# **Regular opens** in formal topology and representation theorem for **overlap algebras**

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# Overview

1. **OVERLAP ALGEBRAS** and open locales

   Key words: positivity (Formal Topology),
   openness (locale theory),
   inhabitedness (set-theory).

   Main notion involved: Overlap Algebras (Sambin)

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   Problem: improve the usual definition of REGULAR elements in point-free topology

   Result: a NEW definition of REGULAR OPEN subset for Formal Topologies and open locales

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3. **Representation** theorem for Overlap Algebras (via Regular elements)

   Connections between parts 1 and 2

# Part I

## Overlap Algebras

and

open locales

# Problem 1:

complete the following table:
(fill in the appropriate ALGEBRAIC COUNTERPART of the notions on the first row according to the UNDERLYING LOGIC (first column))

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# Attempt 1: cHa (Locale)

We need the notion of INHABITED subset.

**How do you define INHABITED (POSITIVE) elements in a cHa ?**

$x \neq 0$ does not work, because:

$$
\begin{array}{ccc}
U \neq \emptyset & \not\Rightarrow & U \text{ inhabited} \\
\Updownarrow & & \Updownarrow \\
\neg\forall a \neg(a \in U) & & \exists a (a \in U)
\end{array}
$$

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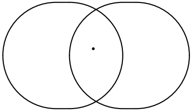
Perfect for Topology
BUT
NOT satisfactory for arbitrary subsets!
(see next slides)

# The overlap relation
(between two subsets)



**Definition:**

$$U \between V \quad \overset{def}{\iff} \quad U \cap V \text{ is inhabited}$$

Consequently:

$$U \text{ is inhabited} \quad \iff \quad U \between U$$

# Overlap relation and Positivity: $\between$ vs Pos
## The "density" property

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## The "density" property

There exists a strict analogy between $U \between V$ and $\mathrm{Pos}(U \wedge V)$

BUT $\between$ is STRONGER than Pos

because it satisfies:

**DENSITY**

$$\forall W \; (U \between W \;\Rightarrow\; V \between W) \quad \underset{\Longleftrightarrow}{\Longrightarrow} \quad U \subseteq V$$

which implies:
$(U \between U \Rightarrow U \subseteq V) \Rightarrow U \subseteq V$ (positivity axiom).

# Overlap Algebras

**Definition (Overlap Algebra)**

Complete lattice

$+$

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- $x \between (y \wedge z) \iff (x \wedge y) \between z$

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- $x \between (\bigvee_{i\in I} y_i) \iff (\exists i \in I)\ (x \between y_i)$
- $x \between (y \wedge z) \iff (x \wedge y) \between z$

- $\forall z\ \big(x \between z \Rightarrow y \between z\big) \Longrightarrow x \leq y$

# Overlap Algebras vs Locales and cBa

## Proposition

*Overlap algebra*

$=$

*open locale*

$+$

$$\forall z\big(\mathrm{Pos}(x \wedge z) \Rightarrow \mathrm{Pos}(y \wedge z)\big) \Longrightarrow x \leq y$$

# Overlap Algebras vs Locales and cBa

**Proposition**

$$\textit{Overlap algebra} = \textit{open locale} + \forall z\big(\mathrm{Pos}(x \wedge z) \Rightarrow \mathrm{Pos}(y \wedge z)\big) \Longrightarrow x \leq y$$

**Classically (Vickers):**

$$\text{Overlap algebra} = \text{cBa}$$

where:

$$x \between y \iff x \wedge y \neq 0$$

## Solution to problem 1:
(Sambin)

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(**o-algebras**) : (cBa) $=$ (intuitionistic logic) : (classical logic)

# Part II

## Regular elements

A problem, a known solution, a new result.

# Problem 2

In classical topology:

a(n open) set is *REGULAR* if it equals the *interior* of its *closure*

$$D \text{ is } REGULAR \quad \Longleftrightarrow \quad D = \operatorname{int} \operatorname{cl} D$$

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$$D \text{ is } REGULAR \quad \Longleftrightarrow \quad D = \operatorname{int} \operatorname{cl} D$$

**Problem:**

define the notion of *REGULAR open subset* constructively
(i.e. in formal topology / locale theory).

# Known solution to problem 2

## Stable elements

Classically: $\quad D$ is regular $\quad\Longleftrightarrow\quad D = \mathrm{int}\,\overbrace{-\underbrace{\mathrm{int}\,-}_{\sim}}^{\mathrm{cl}}D$

where: $\overbrace{(\mathrm{int}\,-)}^{\sim}$ is the PSEUDO-COMPLEMENT in $\{\text{open sets}\}$

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**Usual definition of *stable* (regular) element in a Heyting algebra:**

$$x = \sim\sim x$$

where $\sim$ is the pseudo-complement.

# A new solution

(see: FC - *Regular opens in formal topology and representation theorem for overlap algebras* - in preparation)

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**Definition (of the operator $\mathcal{R} : \mathcal{L} \longrightarrow \mathcal{L}$)**

$$y \leq \mathcal{R}(x) \qquad \stackrel{def}{\Longleftrightarrow} \qquad (\forall z \in \mathcal{L})\big(Pos(y \wedge z) \Rightarrow \mathrm{Pos}(x \wedge z)\big)$$

which gives:

$$\mathcal{R}(x) \;=\; \bigvee\{y \in \mathcal{L} \quad | \quad y \leq \mathcal{R}(x)\}$$

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**New definition of REGULAR elements**

$$x \text{ is REGULAR} \quad \stackrel{def}{\Longleftrightarrow} \quad x = \mathcal{R}(x)$$

# Justification of the new def.

**Remember:**

$$x \text{ regular} \quad \stackrel{def}{\Longleftrightarrow} \quad x = \mathcal{R}(x) \stackrel{def}{=} \bigvee\{y \quad | \quad \forall z\big(Pos(y \wedge z) \Rightarrow \mathrm{Pos}(x \wedge z)\big)\}$$

What happens if $\underbrace{\text{the formal topology (open locale) is } \textit{spatial}}_{=\ \{\text{open subsets}\}\text{ of a topological space}}$?

