}$$

$$\dfrac{(a \vee b) \,\epsilon\, \alpha}{\{a, b\} \between \alpha} \qquad \dfrac{(\exists x a(x)) \,\epsilon\, \alpha}{\{a(t) : t \in \mathrm{Trm}\} \between \alpha} \qquad \dfrac{a \,\epsilon\, \alpha \quad b \,\epsilon\, \alpha}{(a \,\&\, b) \,\epsilon\, \alpha}$$

(where $\top$ is $\bot \to \bot$).

PROOF: Firstly, let us suppose $\alpha \subseteq \mathrm{Frm}$ satisfies the above rules. The first and the last ones force α to be inhabited and convergent. To see that the other rules force α to be a formal closed subset argue as follows: consider the co-induction rule and put $P = \alpha = U$; thus the above rules are just what is needed to get $\alpha \subseteq \mathcal{J}\alpha$.

Conversely, let $\alpha \subseteq \mathrm{Frm}$ be a formal point; in particular $\mathcal{J}\alpha = \alpha$. Because α is inhabited, there exists $\varphi \in \mathrm{Frm}$ such that $\varphi \ltimes \alpha$; but $\varphi \vdash \top$, so $\varphi \lhd \top$ and $\top \,\epsilon\, \alpha$. The last rule is just convergence of α, while the other ones are exactly co-$\bot$, co-$\vdash$, co-$\vee$ and co-$\exists$ respectively because α is closed. q.e.d.

Thus a formal point of the canonical topology is just a Henkin set, that is a model. Keeping this in mind is helpful to understand why the definitions given in this thesis actually work.

2.3 The completeness theorem

Now that the canonical topology has been introduced, in order to complete the canonical model it remains to define both the interpretation for terms and the valuation for formulae. Let us use Trm itself as domain and the identity function as assignment for variables. Finally, for any instance of an atomic formula different from $\perp$, say p, let us put

$$V(p) = \{\varphi \in \mathrm{Frm} : \varphi \vdash p\}.$$

Proposition 2.3.1 (Lemma of the canonical valuation) *For any φ and ψ in* Frm*:*

$$\varphi \lhd V(\psi) \Longleftrightarrow \varphi \vdash \psi$$

(that is, $\mathcal{A}V(\psi) = \{\varphi : \ \varphi \vdash \psi\} = {\downarrow}\psi$).

PROOF: It is enough to prove that $\varphi \lhd V(\psi)$ iff $\varphi \lhd \psi$, that is $\psi =_{\mathcal{A}} V(\psi)$. The proof is by induction on the complexity of ψ.

If $\psi = p$ is atomic then $\psi \,\epsilon\, V(\psi)$, so $\psi \lhd V(\psi)$. Vice versa, $\varphi \,\epsilon\, V(\psi)$ means $\varphi \vdash \psi$ by definition; thus $\varphi \lhd \psi$.

If $\psi = \perp$ then $V(\perp) = \emptyset$; thus $V(\perp) \lhd \perp$. On the other hand, $\perp \lhd \emptyset$ by the $\perp$-rule.

If $\psi = \psi_1 \,\&\, \psi_2$ then argue as follows: $\varphi \lhd \psi_1 \,\&\, \psi_2$ iff $\varphi \vdash \psi_1 \,\&\, \psi_2$ iff both $\varphi \vdash \psi_1$ and $\varphi \vdash \psi_2$ iff both $\varphi \lhd \psi_1$ and $\varphi \lhd \psi_2$ iff (by inductive hypothesis) both $\varphi \lhd V(\psi_1)$ and $\varphi \lhd V(\psi_2)$ iff $\varphi \lhd V(\psi_1){\downarrow}V(\psi_2)$ iff (by definition) $\varphi \lhd V(\psi_1 \,\&\, \psi_2)$.

The case $\psi = \psi_1 \vee \psi_2$ is sketched below.

$$\dfrac{\dfrac{\psi_1 \lhd V(\psi_1)}{\psi_1 \lhd V(\psi_1) \cup V(\psi_2)} \qquad \dfrac{\psi_2 \lhd V(\psi_2)}{\psi_2 \lhd V(\psi_1) \cup V(\psi_2)}}{\psi_1 \vee \psi_2 \lhd V(\psi_1) \cup V(\psi_2)} \qquad\qquad \dfrac{\dfrac{V(\psi_i) \lhd \psi_i \quad \overline{\psi_i \lhd \psi_1 \vee \psi_2}}{V(\psi_i) \lhd \psi_1 \vee \psi_2}}{V(\psi_1) \cup V(\psi_2) \lhd \psi_1 \vee \psi_2}$$

Let us consider now the case $\psi = \psi_1 \to \psi_2$. Thanks to the adjunction between $\downarrow$ and $\to_{\mathcal{A}}$, we know that $\varphi \lhd V(\psi_1) \to_{\mathcal{A}} V(\psi_2)$ is equivalent to $\varphi{\downarrow}V(\psi_1) \lhd V(\psi_2)$; thanks to the inductive hypothesis this is equivalent to $\varphi{\downarrow}\psi_1 \lhd \psi_2$, which is the same as $\varphi \,\&\, \psi_1 \lhd \psi_2$. This is true if and only if $\varphi \,\&\, \psi_1 \vdash \psi_2$, i.e. $\varphi \vdash \psi_1 \to \psi_2$ and then $\varphi \lhd \psi_1 \to \psi_2$.

The proofs for the cases $\psi = \forall x\ \gamma(x)$ and $\psi = \exists x\ \gamma(x)$ are similar to those of $\psi = \psi_1 \,\&\, \psi_2$ and $\psi = \psi_1 \vee \psi_2$, respectively. q.e.d.

At this point, the completeness theorem is obtained as an easy corollary.

Theorem 2.3.2 (completeness) *If* $\Gamma \models \varphi$ *then* $\Gamma \vdash \varphi$.

Proof: If $\Gamma \models \varphi$ then, in particular, $\Gamma \vdash \varphi$ is valid in the canonical model. In other words, we have $V(\gamma_1){\downarrow} \dots {\downarrow} V(\gamma_n) \lhd V(\varphi)$ provided that $\Gamma = \{\gamma_1, \dots, \gamma_n\}$. From the previous proposition we get $\psi \lhd V(\psi)$, for any ψ; so:

$$\dfrac{\dfrac{\dfrac{}{\gamma_1 \mathbin{\&} \dots \mathbin{\&} \gamma_n \lhd V(\gamma_1 \mathbin{\&} \dots \mathbin{\&} \gamma_n)} \quad \dfrac{V(\gamma_1){\downarrow} \dots {\downarrow} V(\gamma_n) \lhd V(\varphi)}{V(\gamma_1 \mathbin{\&} \dots \mathbin{\&} \gamma_n) \lhd V(\varphi)}\,\text{def}}{\dfrac{\gamma_1 \mathbin{\&} \dots \mathbin{\&} \gamma_n \lhd V(\varphi)}{\gamma_1 \mathbin{\&} \dots \mathbin{\&} \gamma_n \vdash \varphi}\,\text{proposition 2.3.1}}\,\text{trans}}{\Gamma \vdash \varphi}\,\text{logic}$$

q.e.d.

Chapter 3

Constructive semantics for non-deducibility

Because of the completeness theorem, saying that a sequent is non-deducible is the same as saying that not all topological interpretations validate it. From a constructive point of view, this statement cannot be immediately transformed into a positive one. Thus the completeness theorem is not enough to deal with non-deducibility: some other theorems are needed. The same happens for other logical notion which are classically defined by means of negation. An example is the notion of a "satisfiable formula", which classically is defined as a formula that does not entail a contradiction. In the next chapter we will see as it is possible to give a constructive notion of satisfiability; in the present one we give some constructive results about non-deducibility.

3.1 Co-valuation: the dual of valuation

The first thing one can prove is that non-deducibility is equivalent to the existence of a counter-model even if we adopt a constructive point of view.

Proposition 3.1.1 $\Gamma \nvdash \varphi$ *if and only if there exists a topological interpretation such that* $\neg(V(\Gamma) \lhd V(\varphi))$.

PROOF: If there is a topology such that $\neg(V(\Gamma) \lhd V(\varphi))$, then $\Gamma \vdash \varphi$ cannot be provable otherwise $V(\Gamma) \lhd V(\varphi)$ should be true, a contradiction.

Vice versa, suppose $\Gamma \nvdash \varphi$. We know that $\Gamma \vdash \varphi$ is equivalent to $V(\Gamma) \lhd V(\varphi)$ in the canonical topology; hence $\neg(V(\Gamma) \lhd V(\varphi))$ holds in the canonical topology (remember that $\varphi \vdash \psi$ implies $\neg\psi \vdash \neg\varphi$ also intuitionistically). q.e.d.

However, we would like to find out a possibly more direct characterization of non-deducibility. Since a semantics for deducibility is carried out by means of the open subsets of a topology, one could expect the closed subsets to work with respect to non-deducibility (see [15]). From the point of view of the classical topological semantics this is trivial, of course. In order to find the right constructive definition, let us do a bit of classical calculation: $\Gamma \nvdash \varphi$ iff there exists a topology such that $\neg(V(\Gamma) \subseteq \mathcal{A}V(\varphi))$ iff $V(\Gamma) \between -\mathcal{A}V(\varphi)$ iff $V(\Gamma) \between \mathcal{J} - V(\varphi)$ iff $V(\Gamma) \ltimes -V(\varphi)$. Thus $\Gamma \nvdash \varphi$ is equivalent to the existence of an interpretation such that $V(\Gamma) \ltimes -V(\varphi)$ provided $-$ is the classical complement. The notion of a co-valuation is a constructive way to simulate the complement of a valuation.

Definition 3.1.2 *Let $V(\varphi)$ be a valuation of a formula φ in a topology. A* co-valuation *of φ (with respect to the given valuation) is a subset $F(\varphi)$ such that*

$$\mathcal{J}F(\varphi) \cap \mathcal{A}V(\varphi) = \emptyset.$$

For instance, $F(\varphi) = -V(\varphi)$ and $F(\varphi) = -\mathcal{A}V(\varphi)$ are natural examples of co-valuations. Besides these elementary definitions which do not give much constructive information, there exists also a recursive way for defining a co-valuation, as described below.