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- $\mathcal{R}(x) = \bigcup\{y : y \subseteq \mathrm{cl}\, x\} = \mathrm{int}\ \mathrm{cl}\, x$
- $x = \mathcal{R}(x)$ iff $x = \mathrm{int}\ \mathrm{cl}\, x$ iff $x$ is regular in the usual sense.

# Properties of $\mathcal{R}$

**Remember:**

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- $\{x : x = \mathcal{R}(x)\} = \{\text{regular elements}\}$ is an OVERLAP ALGEBRA where:

$$\mathcal{R}(x) \between \mathcal{R}(y) \qquad \stackrel{def}{\iff} \qquad \mathrm{Pos}(x \wedge y)$$

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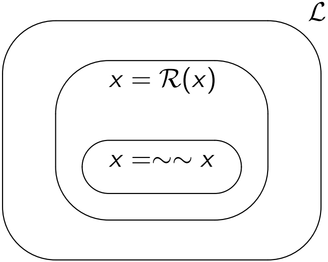

# Digression:

## apply the same idea to the logic itself

**Problem:**

define a unary connective (modality) $\mathcal{R}$ on formulae such that:

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Hint: you need some sort of OVERLAP RELATION on formulae.

# Digression:

## apply the same idea to the logic itself

**Problem:**

define a unary connective (modality) $\mathcal{R}$ on formulae such that:

$$\varphi \quad \vdash \quad \mathcal{R}(\varphi) \quad \vdash \quad \neg\neg\varphi$$

Hint: you need some sort of OVERLAP RELATION on formulae.

**Solution:**

take the (constructive) SATISFIABILITY relation
(see: FC, *Constructive Satisfiability*, PhD Thesis)

# Part III

## Representation of Overlap Algebras

Let $\mathcal{L}$ be an open locale and $\mathcal{R}$ the operator defined above.

**Theorem**

1. $\{x \in \mathcal{L} : x = \mathcal{R}(x)\}$ *is an OVERLAP ALGEBRA*

**Proof.**

1. where: $\mathrm{Pos}\big(\mathcal{R}(x)\big) = \mathrm{Pos}(x)$ and $\mathcal{R}(x) \between \mathcal{R}(y) \Leftrightarrow \mathrm{Pos}(x \wedge y)$

Let $\mathcal{L}$ be an open locale and $\mathcal{R}$ the operator defined above.

**Theorem**

1. $\{x \in \mathcal{L} : x = \mathcal{R}(x)\}$ *is an OVERLAP ALGEBRA*
2. *every overlap algebra can be represented in this way*

**Proof.**

1. where: $\mathrm{Pos}\big(\mathcal{R}(x)\big) = \mathrm{Pos}(x)$ and $\mathcal{R}(x) \between \mathcal{R}(y) \Leftrightarrow \mathrm{Pos}(x \wedge y)$
2. overlap algebra $=$ open locale $+$ $\underbrace{\text{all elements are regular}}_{\forall x \left(x=\mathcal{R}(x)\right)}$

   (each overlap algebra coincides with the overlap algebra of its regular elements)

$\square$

Incidentally, compare:

The regular elements of an open locale form an Overlap Algebra.

with the classical result:

The regular open sets of a topological space form a complete Boolean algebra.

which is so basic for the classical theory of Boolean valued models!

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Se also: FC, G. Sambin - *The overlap algebra of regular opens* - submitted, where another constructive version of this theorem is given even if in a concrete (as opposed to formal) topological framework. See also my talk in Padua last October.

# About morphisms
(j.w.w. Milly Maietti and Paola Toto)

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- Sambin's Basic Picture needs the category $\mathrm{Rel}$ of sets and relations.
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- Consequently, he introduced morphisms (*o-relations*) which are reminiscent of the four operators ext , rest , $\Box$ and $\Diamond$ of its Basic Picture.

**Proposition**

$$\underbrace{\textit{O-relations}}_{\substack{\textit{(morphisms}\\ \textit{between}\\ \textit{overlap}\\ \textit{algebras)}}} \quad \cong \quad \underbrace{\textit{morphisms between Basic Topologies}}_{\substack{= \\ \textit{morphisms of } \underline{\textit{sup-lattices}} \\ + \\ \textit{a condition on } \underline{\textit{formal closed}} \textit{ subsets}}}$$

# Future work
## (with Nicola Gambino)

Study the following categories:

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Study the following categories:

- Overlap algebras & morphisms of locales (+ some natural condition for $\Join$)
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# Point-free set-theory (?!)

$$x \subseteq y \iff \forall z(z \in x \Rightarrow z \in y) \quad \text{(POINT-WISE definition)}$$
$$x \subseteq y \iff \forall z(z \between x \Rightarrow z \between y) \quad \text{(POINT-FREE definition)}$$

**Questions:**

- Is it possible to develop set-theory in a "point-free" way?
- (That is, in a language with $\between$ instead of $\in$.)

- Would it be "more constructive"?
- Would it allow for new constructive versions of classical set-theoretical results?
- (As for point-free topology with respect to point-wise topology.)

# **Regular opens** in formal topology and representation theorem for **overlap algebras**

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*THANK YOU!*