For each atomic formula p, different from $\bot$, fix a subset $F(p)$ such that $\mathcal{J}F(p) \cap \mathcal{A}V(p) = \emptyset$; then consider the following recursive definition.

- $F(\bot) = S$;
- $F(\varphi \,\&\, \psi) = \{a \in S : \ a \ltimes F(\varphi) \vee a \ltimes F(\psi)\} = \mathcal{J}F(\varphi) \cup \mathcal{J}F(\psi)$; [1]
- $F(\varphi \vee \psi) = F(\varphi) \cap F(\psi)$;
- $F(\varphi \to \psi) = \{a \in S : \ a{\downarrow}V(\varphi) \ltimes F(\psi)\}$;
- $F(\forall x\ \varphi(x)) = \{a \in S : \ (\exists d \in D)(a \ltimes F(\varphi(d)))\} = \bigcup_{d \in D} \mathcal{J}F(\varphi(d))$;
- $F(\exists x\ \varphi(x)) = \bigcap_{d \in D} F(\varphi(d))$.

Proposition 3.1.3 *The function F defined above is a co-valuation.*

[1]If the topology is a bi-convergent one then put $F(\varphi \,\&\, \psi) = F(\varphi){\Downarrow}F(\psi)$.

Proof: All cases are quite simple. For instance, let us consider the case of $\exists x\ \varphi(x)$ and suppose there exists $a \epsilon \mathcal{J}F(\exists x\varphi(x)) \cap \mathcal{A}V(\exists x\varphi(x))$. This implies $\mathcal{J}F(\exists x\varphi(x)) \between \mathcal{A}V(\exists x\varphi(x))$ and then $\mathcal{J}F(\exists x\varphi(x)) \between V(\exists x\varphi(x))$ by compatibility. Hence, without loss of generality, we can take $a \epsilon \mathcal{J}F(\exists x\varphi(x)) \cap V(\exists x\varphi(x))$. The definition of $V(\exists x\varphi(x))$ implies $a \epsilon V(\varphi(d))$ for some d. On the other hand $a \ltimes F(\exists x\varphi(x))$ yields $a \epsilon F(\varphi(d))$ and then $a \ltimes F(\varphi(d))$ by co-transitivity. Hence a should belong to $\mathcal{J}F(\varphi(d)) \cap V(\varphi(d))$ thus contradicting the inductive hypothesis.

As for another example, let us consider the case of implication. We will be able to prove a stronger fact, namely $F(\varphi \to \psi) \cap V(\varphi \to \psi) = \emptyset$. If there existed an element, say a, belonging to both the valuation and co-valuation of $\varphi \to \psi$, then both $a{\downarrow}V(\varphi) \lhd V(\psi)$ and $a{\downarrow}V(\varphi) \ltimes F(\psi)$ would be true; hence $V(\psi) \ltimes F(\psi)$ by compatibility. This is exactly $V(\psi) \between \mathcal{J}F(\psi)$: a contradiction. q.e.d.

In the case of the canonical topology, we need to define a class of particular co-valuations.

Definition 3.1.4 *A* canonical co-valuation *is a map* $F :\ \mathrm{Frm} \to \mathcal{P}(\mathrm{Frm})$ *such that*

$$\psi \ltimes F(\varphi) \implies \psi \nvdash \varphi \implies \psi \epsilon F(\varphi)$$

for any $\varphi, \psi \in \mathrm{Frm}$.

Note that a canonical co-valuation is really a co-valuation; indeed, if ψ belongs to $\mathcal{J}F(\varphi) \cap \mathcal{A}V(\varphi)$ then $\psi \ltimes F(\varphi)$ and $\psi \lhd V(\varphi)$, thus $\psi \nvdash \varphi$ and $\psi \vdash \varphi$, a contradiction.

Canonical co-valuations actually exist; for instance

$$F(\varphi) = -\mathcal{A}V(\varphi) = \{\psi \in \mathrm{Frm} :\ \psi \nvdash \varphi\}$$

is canonical. Note that neither $F(\varphi) = -V(\varphi)$ nor any recursive co-valuation work as canonical co-valuation. This is mostly due to the weak foundation we are using and, in particular, to the underlying logic (if one adopts a classical metalanguage then both of them work). For instance, let us consider a recursive co-valuation and suppose it was canonical. Moreover, let us assume $\gamma \nvdash \varphi \;\&\; \psi$ for some γ, φ and ψ in Frm. This would imply $\gamma \epsilon F(\varphi \;\&\; \psi)$; hence either $\gamma \ltimes F(\varphi)$ or $\gamma \ltimes F(\psi)$ which in turn would yield either $\gamma \nvdash \varphi$ or $\gamma \nvdash \psi$. Summing up, if the recursive co-valuation was canonical then we would be able to choose wether $\neg(\gamma \vdash \varphi)$ or $\neg(\gamma \vdash \psi)$ from the assumption $\neg((\gamma \vdash \varphi) \;\&\; (\gamma \vdash \psi))$; this is constructively unacceptable (see Bishop's LLPO).

3.2 Semantics for non-deducibility

To prove the following fundamental lemma two results from proof-theory are needed, namely the disjunction (and existence) property and the consistency of the calculus (see section 1.1).

Lemma 3.2.1 *A formula φ is unprovable, written $\nvdash \varphi$, if and only if $\top \ltimes F(\varphi)$ for any canonical co-valuation F.*

Proof: $\Leftarrow$) $\top \ltimes F(\varphi) \Rightarrow \top \nvdash \varphi \Rightarrow \nvdash \varphi$ because F is a canonical co-valuation.

$\Rightarrow$) Let us consider the subset $P \subseteq \mathrm{Frm}$ defined by $\gamma \epsilon P \equiv \vdash \gamma$, put $U = F(\varphi)$ and use co-induction on the canonical positivity relation (see section 2.2): from

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\gamma \epsilon P}{\vdash \gamma} \quad [\gamma \vdash \varphi]'}{\vdash \varphi} \qquad \nvdash \varphi}{\bot}\,'}{\gamma \nvdash \varphi}}{\gamma \epsilon F(\varphi)} \qquad \dfrac{\dfrac{\dfrac{\gamma \epsilon P}{\vdash \gamma} \quad \gamma \vdash \delta}{\vdash \delta}}{\delta \epsilon P} \qquad \dfrac{\dfrac{}{\nvdash \bot}\,\text{consistency}}{\neg(\bot \epsilon P)}$$

$$\dfrac{\dfrac{\dfrac{(\gamma \vee \delta) \epsilon P}{\vdash \gamma \vee \delta}}{(\vdash \gamma) \vee (\vdash \delta)}\,\text{disjunction}}{\{\gamma, \delta\} \between P} \qquad \dfrac{\dfrac{\dfrac{\exists x \gamma(x) \epsilon P}{\vdash \exists x \gamma(x)}}{\vdash \gamma(t) \text{ for some } t}\,\text{existence}}{\{\gamma(t) : t \in \mathrm{Trm}\} \between P}$$

$\top \ltimes F(\varphi)$ follows, as $\top \epsilon P$. q.e.d.

As a corollary one gets the following constructive semantic characterization of non-deducibility.

Theorem 3.2.2 *For any $\varphi \in \mathrm{Frm}$, $\nvdash \varphi$ iff there exists a topology $\mathcal{S}$ and a co-valuation F on it such that $S \ltimes F(\varphi)$.*

Proof:

$\Rightarrow$) Consider the canonical topology (with a canonical co-valuation) and use the previous lemma to get $\top \ltimes F(\varphi)$. Hence $\mathrm{Frm} \ltimes F(\varphi)$ (since $\top \epsilon \mathrm{Frm}$).

$\Leftarrow$) If φ was provable then $S \lhd V(\varphi)$ would be true in every topology (by the soundness theorem). By hypothesis a certain topological interpretation exists such that $S \ltimes F(\varphi)$. In this topology both $S \lhd V(\varphi)$ and $S \ltimes F(\varphi)$ would be true contradicting the definition of co-valuation. q.e.d.

Note that if there exists a topology such that $V(\Gamma) \ltimes F(\varphi)$ then $\Gamma \nvdash \varphi$; otherwise, if $\Gamma \vdash \varphi$ was deducible then $V(\Gamma) \lhd V(\varphi)$, contradicting the requested condition on the co-valuation of φ. The opposite implication is classically true.

3.3 Some proofs of non-deducibility

From now till the end of the chapter we are giving some concrete applications of the results above. In all the following examples, the topology we will use is $\mathcal{Q}$, the (bi-convergent) topology of the rational numbers; moreover, all the co-valuations we are going to define actually are inductive ones. So a natural question arises: is $\mathcal{Q}$ (with all possible inductive co-valuation on it) enough to give a semantics for non-deducibility? In other words, is it possible to prove (in a constructive way) that $\nvdash \varphi$ is equivalent to the existence of an inductive co-valuation on $\mathcal{Q}$ such that $S \ltimes_{\mathcal{Q}} F(\varphi)$? This remains as an open problem.[2]

Since $\mathcal{Q}$ is a representable topology, we can transfer things on the concrete side. One has to recall (see paragraph 1.2.4) that the functors $r^- : Sat(\mathcal{A}) \to Red(int)$ and $r^* : Red(\mathcal{J}) \to Sat(cl)$ are two isomorphisms between the formal and the concrete objects. So the functions r^-V and r^*F are the counterparts of the valuating and co-valuating functions, respectively. Moreover, the conditions defining an inductive co-valuation can be read in terms of (concrete) closed subsets. For instance, $r^*F(\forall x\ \varphi(x))$ results in $cl \bigcup_{d \in D} r^*F(\varphi(d))$.

Corollary 3.3.1 *The formula $\varphi \vee \neg\varphi$ is generally non-deducible.*

Proof: Let φ be an atomic formula and put

$$V(\varphi) = \{(a,b) :\ 0 < a < b\} \quad \text{and} \quad F(\varphi) = \{(a,b) :\ a < 0\}$$

which correspond to the intervals $(0, +\infty)$ and $(-\infty, 0]$ respectively. We want to prove that $\mathcal{J}F(\varphi) \cap \mathcal{A}V(\varphi) = \emptyset$; this is equivalent to check $r^-V(\varphi) \cap r^*F(\varphi) = \emptyset$ which is trivial since it is exactly $(0, +\infty) \cap (-\infty, 0] = \emptyset$. In order to find out what the co-valuation of $\varphi \vee \neg\varphi$ is, let us firstly calculate $F(\neg\varphi)$. The basic open (x,y) is in $F(\varphi \to \bot)$ if and only if $(x,y){\downarrow}V(\varphi) \ltimes S$; this is equivalent to $r^-(x,y) \cap r^-V(\varphi) \between r^*S$ that in turn is $r^-(x,y) \between r^-V(\varphi)$ because $r^*S = \mathbb{Q}$. Hence

$$F(\neg\varphi) = \{(a,b) :\ (a < b)\ \&\ (b > 0)\}$$

which is the formal counterpart of the closed subset $[0, +\infty)$; thus

$$F(\varphi \vee \neg\varphi) = F(\varphi) \cap F(\neg\varphi) = \{(a,b) :\ a < 0 < b\}$$

[2] It is well known (see [18]) that intuitionistic *propositional* logic is complete with respect to both the real and the rational line. On the contrary, completeness with respect to the reals fails when first order formulae are considered, as it is shown by some counterexamples we know from Pawel Urzyczyn. The interesting thing is that the formulae used in those counterexamples are just valid on the rationals.

and $r^*F(\varphi \vee \neg\varphi) = \{0\}$. Finally, $S \ltimes F(\varphi \vee \neg\varphi)$ is obvious since it is the same as $\mathbb{Q} \between \{0\}$. q.e.d.

Corollary 3.3.2 *The formula $\neg\neg\varphi \rightarrow \varphi$ is generally non-deducible.*

PROOF: Let φ be an atomic formula and put

$$r^-V(\varphi) = (-\infty, 0) \cup (0, \infty) \quad \text{and} \quad r^*F(\varphi) = \{0\}$$

which, of course, are disjoint subsets. It is easy to check that

$$r^-V(\neg\varphi) = \emptyset \quad \text{and} \quad r^-V(\neg\neg\varphi) = \mathbb{Q}.$$

Remember that $\Gamma \nvdash \varphi$ is implied by $V(\Gamma) \ltimes F(\varphi)$ thus it is enough to prove that

$$V(\neg\neg\varphi) \ltimes F(\varphi)$$

that is $S \ltimes \{(a,b) : \ a < 0 < b\}$; this is just what is proved in the final part of the previous proof. q.e.d.

Corollary 3.3.3 (see [10]) *In general, $\forall x[\varphi \vee \psi(x)] \nvdash [\varphi \vee \forall x\psi(x)]$.*

PROOF: Let φ and $\psi(x)$ be atomic formulae. Let us choose $D = \mathbb{N}$, the set of natural numbers, as domain for interpreting terms. Finally put:

$$r^-V(\varphi) = (-\infty, 0) \cup (0, +\infty), \qquad r^-V(\psi(n)) = (-\frac{1}{n}, \frac{1}{n});$$

$$r^*F(\varphi) = \{0\}, \quad r^*F(\psi(n)) = (-\infty, -\frac{1}{n}] \cup [\frac{1}{n}, +\infty).$$

A little bit of calculation is needed to check that:

$$V(\forall n(\varphi \vee \psi(n))) =_{\mathcal{A}} S, \qquad F(\varphi \vee \forall x\psi(x)) = \{(a,b) : \ a < 0 < b\};$$

then one proceeds as in the previous proofs. q.e.d.

Corollary 3.3.4 $\neg\forall x\varphi(x) \nvdash \exists x\neg\varphi(x)$

PROOF: [Hint] Let $D = \{q \in \mathbb{Q} : \ q > 0\}$ and put

$$V(\varphi(q)) = \{(a,b) : \ -q < a < b < q\}$$

that intuitively is the open interval $(-q, q)$ on the rational line. q.e.d.

Chapter 4

Satisfiability

Usually, a formula, say φ, is said to be satisfiable if $\nvdash \neg\varphi$. In fact, if this holds then a classical argument shows how to *construct* (by using Zorn's lemma!) a Henkin set, i.e. a model, containing φ. Of course, this argument is not acceptable in our context. Moreover, we cannot even define a formula to be satisfiable if it belongs to some Henkin set, because of the impredicative nature this definition would have. Thus, the only thing we can do is to look for a more abstract definition of satisfiability which, when adopting a classical metatheory, would turn out to be equivalent to the usual notion.

4.1 Semantic satisfiability

Condition $\nvdash \neg\varphi$, i.e. $\varphi \nvdash \bot$, is equivalent to $\neg(\varphi \lhd \emptyset)$ in the canonical topology, as $V(\bot)$ is just $\emptyset$. Under a classical reading, $\neg(\varphi \lhd \emptyset)$ is the same as $\varphi \ltimes \mathrm{Frm}$; indeed, as $\ltimes$ and $\lhd$ are generated by the same axiom-set, $\mathcal{J}$ is the maximum reduction operator compatible with $\mathcal{A}$; from a classical point of view $-\mathcal{A}-$ is a reduction operator too and, actually, the maximum compatible one, i.e. it is just $\mathcal{J}$. So, classically speaking, satisfiability of φ can be expressed by the condition: $\varphi \ltimes \mathrm{Frm}$. On the contrary, from our foundational standpoint, this requirement implies the usual $\nvdash \neg\varphi$. Indeed, if $\vdash \neg\varphi$ then $\varphi \lhd \emptyset$; this together with $\varphi \ltimes \mathrm{Frm}$ gives $\emptyset \ltimes \mathrm{Frm}$ by compatibility, a contradiction. So $\varphi \ltimes \mathrm{Frm}$ implies $\nvdash \neg\varphi$, as claimed. The above discussion suggests to identify satisfiability of φ with the condition $\varphi \ltimes \mathrm{Frm}$.[1] However, this condition refers to the canonical model only,

[1]This choice is strengthened by the fact that the intended meaning of $\varphi \ltimes \mathrm{Frm}$ is that there exists a formal point (i.e. a model) "contained" in φ.

while it could be desirable to use a wider semantic characterization which would involve the entire class of models.

Theorem 4.1.1 *The condition* $\varphi \ltimes \mathrm{Frm}$ *holds in the canonical topology if and only if* $V(\varphi) \ltimes S$ *holds in some topological interpretation.*

Proof: $\Rightarrow$) The interpretation we are looking for is the canonical one. Remember that $\varphi \lhd V(\varphi)$ holds in the canonical topology and use compatibility.

$\Leftarrow$) Let $(S, \lhd_S, \ltimes_S)$ be the topology which exists by hypothesis. In order to prove that $\varphi \ltimes \mathrm{Frm}$ let us use the co-induction rule for the canonical $\ltimes$ (see page 24). Put $\psi \in P \equiv V(\psi) \ltimes_S S$, choose $U = \mathrm{Frm}$ and verify all the premises of the rule.

$P \subseteq \mathrm{Frm}$ is obvious.

Suppose $\psi \in P$ and $\psi \vdash \gamma$; then $V(\psi) \ltimes_S S$ and $V(\psi) \lhd_S V(\gamma)$ by the soundness theorem; thus $\gamma \in P$ by compatibility.

$V(\bot) \ltimes_S S$ cannot hold, otherwise $\emptyset \between S$; so $\neg(\bot \in P)$.

Let $(\psi \vee \gamma) \in P$, that is, $V(\psi \vee \gamma) = (V(\psi) \cup V(\gamma)) \ltimes_S S$; in other words, $(V(\psi) \cup V(\gamma)) \between \mathcal{J}_S S$ which is the same as $(V(\psi) \between \mathcal{J}_S S) \vee (V(\gamma) \between \mathcal{J}_S S)$, i.e. $\{\psi, \gamma\} \between P$.

In a quite similar way, if $(\exists x\ \psi(x)) \in P$ then $(\bigcup_{t \in \mathrm{Trm}} V(\psi(t))) \between \mathcal{J}_S S$; hence there exists a term t such that $V(\psi(t)) \ltimes_S S$, i.e. $\{\psi(t) : t \in \mathrm{Trm}\} \between P$.

Since all premises of the co-induction rule are fulfilled, we can derive $\varphi \ltimes \mathrm{Frm}$ from $\varphi \in P$ (i.e. $V(\varphi) \ltimes_S S$). q.e.d.

We know that the canonical topology actually is a canonical model with respect to deducibility. The theorem above asserts that it is a canonical model for satisfiability too. What we have said till now justifies the following.

Definition 4.1.2 *A formula, say* φ*, is* satisfiable *if* $V(\varphi) \ltimes S$ *holds in some topological model.*

Of course, this definition introduces a semantic notion of satisfiability; a syntactic version will be introduced in the next section. From the previous discussion also the following fact follows.

Corollary 4.1.3 *If a formula is satisfiable then its negation is intuitionistically non-deducible.*

From now till the end of the section, we are giving some simple results about satisfiability.

Corollary 4.1.4 *Every atomic formula different from $\bot$ is satisfiable.*

PROOF: Let p be an atomic formula, $(S, \lhd, \ltimes)$ be a topology such that $S \ltimes S$ and put $V(p) = S$. q.e.d.

Proposition 4.1.5 *If φ is provable in classical logic then φ is (intuitionistically) satisfiable.*

PROOF: Let $S = \{1\}$ be a set with only one element and let both $\mathcal{A}$ and $\mathcal{J}$ be the identity operator. It is easily seen that $(S, \mathcal{A}, \mathcal{J})$ is a convergent basic topology; moreover, $U{\downarrow}V = U \cap V$. Let us write $\mathcal{P}_\omega S$ for the collection of finite subsets of S (see [4])[2]; $\mathcal{P}_\omega S = \{\emptyset, S\}$ follows from two facts: it is decidable if a finite subsets is either empty or inhabited and an inhabited subset of S has to coincide with S itself.

Let $D = S$ be the domain for interpreting terms and choose $V(p)$ in $\mathcal{P}_\omega S$ for each atomic formula p. Actually $V(\varphi)$ is finite for any formula φ. The proof is by induction and it is quite trivial; as an example, look at the case of implication: if $V(\varphi) = S$ and $V(\psi) = \emptyset$ then

$$V(\varphi \to \psi) = \{a : \ a{\downarrow}V(\varphi) \lhd V(\psi)\} = \{a : \ \{a\} \cap S \subseteq \emptyset\} = \{a : \ a \,\epsilon\, \emptyset\} = \emptyset;$$

otherwise $V(\varphi \to \psi) = S$.

Summing up, the above interpretation is a truth values assignment, in the classical sense. Since φ is classically provable then the soundness theorem for classical logic forces it to be valid in all assignments. Thus $V(\varphi) = S$ and $V(\varphi) \ltimes S$ because $\mathcal{J}$ is the identity operator. q.e.d.

Proposition 4.1.6 *There exists a formula which is intuitionistically satisfiable, but classically refutable.*

PROOF: Let us consider the formula

$$\psi = \neg(\forall x \neg\neg\varphi(x) \to \neg\neg\forall x \varphi(x))$$

which, of course, is provably false in classical logic. Let $\mathcal{Q}$ be the topology of the rational numbers and take $D = \{x \in \mathbb{Q} : \ x \geq 1\}$ as the domain for terms. For any $x \in D$, let us define $V(\varphi(x))$ to be

$$\{(a, b) \in \mathbb{Q} \times \mathbb{Q} : (b \leq -x) \vee (-x \leq a < b \leq -1) \vee (1 \leq a < b \leq x) \vee (x \leq a)\}$$

[2]According to [4], a finite subset is $\{x \in S : (x = a_1) \vee \ldots \vee (x = a_n)\}$ for some finite list $\{a_1, \ldots, a_n\}$ of elements in S.

which, informally, is the open subset $(-\infty, -x) \cup (-x, -1) \cup (1, x) \cup (x, +\infty)$. After some calculation one finds out that $V(\psi)$ informally corresponds to

$$(-\infty, -1) \cup (1, +\infty)$$

which surely contains a point; thus $V(\psi) \ltimes_{\mathcal{Q}} S$. q.e.d.

Corollary 4.1.7 (see [15] and [12]) *The formula*

$$\neg\neg(\forall x \neg\neg\varphi(x) \to \neg\neg\forall x \varphi(x))$$

is intuitionistically non-deducible.

PROOF: Note that this formula is $\neg\psi$ where ψ is as in the previous proof and apply corollary 4.1.3. q.e.d.

Quite similarly it is possible to prove that

$$\neg\forall x(\varphi(x) \vee \neg\varphi(x))$$

is intuitionistically satisfiable, although classically false; hence its negation is intuitionistically non-deducible.

4.2 Syntactic satisfiability

In this section we give a syntactic characterization of the semantic notion of satisfiability defined in the previous pages. One could expect a sort of calculus and, in fact, we had been looking for it for a certain time; but, finally, we realized that the right expectation was to find a "co-inductive calculus". In other words, syntactic satisfiability has to be characterized as the *largest* notion fulfilling some suitable rules. Provided Γ and Δ are (possibly empty) finite lists of formulae, we use the notation

$$\Gamma \bowtie \Delta$$

to intend that & Γ, the conjunction of all the formulae in Γ, is consistent with the conjunction of those in Δ, i.e. & Δ; in other words, the conjunction of all formulae in Γ and Δ is satisfiable. The new symbol $\bowtie$ is a metalinguistic link, exactly like $\vdash$ (by the way, note that the horizontal line in a rule of a calculus is just a meta-metalinguistic link). Firstly, we want to express the fact that Γ and Δ must be thought modulo order and repetitions; thus we give the following first group of rules:

exchange rules $\dfrac{\Gamma, \varphi, \varphi', \Gamma' \bowtie \Delta}{\Gamma, \varphi', \varphi, \Gamma' \bowtie \Delta} \qquad \dfrac{\Gamma \bowtie \Delta, \varphi, \varphi', \Delta'}{\Gamma \bowtie \Delta, \varphi', \varphi, \Delta'}$

contraction rules $\dfrac{\Gamma, \varphi, \Gamma' \bowtie \Delta}{\Gamma, \varphi, \varphi, \Gamma' \bowtie \Delta} \qquad \dfrac{\Gamma \bowtie \Delta, \varphi, \Delta'}{\Gamma \bowtie \Delta, \varphi, \varphi, \Delta'}$

A second more interesting group of rules is the following:

weakening rules $\dfrac{\Gamma, \varphi \bowtie \Delta}{\Gamma \bowtie \Delta} \qquad \dfrac{\Gamma \bowtie \varphi, \Delta}{\Gamma \bowtie \Delta}$

transfer rules $\dfrac{\Gamma, \varphi \bowtie \Delta}{\Gamma \bowtie \varphi, \Delta} \qquad \dfrac{\Gamma \bowtie \varphi, \Delta}{\Gamma, \varphi \bowtie \Delta}$

(note that transfer with exchange implies commutativity of $\bowtie$). These given until now are the structural rules of the (co-)calculus. They allow one to write $\varphi \bowtie \Omega$ instead of $\Gamma \bowtie \Delta$, provided either Γ or Δ is non empty. Next we give three rules involving the connective $\vee$, the quantifier $\exists$ and the constant $\perp$:

$\vee$ **-rule** $\dfrac{\varphi \vee \psi \bowtie \Gamma}{\varphi \bowtie \Gamma \quad \psi \bowtie \Gamma}$

$\exists$ **-rule** $\dfrac{\exists x\ \varphi(x) \bowtie \Gamma}{\varphi(t) \bowtie \Gamma \quad (\text{some } t \in \mathrm{Trm})}$

$\perp$ **-rule** $\dfrac{\perp \bowtie \Gamma}{}$

where the space in the lower part of the $\vee$-rule has to be read as an "or" at the metalanguage; similarly the lower part of the $\exists$-rule has to be thought as an infinite disjunction of all $\varphi(t) \bowtie \Gamma$ for $t \in \mathrm{Trm}$. The empty lower part in the $\perp$-rule stands for a contradiction; hence this rule says that $\perp$ cannot be consistent with anything.[3] These three rules are examples of co-inductive rules: in the upper part is only one object containing the logical constant that disappears in the objects in the lower part.[4] The last rule in the calculus is the following.

$\vdash$ **-rule** $\dfrac{\Gamma \bowtie \Delta \quad \Gamma \vdash \varphi}{\varphi \bowtie \Delta}$

This rule could appear clashing with the other ones because it introduces an inductive element in the calculus, namely the derivation of $\Gamma \vdash \varphi$. Thus it could be desirable to substitute it in some way, as we will in fact do for the propositional case at the end of the chapter. However, as we will see in the next chapter, the

[3] The conventions about the upper and the lower parts of a rule are very similar to the standard ones about a single sequent.

[4] The absence of any rule involving &, $\rightarrow$ and $\forall$ is justified by Proposition 4.2.3.

strength of $\bowtie$ is greatly seen when it is used together with the entailment relation $\vdash$; in such a context the $\vdash$-rule is a necessary link between them.

As $\vdash$ is the smallest relation satisfying its rules, so $\bowtie$ has to be thought as the *largest* relation fulfilling the above rules; formally this fact can be expressed as

$$\frac{\Gamma R \Delta \quad R \text{ is a relation satisfying all the rules}}{\Gamma \bowtie \Delta} \ \text{co} - \text{induction.}$$

Of course, if a classical metatheory was adopted then $\Gamma \bowtie \Delta$ would become equivalent to $\Gamma, \Delta \nvdash \bot$. This is not the case from a constructive point of view. In fact the rule

$$\frac{\varphi \vee \psi, \Gamma \nvdash \bot}{\varphi, \Gamma \nvdash \bot \quad \psi, \Gamma \nvdash \bot}$$

cannot be justified. In the end, the reason is that $\neg A \vee \neg B$ does not follows from $\neg(A \ \& \ B)$ (see Bishop's LLPO).

The usual notion of derivation in a calculus has here its dual, namely the notion of "refutation", which is formally introduced in the following definition. To simplify the definition we think of the $\vdash$-rule as expressed by the following (non decidable!) scheme:

$$\frac{\Gamma \bowtie \Delta}{\varphi \bowtie \Delta} \quad \text{for any provable } \Gamma \vdash \varphi.$$

Definition 4.2.1 *A* refutation *of* $\Gamma \bowtie \Delta$ *is a finite tree whose nodes are instances of satisfiability such that:*

- $\Gamma \bowtie \Delta$ *is the root (think of it at the top);*
- *if* $\Gamma'' \bowtie \Delta''$ *is an immediate successor of* $\Gamma' \bowtie \Delta'$ *then there exists a rule in which the latter is the upper part and the former is one of the lower ones;*
- *every leaf is empty (that is the knots that are immediately above the leaves are all of the kind* $\bot \bowtie \Gamma$ *and each of them is followed by an application of the* $\bot$*-rule).*

The fact that the calculus is a co-inductive one implies that it can be used either for refuting an hypothesis of satisfiability or for inferring an instances of satisfiability from another; on the contrary, the proof that something is just satisfiable needs a metalinguistic investigation by means of co-induction. Let us give some examples showing all these cases.

Proposition 4.2.2 *There exists a refutation of $\varphi \bowtie \neg\varphi$.*

PROOF:

$$\dfrac{\dfrac{\varphi \bowtie \neg\varphi}{\varphi, \neg\varphi \bowtie}\text{ transfer} \qquad \varphi, \neg\varphi \vdash \bot}{\dfrac{\bot \bowtie}{}\ \bot\text{−rule}}\ \vdash\text{−rule}$$

q.e.d.

Proposition 4.2.3 *The following derived rules can be proved:*

$$\frac{\varphi \,\&\, \psi \bowtie \Gamma}{\varphi \bowtie \Gamma} \quad \frac{\varphi \,\&\, \psi \bowtie \Gamma}{\psi \bowtie \Gamma} \qquad \frac{\varphi, \varphi \to \psi \bowtie \Gamma}{\psi \bowtie \Gamma} \qquad \frac{\forall x\, \varphi(x) \bowtie \Gamma}{\varphi(t) \bowtie \Gamma}\ \forall\, t \in \mathrm{Trm}.$$

PROOF: By applications of the $\vdash$-rule. q.e.d.

Proposition 4.2.4 *The following instances of satisfiability are valid:*

1. $\top \bowtie \top$;
2. $p \bowtie p$ *for any atomic formula* p *different from* $\bot$.

PROOF: 1) Consider the relation $\Gamma R \Delta \equiv\ \vdash (\&\, \Gamma) \,\&\, (\&\, \Delta)$ and check that all rules are satisfied (disjunction and existence properties are needed together with non-deducibility of $\bot$); thus $\top \bowtie \top$ because $\top R \top$.

2) Put $\Gamma R \Delta \equiv p \vdash (\&\, \Gamma) \,\&\, (\&\, \Delta)$ and use co-induction again. As p is atomic, all the requested hypotheses follow by "cut-elimination". q.e.d.

4.3 Meta-theoretical theorems

In this section we will prove a few metatheoretical facts about satisfiability. The following theorem states the equivalence between syntactic and semantic satisfiability; this is the analogous of a soundness and completeness theorem.

Theorem 4.3.1 *For any* Γ *and* Δ*, the assertion* $\Gamma \bowtie \Delta$ *is true if and only if* $(\&\, \Gamma) \,\&\, (\&\, \Delta)$ *is (semantically) satisfiable.*

PROOF: Note that the following rules are derivable

$$\frac{\varphi, \psi \bowtie \Gamma}{\varphi \,\&\, \psi \bowtie \Gamma} \qquad \frac{\varphi \bowtie \Gamma}{\varphi \bowtie \varphi}$$

so $\Gamma \bowtie \Delta$ is the same as $(\& \,\Gamma) \,\&\, (\& \,\Delta) \bowtie (\& \,\Gamma) \,\&\, (\& \,\Delta)$. This allows us to restrict our attention to the case of a single formula. Thanks to theorem 4.1.1, proving the following statement will be enough:

$$\varphi \bowtie \ \text{ is true } \iff \varphi \ltimes \mathrm{Frm} \ \text{ holds in the canonical topology.}$$

$\Rightarrow$) We prove that $\varphi \ltimes \mathrm{Frm}$ by co-induction on the canonical $\ltimes$. Put $U = \mathrm{Frm}$ and

$$\gamma \,\epsilon\, P \equiv \gamma \bowtie$$

and check that all the hypothesis of the co-induction rule are fulfilled.

$\Leftarrow$) Let R be the relation defined by

$$\Gamma R \Delta \equiv (\& \,\Gamma) \,\&\, (\& \,\Delta) \ltimes \mathrm{Frm}$$

and check that R satisfies all the rules of the calculus. q.e.d.

The previous theorem can be restated in the following form:

$$\Gamma \bowtie \Delta \ \textit{holds} \iff V(\Gamma){\downarrow}V(\Delta) \ltimes S \ \textit{for some topological interpretation.}$$

It is natural to look at it as a soundness and completeness theorem. But which implication gives soundness and which one completeness? Classically this question is easily answered. As $\Gamma \bowtie \Delta$ is classically equivalent to $\Gamma, \Delta \nvdash \bot$, the following definitions are just a rereading of those regarding $\vdash$. An object like $\Gamma \bowtie \Delta$ is *valid in an interpretation* if $V(\Gamma){\downarrow}V(\Delta) \ltimes S$ holds in that interpretation; $\Gamma \bowtie \Delta$ is *valid* if it is valid in *some* interpretation. The two parts of the soundness and completeness theorem are:

$$(\textit{soundness}) \qquad \textit{if } \Gamma \bowtie \Delta \textit{ is valid then it holds};$$

$$(\textit{completeness}) \qquad \textit{if } \Gamma \bowtie \Delta \textit{ holds then it is valid.}$$

Moreover, one has to consider that completeness is the hardest part in the theorem, as it needs the construction of a canonical model. On the contrary, soundness is proved by a simple application of the co-inductive definition of $\bowtie$.

The following proposition shows a duality between the notion of "refutation" and that of "derivation".

Proposition 4.3.2 *There exists a refutation of* $\Gamma \bowtie \Delta$ *if and only if there exists a derivation of* $\Gamma, \Delta \vdash \bot$.

PROOF: The "if" part is sketched below.

$$\dfrac{\dfrac{\begin{array}{c}\Gamma \bowtie \Delta \\ \vdots\ \text{transfers} \\ \Gamma, \Delta \bowtie\end{array} \qquad \begin{array}{c}\vdots \\ \Gamma, \Delta \vdash \bot\end{array}}{\bot \bowtie}\ \vdash -\text{rule}}{}\ \bot-\text{rule}$$

With regard to the other implication, the idea is to reverse the refutation of $\Gamma \bowtie \Delta$ and substitute each rule with a suitable derivation. We write the original rule on the left-hand side while the corresponding derivation is written on the right:

$$\frac{\varphi \vee \psi \bowtie \Gamma}{\varphi \bowtie \Gamma \quad \psi \bowtie \Gamma} \quad \rightsquigarrow \quad \frac{\Gamma, \varphi \vdash \bot \quad \Gamma, \psi \vdash \bot}{\Gamma, \varphi \vee \psi \vdash \bot}$$

$$\frac{\exists x\ \varphi(x) \bowtie \Gamma}{\varphi(t) \bowtie \Gamma \quad (\text{some t} \in \text{Trm})} \quad \rightsquigarrow \quad \frac{\Gamma, \varphi(t) \vdash \bot \quad (\text{for all t} \in \text{Trm})}{\Gamma, \exists x\ \varphi(x) \vdash \bot}$$

$$\frac{\bot \bowtie \Gamma}{} \quad \rightsquigarrow \quad \frac{\begin{array}{c}\dfrac{}{\bot \vdash \bot} \\ \vdots\end{array}}{\Gamma, \bot \vdash \bot}$$

$$\frac{\Gamma \bowtie \Delta \quad \Gamma \vdash \varphi}{\varphi \bowtie \Delta} \quad \rightsquigarrow \quad \frac{\Gamma \vdash \varphi \quad \Delta, \varphi \vdash \bot}{\Gamma, \Delta \vdash \bot}$$

Moreover, each structural rule is transformed according to the following scheme:

$$\frac{\Gamma \bowtie \Delta}{\Gamma' \bowtie \Delta'} \quad \rightsquigarrow \quad \frac{\Gamma', \Delta' \vdash \bot}{\Gamma, \Delta \vdash \bot}$$

(the right-hand side is valid thanks to the corresponding structural rules of the sequent calculus, apart from the case of transfer which is trivial). q.e.d.

Thanks to this proposition, the existence of a refutation of $\varphi \bowtie$ is exactly the same as asserting refutability of φ, according to the usual definition of refutability of a formula.

Note that, in the previous proof, the $\vdash$-rule corresponds to the cut-rule. Thus a natural question is whether it is possible to replace the $\vdash$-rule by a list of "better" rules in such a way that the collection of refutable objects remains the same. The answer is affirmative at least in the propositional case. Indeed, it is known that there exists a Hilbert style formulation for intuitionistic propositional logic. In other words, the set of all intuitionistically true propositional formulae can be characterized as the smallest one that contains some axioms and is closed under modus ponens. Thus the $\vdash$ -rule can be substituted by the following (derivable) rules:

$$\frac{\Gamma \bowtie \Delta}{\Gamma, A \bowtie \Delta} \text{ for any axiom A }, \qquad \frac{\varphi, \varphi \to \psi \bowtie \Gamma}{\psi \bowtie \Gamma} \ .$$

The extension of this argument to the predicative case looks quite difficult. In fact, one would need something corresponding to the generalization rule $(\varphi(t)/\forall x\, \varphi(x))$; on the contrary the natural rule

$$\frac{\varphi(t) \bowtie \Gamma \quad \forall\, t \in \mathrm{Trm}}{\forall x\, \varphi(x) \bowtie \Gamma}$$

is not valid. In fact, by assuming a classical metalanguage the rule would be equivalent to the statement: "if $\Gamma \vdash \neg\forall x\, \varphi(x)$ then $\Gamma \vdash \neg\varphi(t)$ for some $t \in \mathrm{Trm}$" which is not true for LJ. The problem could be solved by allowing Γ and Δ in the expression $\Gamma \bowtie \Delta$ to be infinite and then using the rule:

$$\frac{\{\varphi(t) :\ t \in \mathrm{Trm}\} \bowtie \Gamma}{\forall x\, \varphi(x) \bowtie \Gamma} \ .$$

4.4 Deducibility and satisfiability together

Let $\widetilde{LJ}$ be the formal system obtained by enriching LJ with the new metalinguistic link $\bowtie$ and all its rules. Theorems 2.1.2, 2.3.2 and 4.3.1 together can be seen as a soundness and completeness theorem for $\widetilde{LJ}$ with respect to the semantics given by topologies.

Semantics for non-deducibility as developed in the previous chapter makes essential use of the positivity relation in all is strength. On the contrary, semantics for satisfiability only involves positivity in statements of the form $U{\downarrow}V \ltimes S$. Thus, what is really needed to study satisfiability is the existential operator $\mathcal{E}$ (see paragraph 1.3.2). For this reason and in view of paragraph 1.4.1 it seems natural to extend the semantics to o-algebras.

Definition 4.4.1 *Let $(P, =, \leq, \asymp, \wedge, \bigvee, \to, 0, 1)$ be an o-algebra and D be a domain for interpreting terms. Moreover, let $V(p)$ be an element of P for any atomic formula p. The valuating function V is extended to arbitrary formulae by the following recursive clauses:*

- $V(\bot) = 0$*;*

- $V(\varphi \,\&\, \psi) = V(\varphi) \wedge V(\psi)$;
- $V(\varphi \vee \psi) = V(\varphi) \vee V(\psi)$;
- $V(\varphi \rightarrow \psi) = V(\varphi) \rightarrow V(\psi)$;
- $V(\forall x\ \varphi(x)) = \bigwedge_{d \in D} V(\varphi(d))$;
- $V(\exists x\ \varphi(x)) = \bigvee_{d \in D} V(\varphi(d))$.

A formula, say φ, is valid in P if $V(\varphi) = 1$ (the top element). Moreover, the sequent $\Gamma \vdash \varphi$ is valid in P if $V(\Gamma) \leq V(\varphi)$; while $\Gamma \bowtie \Delta$ is valid in P if $V(\Gamma) \between V(\Delta)$. Finally, a rule (either inductive or co-inductive) is valid in P if the validity in P of all its premises implies the validity in P of some of its consequences.

Definition 4.4.2 *A formula is* valid *if it is valid in all o-algebras. A sequent is* valid *if it is valid in all o-algebras. An assertion like $\Gamma \bowtie \Delta$ is* valid *if it is valid in* some *o-algebra. A rule is* valid *if it is valid in all o-algebras.*

Theorem 4.4.3 *$\widetilde{LJ}$ is sound and complete with respect to the semantics given by o-algebras.*

PROOF: By constructing a canonical o-algebra which, of course, is that represented (according to paragraph 1.4.1) by the canonical topology. q.e.d.

Remark - Paragraph 1.4.1 suggests to read $U \between V$ as $U{\downarrow}V \ltimes \mathrm{Frm}$, providing $U, V \subseteq \mathrm{Frm}$. This is equivalent to say that $\exists\varphi \in U$ and $\exists\psi \in U$ such that $\varphi{\downarrow}\psi \ltimes \mathrm{Frm}$. Thanks to the canonical valuation lemma, this is just $V(\varphi){\downarrow}V(\psi) \ltimes \mathrm{Frm}$ and hence $\varphi \bowtie \psi$. Summing up we have

$$U \between V \equiv (\exists\varphi \in U)(\exists\psi \in V)(\varphi \bowtie \psi)$$

thus defining the canonical overlap relation in terms of syntactic satisfiability.

Chapter 5

An application: constructive tense logic

The new metalinguistic link, written $\bowtie$, introduced in the previous chapter is needed to solve the following problem: to find a complete axiomatization of the minimal intuitionistic tense logic, provided only constructive reasonings are allowed at the metalinguistic level.

5.1 The calculus CMT

Consider the $\widetilde{LJ}$ calculus (cut rule included) enriched with the following three equivalences (six rules) about the four modalities $\Diamond$, $\Box$, $\blacklozenge$ and $\blacksquare$:

$$\Diamond\varphi \vdash \psi \quad \equiv \quad \varphi \vdash \blacksquare\psi$$

$$\blacklozenge\varphi \vdash \psi \quad \equiv \quad \varphi \vdash \Box\psi$$

$$\Diamond\varphi \bowtie \psi \quad \equiv \quad \varphi \bowtie \blacklozenge\psi$$

(read the four modalities as possibility and necessity in the future and in the past, respectively). The name we propose for the resulting logic is "constructive minimal tense logic", briefly CMT; justifications for using the word "minimal" are being given during the chapter.

Of course, if one uses a classical metalanguage, it is possible to read $\Gamma \bowtie \Delta$ as standing for $\Gamma, \Delta \nvdash \bot$ and then the third equation can be changed into

$$\Diamond\varphi, \psi \vdash \bot \equiv \varphi, \blacklozenge\psi \vdash \bot$$

thus allowing the derivation of the standard equivalence $\Diamond\varphi \equiv \neg\Box\neg\varphi$ (and similarly for the "black" modalities). Note that, in a classical setting, the third equation can be rewritten as

$$\vdash \Box\varphi \vee \psi \equiv \vdash \varphi \vee \blacksquare\psi$$

which is exactly the syntactic version of a well known condition about bi-modal algebras (see [11]). On the contrary, from a constructive point of view, $\Gamma \bowtie \Delta$ is not the same as $\Gamma, \Delta \nvdash \bot$; actually, the former is stronger than the latter.

Next is a brief list of theorems (and metatheorems) of CMT. Because of the symmetry of the calculus, the symmetric of any theorem in the following list is also provable (the symmetric assertion is obtained by simultaneous exchange of $\Box$ with $\blacksquare$ and $\Diamond$ with $\blacklozenge$).

Proposition 5.1.1 *The following facts hold for any $\varphi, \psi \in \mathrm{Frm}$:*

1. $\dfrac{\vdash \varphi}{\vdash \Box\varphi}$, $\dfrac{\Diamond\varphi \bowtie}{\varphi \bowtie}$;
2. $\dfrac{\varphi \vdash \psi}{\Box\varphi \vdash \Box\psi}$, $\dfrac{\varphi \vdash \psi}{\Diamond\varphi \vdash \Diamond\psi}$;
3. $\Box(\varphi \,\&\, \psi) \equiv (\Box\varphi) \,\&\, (\Box\psi)$, $\Box\forall x\ \varphi(x) \equiv \forall x\ \Box\varphi(x)$;
4. $\Diamond(\varphi \vee \psi) \equiv (\Diamond\varphi) \vee (\Diamond\psi)$, $\Diamond\exists x\ \varphi(x) \equiv \exists x\ \Diamond\varphi(x)$;
5. $\Box(\varphi \to \psi) \vdash \Box\varphi \to \Box\psi$;
6. $\top \equiv \Box\top$, $\bot \equiv \Diamond\bot$;
7. $\dfrac{\Diamond\varphi \bowtie \Box\psi}{\varphi \bowtie \psi}$.

Proof: 1.

$$\cfrac{\cfrac{\cfrac{\vdash \varphi}{\blacklozenge\top \vdash \varphi}}{\top \vdash \Box\varphi}}{\vdash \Box\varphi} \qquad \cfrac{\cfrac{\cfrac{\Diamond\varphi \bowtie \qquad \overline{\vdash \top}}{\Diamond\varphi \bowtie \top}}{\varphi \bowtie \blacklozenge\top}}{\varphi \bowtie}$$

2.

$$\cfrac{\cfrac{\cfrac{\overline{\Box\varphi \vdash \Box\varphi}}{\blacklozenge\Box\varphi \vdash \varphi} \quad \varphi \vdash \psi}{\blacklozenge\Box\varphi \vdash \psi}}{\Box\varphi \vdash \Box\psi} \qquad \cfrac{\cfrac{\varphi \vdash \psi \quad \cfrac{\overline{\Diamond\psi \vdash \Diamond\psi}}{\psi \vdash \blacksquare\Diamond\psi}}{\varphi \vdash \blacksquare\Diamond\psi}}{\Diamond\varphi \vdash \Diamond\psi}$$

3.

$$\dfrac{\dfrac{\dfrac{}{\varphi \,\&\, \psi \vdash \varphi}}{\Box(\varphi \,\&\, \psi) \vdash \Box\varphi} \quad \dfrac{\dfrac{}{\varphi \,\&\, \psi \vdash \psi}}{\Box(\varphi \,\&\, \psi) \vdash \Box\psi}}{\Box(\varphi \,\&\, \psi) \vdash (\Box\varphi) \,\&\, (\Box\psi)} \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{\Box\varphi \,\&\, \Box\psi \vdash \Box\varphi}}{\blacklozenge(\Box\varphi \,\&\, \Box\psi) \vdash \varphi} \quad \dfrac{\dfrac{}{\Box\varphi \,\&\, \Box\psi \vdash \Box\psi}}{\blacklozenge(\Box\varphi \,\&\, \Box\psi) \vdash \psi}}{\blacklozenge(\Box\varphi \,\&\, \Box\psi) \vdash \varphi \,\&\, \psi}}{(\Box\varphi) \,\&\, (\Box\psi) \vdash \Box(\varphi \,\&\, \psi)}$$

and similarly for the other equivalence.

4.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\Diamond\varphi \vdash \Diamond\varphi \vee \Diamond\psi}}{\varphi \vdash \blacksquare(\Diamond\varphi \vee \Diamond\psi)} \quad \dfrac{\dfrac{}{\Diamond\psi \vdash \Diamond\varphi \vee \Diamond\psi}}{\psi \vdash \blacksquare(\Diamond\varphi \vee \Diamond\psi)}}{\varphi \vee \psi \vdash \blacksquare(\Diamond\varphi \vee \Diamond\psi)}}{\Diamond(\varphi \vee \psi) \vdash (\Diamond\varphi) \vee (\Diamond\psi)} \qquad \dfrac{\dfrac{\dfrac{}{\varphi \vdash \varphi \vee \psi}}{\Diamond\varphi \vdash \Diamond(\varphi \vee \psi)} \quad \dfrac{\dfrac{}{\psi \vdash \varphi \vee \psi}}{\Diamond\psi \vdash \Diamond(\varphi \vee \psi)}}{(\Diamond\varphi) \vee (\Diamond\psi) \vdash \Diamond(\varphi \vee \psi)}$$

and similarly for the case of $\exists$.

5.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\varphi \,\&\, (\varphi \to \psi) \vdash \psi}}{\Box(\varphi \,\&\, (\varphi \to \psi)) \vdash \Box\psi}}{\Box\varphi \,\&\, \Box(\varphi \to \psi) \vdash \Box\psi}}{\Box(\varphi \to \psi) \vdash \Box\varphi \to \Box\psi}$$

6.

$$\dfrac{\dfrac{}{\blacklozenge\top \vdash \top}}{\top \vdash \Box\top} \qquad \dfrac{\dfrac{}{\bot \vdash \blacksquare\bot}}{\Diamond\bot \vdash \bot}$$

7.

$$\dfrac{\dfrac{\Diamond\varphi \bowtie \Box\psi}{\varphi \bowtie \blacklozenge\Box\psi} \quad \dfrac{\dfrac{}{\Box\psi \vdash \Box\psi}}{\blacklozenge\Box\psi \vdash \psi}}{\varphi \bowtie \psi}$$

q.e.d.

Item 7. can be used to prove the following two facts

$$\neg(\Diamond\varphi \bowtie \Box\neg\varphi) \qquad \neg(\Diamond\neg\varphi \bowtie \Box\varphi)$$

which can be read as the constructive counterparts of the following (maybe expected) ones:

$$\Diamond\varphi \vdash \neg\Box\neg\varphi\ , \quad \Box\varphi \vdash \neg\Diamond\neg\varphi\ , \quad \Diamond\neg\varphi \vdash \neg\Box\varphi\ , \quad \Box\neg\varphi \vdash \neg\Diamond\varphi\ .$$

If one wants to explicitly obtain the latter group from the former, the extension to CMT of proposition 4.3.2 is needed. To make its proof work it is enough to enrich the calculus with: $\Diamond\varphi, \psi \vdash \bot \equiv \varphi, \blacklozenge\psi \vdash \bot$. This enriched CMT allows the proof of $\neg\Diamond\varphi \vdash \Box\neg\varphi$ and hence $\Box\neg\varphi$ becomes equivalent to $\neg\Diamond\varphi$. Note, however, that $\neg\Box\varphi \vdash \Diamond\neg\varphi$ does not hold in the enriched CMT either; in fact it need classical reasonings to be proved.

5.2 Soundness theorem for CMT

We would like to show that CMT is the minimal tense logic in the sense that it is valid in all Kripke frames. Actually, because of the constructive foundation we want to use, we will be able to characterize CMT as the logic of more abstract algebraic structures, namely o-Kripke frames. So there exist some statements which hold in all Kripke frames but are not true in CMT, e.g. Fisher-Servi's axioms. In this precise sense, CMT is even more basic than the other tense logics. We start with the soundness theorem with respect to Kripke frames.

Definition 5.2.1 *Let (X, r) be a Kripke frame and think of it as the basic pair (X, r, X) (see paragraph 1.2.4). Let $V(p)$ be a subset of X for each atomic formula p in the language of CMT. Finally let D be a domain for interpreting terms, as usual. The valuating function V is extended to non atomic formulae by using the following inductive clauses:*

- $V(\bot) = \emptyset$;
- $V(\varphi \,\&\, \psi) = V(\varphi) \cap V(\psi)$;
- $V(\varphi \vee \psi) = V(\varphi) \cup V(\psi)$;
- $V(\varphi \to \psi) = \{x : \; x \,\epsilon\, V(\varphi) \to x \,\epsilon\, V(\psi)\}$;
- $V(\forall x \; \varphi(x)) = \bigcap_{d \in D} V(\varphi(d))$;
- $V(\exists x \; \varphi(x)) = \bigcup_{d \in D} V(\varphi(d))$;
- $V(\Box\varphi) = r^* V(\varphi) = \{x : \; rx \subseteq V(\varphi)\}$;
- $V(\Diamond\varphi) = r^- V(\varphi) = \{x : \; rx \between V(\varphi)\}$;
- $V(\blacksquare\varphi) = r^{-*} V(\varphi)$;
- $V(\blacklozenge\varphi) = rV(\varphi)$.

A formula, say φ, is valid in (X, r) (under the valuation V) if $V(\varphi) = X$. Moreover, the sequent $\Gamma \vdash \varphi$ is valid in (X, r) if $V(\Gamma) \subseteq V(\varphi)$; while $\Gamma \bowtie \Delta$ is valid in (X, r) if $V(\Gamma) \between V(\Delta)$. Finally, a rule (either inductive or co-inductive) is valid in that frame if the validity in the frame of all its premises implies the validity in the frame of some of its consequences.

Definition 5.2.2 *A formula is* valid *if it is valid in all interpretations. A sequent is* valid *if it is valid in all interpretations. An assertion like $\Gamma \bowtie \Delta$ is* valid *if it is valid in* some *interpretation. A rule is* valid *if it is valid in all interpretations.*

Lemma 5.2.3 *All the rules of CMT are valid.*

PROOF: The proof is quite trivial. As an example let us consider the following:

$$\frac{\Diamond\varphi \bowtie \psi}{\varphi \bowtie \blacklozenge\psi}\ .$$

Let (X, r) be an arbitrary Kripke frame and suppose that the upper part of the rule is valid in X, i. e. $r^{-}V(\varphi) \between V(\psi)$. Because $r \cdot\!|\!\cdot r^{-}$ (r and r^{-} are symmetric; see paragraph 1.2.4) one gets $V(\varphi) \between rV(\psi)$ which exactly is validity in (X, r) of the lower part of the rule. So the rule is valid in (X, r) and then it is valid. q.e.d.

Proposition 5.2.4 (Soundness of CMT with respect to Kripke frames) *If a sequent is provable in CMT then it is valid. If $\Gamma \bowtie \Delta$ is valid then it is true in CMT.*

PROOF: The first part follows from the previous lemma by using induction on the derivation of the sequent. As for the second part, remember that $\bowtie$ is the greatest relation which satisfies all the given rules. In other words, we can use co-induction to prove that $\Gamma \bowtie \Delta$ holds in CMT. By hypothesis, $\Gamma \bowtie \Delta$ is valid; this means it is valid in at least one interpretation. Thus, there exists a Kripke frame (X, r) and a valuating function $V : \mathrm{Frm} \to \mathcal{P}X$ such that $V(\Gamma) \between V(\Delta)$ holds. Let R be the relation defined by

$$\Gamma\, R\, \Delta \quad \equiv \quad V(\Gamma) \between V(\Delta)$$

and note that the previous lemma forces R to satisfy all rules for $\bowtie$. q.e.d.

As we have already said, we would like to consider a generalization of the notion of a Kripke frame.

Definition 5.2.5 *An* o-Kripke frame *is an o-algebra, say $\mathcal{P}$, endowed with an o-relation $(F, F', G, G') : \mathcal{P} \to \mathcal{P}$ (see definition 1.4.5).*

By reading F, F', G and G' instead of r, r^-, r^* and r^{-*} respectively, it is easy to note that both the notion of validity and the soundness theorem for CMT can be extended to o-Kripke frames in a natural way. This theorem is the tool we are going to use in the following corollary in order to establish some non-deducibility results.

Corollary 5.2.6 *The following sequents are* not *provable in CMT:*

1. $\Diamond(\varphi \to \psi) \vdash \Box\varphi \to \Diamond\psi$;

2. $\Diamond\varphi \to \Box\psi \vdash \Box(\varphi \to \psi)$;

3. $\Box(\varphi \to \psi) \vdash \Diamond\varphi \to \Diamond\psi$.

Moreover the equivalence $\Diamond\varphi, \psi \vdash \bot \equiv \varphi, \blacklozenge\psi \vdash \bot$ is not valid.

PROOF: Let $\mathcal{P} = \{0, a, 1\}$ be a linear order and consider the corresponding (complete) Heyting algebra. The adjunction

$$z \wedge x \leq y \equiv z \leq x \to y$$

forces $\to$ to satisfy the following items:

$$x \to 1 \ = \ 1 \ , \quad 0 \to x \ = \ 1 \ , \quad x \to x \ = \ 1 \ ,$$

$$1 \to 0 \ = \ 0 \ , \quad a \to 0 \ = \ 0 \ , \quad 1 \to a \ = \ a \ .$$

1). Let $x \between y$ be $x \wedge y \neq 0$. Hence $x \between y$ fails if either x or y is 0 and is true otherwise. This makes $\mathcal{P}$ an o-algebra. In order to obtain an o-frame we need to define an o-relation. Let $F' = id = G'$ and put:

$$F(0) \ = \ 0 \ , \quad F(a) \ = \ a \ , \quad F(1) \ = \ a \ ,$$

$$G(0) \ = \ 0 \ , \quad G(a) \ = \ 1 \ , \quad G(1) \ = \ 1 \ .$$

Checking they form a symmetric pair of adjunctions is a straightforward calculation. Now put

$$V(\varphi) = a = V(\psi)$$

and compute the valuation of the compound formulae to get $1 \leq a$ as the valuation of the sequent. This is false, of course.

2). Use the same o-algebra as before but exchange F and G with F' and G' respectively. Then put $V(\varphi) = 1$ and $V(\psi) = a$; what one eventually gets is $1 \leq a$ again.

3). Item 3 implies $\Box\neg\varphi \vdash \neg\Diamond\varphi$ since $\Diamond\bot \equiv \bot$. So, in order to obtain non-deducibility of 3, it is enough to prove non-deducibility of the latter. Let us consider the same Heyting algebra as before but a different overlap relation, namely let $x \between y$ be always false apart from $1 \between 1$ which we ask to be true. Again let us choose F' and G' to be the identity map and put:

$$\begin{array}{lll} F(0) = 0\ , & F(a) = 0\ , & F(1) = 1\ , \\ G(0) = a\ , & G(a) = a\ , & G(1) = 1\ . \end{array}$$

Finally let $V(\varphi)$ be equal to a and eventually get $a \leq 0$. q.e.d.

5.3 The completeness theorem for CMT

Now we want to prove that the semantics given by o-Kripke frames is also complete. The proof is by constructing a canonical o-Kripke frame from the syntax.

5.3.1 The canonical o-Kripke frame

Let Frm be the set of all formulae in the language of CMT and let $\lhd$ be the cover on Frm generated like in section 2.2 provided that $\vdash$ is now read as the entailment relation in CMT. Of course the proof of lemma 2.2.1, corollary 2.2.2, lemma 2.2.3, lemma 2.2.4 and proposition 2.2.5 work in the CMT case as well. Like for the case of LJ, the structure

$$(\mathcal{P}\mathrm{Frm}, =_{\mathcal{A}}, \lhd, \downarrow, \bigcup, \rightarrow_{\mathcal{A}}, \emptyset, \mathrm{Frm})$$

is a complete Heyting algebra. In order to obtain an o-algebra, we need to define an overlap relation. The remark at the very end of the previous chapter suggests to put:

$$U \between V \quad \equiv \quad (\exists\varphi \,\epsilon\, U)(\exists\psi \,\epsilon\, V)(\varphi \bowtie \psi)\ .$$

First of all we have to prove that it respects $=_{\mathcal{A}}$. For this purpose, it is enough to prove the following fact:

$$\frac{\varphi \bowtie \psi \quad \varphi \lhd U}{(\exists\gamma \,\epsilon\, U)(\gamma \bowtie \psi)}\ .$$

The argument is an induction on the proof of $\varphi \lhd U$. The case $\varphi \epsilon U$ is trivial. If $\varphi \vdash \bot$, then from $\varphi \bowtie \psi$ one gets $\bot \bowtie \psi$ which is a contradiction; so everything follows. If, instead, $\varphi \vdash \gamma$ and $\gamma \lhd U$ then $\gamma \bowtie \psi$ and one can use the inductive hypothesis. Moreover, if $\varphi = \varphi_1 \vee \varphi_2$ and $\varphi_i \lhd U$ then either $\varphi_1 \bowtie \psi$ or $\varphi_2 \bowtie \psi$ by the $\vee$-rule for $\bowtie$. In both case the thesis follows by the inductive hypothesis. Similarly if $\varphi = \exists x\, \gamma(x)$.

The other facts needed to obtain an o-algebra are quite easy to check. As an example, we want to prove the 2nd rule for $\between$, namely

$$\frac{U \between V}{U{\downarrow}V \between V}\ .$$

The premise means there exist $\varphi \epsilon U$ and $\psi \epsilon V$ such that $\varphi \bowtie \psi$; this implies $\varphi \,\&\, \psi \bowtie \psi$. Trivially $\varphi \,\&\, \psi \lhd \varphi$ and $\varphi \,\&\, \psi \lhd \psi$; thus $\varphi \,\&\, \psi$ belongs to $\varphi{\downarrow}\psi$ and then to $U{\downarrow}V$ too.

In order to obtain an o-Kripke frame, let us consider the four operators defined by:

$$F'(U) = \{\Diamond\varphi :\ \varphi \lhd U\}, \quad G'(U) = \{\varphi :\ \Diamond\varphi \lhd U\},$$
$$F(U) = \{\blacklozenge\varphi :\ \varphi \lhd U\}, \quad G(U) = \{\varphi :\ \blacklozenge\varphi \lhd U\}.$$

The ideas underlying these definitions should be clear after the following lemma.

Lemma 5.3.1 *For every $\varphi \in \mathrm{Frm}$ and $U \subseteq \mathrm{Frm}$ the following hold:*

1. $F'(\varphi) =_{\mathcal{A}} \Diamond\varphi, \quad F(\varphi) =_{\mathcal{A}} \blacklozenge\varphi;$

2. $G(\varphi) =_{\mathcal{A}} \Box\varphi, \quad G'(\varphi) =_{\mathcal{A}} \blacksquare\varphi;$

3. $\varphi \lhd G'(U) \Rightarrow \varphi \epsilon G'(U), \quad \varphi \lhd G(U) \Rightarrow \varphi \epsilon G(U).$

Proof: To prove 1. it is enough to check that $F'(\varphi) \lhd \Diamond\varphi$. So let $\Diamond\psi$ be an element of $F'(\varphi)$, that is $\psi \lhd \varphi$. This is the same as $\psi \vdash \varphi$ that entails $\Diamond\psi \vdash \Diamond\varphi$, i.e. $\Diamond\psi \lhd \Diamond\varphi$.

In order to prove 2., take $\psi \epsilon G(\varphi)$; then $\blacklozenge\psi \lhd \varphi$ and so $\blacklozenge\psi \vdash \varphi$. In CMT this is the same as $\psi \vdash \Box\varphi$, i.e. $\psi \lhd \Box\varphi$; so $G(\varphi) \lhd \Box\varphi$. Vice versa, $\Box\varphi \epsilon G(\varphi)$ because $\Box\varphi \vdash \Box\varphi$ yields $\blacklozenge\Box\varphi \vdash \varphi$ and then $\blacklozenge\Box\varphi \lhd \varphi$.

As regarding 3., let us check that $\varphi \lhd G'(U) \Rightarrow \Diamond\varphi \lhd U$ by induction on the last rule used in the proof of $\varphi \lhd G'(U)$. If it is the reflexivity rule then all is trivial. If $\varphi \vdash \bot$ then $\Diamond\varphi \vdash \Diamond\bot$ and the $\Diamond\varphi \vdash \bot$ as $\Diamond\bot \equiv \bot$; thus $\Diamond\varphi \lhd U$ by the $\bot$-rule. If, instead, $\varphi \vdash \psi$ and $\psi \lhd G'(U)$ then $\Diamond\varphi \vdash \Diamond\psi$ and $\Diamond\psi \lhd U$ by inductive hypothesis; the thesis follows by the $\vdash$-rule. If the last step is an application of the $\vee$-rule then $\varphi = \varphi_1 \vee \varphi_2$ and $\Diamond\varphi_i \lhd U$ by inductive hypothesis; on the other hand, $\Diamond\varphi \equiv \Diamond\varphi_1 \vee \Diamond\varphi_2$, so $\vee$-rule can be applied. Similarly for the last case involving $\exists$. q.e.d.

So F' is the same of $\Diamond$ when it is restricted to single formulae. This is exactly what we want since F' must be used to model $\Diamond$. Moreover, as G' must be the right adjoint of F', then $G'(\varphi)$ must correspond to $\blacksquare$.

Proposition 5.3.2 *The operators F, G, F' and G' defined above form a symmetric pair of adjunctions.*

PROOF: Firstly, let us prove $F' \dashv G'$. Suppose $F'(U) \lhd V$; this means $\Diamond\varphi \lhd V$ for every $\varphi \lhd U$. Take $\varphi \,\epsilon\, U$, then $\varphi \lhd U$ and so $\Diamond\varphi \lhd V$ by hypothesis. This is just $\varphi \,\epsilon\, G'(V)$; so $U \subseteq G'(V)$ and $U \lhd G'(V)$. Now start from $U \lhd G'(V)$ and take $\Diamond\varphi \,\epsilon\, F'(U)$ i.e. $\varphi \lhd U$. Transitivity implies $\varphi \lhd G'(V)$ and then $\varphi \,\epsilon\, G'(V)$ by item 3. of the previous lemma. So $\Diamond\varphi \lhd V$ by definition and then $F'(U) \lhd V$.

The proof of $F \dashv G$ is symmetric. Lastly, we must prove $F \cdot | \cdot F'$. Suppose $F'(U) \between V$; so there exist $\varphi \lhd U$ and $\psi \,\epsilon\, V$ such that $\Diamond\varphi \bowtie \psi$. In CMT this is just the same as $\varphi \bowtie \blacklozenge\psi$. Since $\psi \lhd V$ (by reflexivity) one gets $\varphi \between F(V)$. Thanks to the properties of an o-algebra, from $\varphi \between F(V)$ and $\varphi \lhd U$ it follows that $U \between F(V)$. The other direction of $F \cdot | \cdot F'$ has a symmetric proof. q.e.d.

5.3.2 The completeness proof

As usual, the canonical valuation is obtained by considering the set Trm as domain for interpreting terms and by putting

$$V(p) = \{\varphi \in \mathrm{Frm} : \ \varphi \vdash p\}$$

for each instance p of an atomic formula. As before, the following lemma is fundamental.

Lemma 5.3.3 (of the canonical valuation) *For every $\varphi, \psi \in \mathrm{Frm}$,*

$$\varphi \lhd V(\psi) \equiv \varphi \vdash \psi.$$

PROOF: Since $\varphi \vdash \psi$ is the same as $\varphi \lhd \psi$, the lemma will be proved if we check that $V(\psi) =_{\mathcal{A}} \psi$. The proof is by induction on the structure of ψ. Thanks to the proof of proposition 2.3.1, we only have to analyze the case involving $\Diamond$, $\Box$, $\blacklozenge$ and $\blacksquare$. By symmetry the proofs for $\Diamond$ and $\Box$ will be enough.

Let ψ be $\Diamond\gamma$ for some formula γ. By inductive hypothesis $\gamma =_{\mathcal{A}} V(\gamma)$ and hence $F'\gamma =_{\mathcal{A}} F'V(\gamma)$. From item 1. of lemma 5.3.1 and the definition of $V(\Diamond\gamma)$, it follows that $\Diamond\gamma =_{\mathcal{A}} V(\Diamond\gamma)$.

If $\psi = \Box\gamma$ the proof is similar. From $\gamma =_{\mathcal{A}} V(\gamma)$ one gets $G\gamma =_{\mathcal{A}} GV(\gamma)$ that is $\Box\gamma =_{\mathcal{A}} V(\Box\gamma)$ by item 2. of lemma 5.3.1. q.e.d.

Proposition 5.3.4 (Completeness theorem for CMT**)** *If a sequent is valid then it holds in* CMT*. If an assertion like* $\Gamma \bowtie \Delta$ *holds in* CMT *then it is valid.*

PROOF: The proof of the first assertion in the theorem is exactly the same as in theorem 2.3.2. For what regards the second part, we have to prove that if $\Gamma \bowtie \Delta$ holds in the calculus CMT then it is valid in some interpretation, namely the canonical one. So suppose $\Gamma \bowtie \Delta$; by the rules for $\bowtie$ this is equivalent to $(\&\ \Gamma) \bowtie (\&\ \Delta)$ and hence to $(\&\ \Gamma) \lessgtr (\&\ \Delta)$. By the canonical valuation lemma and by the fact that $\lessgtr$ respects $=_{\mathcal{A}}$, one can get $V(\&\ \Gamma) \lessgtr V(\&\ \Delta)$, that is, $V(\Gamma) \lessgtr V(\Delta)$; so $\Gamma \bowtie \Delta$ is valid in the canonical interpretation. q.e.d.

As a by-product of the above discussions one has that a sequent holds in CMT if and only if it is valid in the canonical interpretation. Similarly, $\Gamma \bowtie \Delta$ holds in CMT if and only if $V(\Gamma) \lessgtr V(\Delta)$ in the canonical o-Kripke frame.

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