The Kuratowski's Problem in Pointfree Topology

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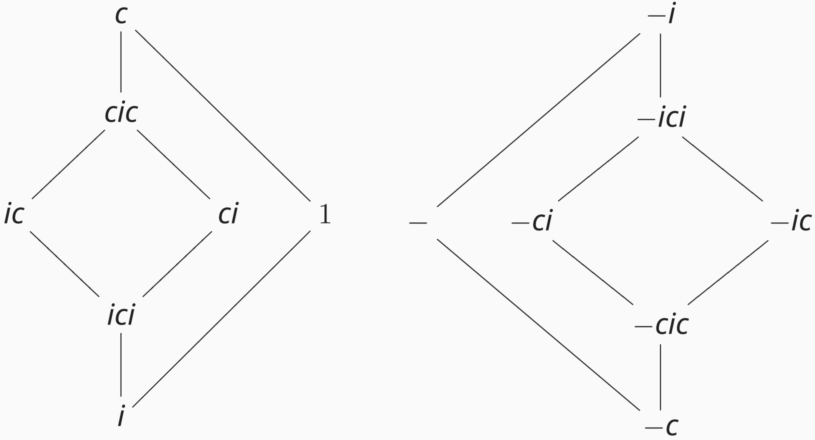
Kuratowski's theorem (1922)

Let i and c be the interior and closure operators on the subsets of a topological space.

Then
there are at most $\boxed{14}$ possible combinations of $i, c, -$
(where $-$ is the set-theoretic complement).

Application: number of modalities in $S4$

The general Kuratowski's ordered monoid

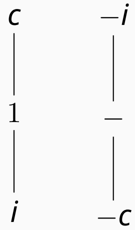


A 14-set of reals

$$\{-2\} \cup (-1, 0) \cup (0, 1) \cup \big((2, 3) \cap \mathbb{Q}\big)$$

Kuratowski's monoids: an example

If the frame of opens is Boolean (compare with $S5$)



$\boxed{i = ici = ci}$ and $\boxed{ic = cic = c}$ and...

[Gardner & MJackson, The Kuratowski closure-complement theorem, New Zealand J. Math. 38 (2008)]

Kuratowski’s result is far more general:
it holds for operators on a Boolean algebra,
and even further…

i and c need not be topological

closure operator:

$$x \leq c(y) \quad \textit{iff} \quad c(x) \leq c(y)$$

that is, monotone and $1 \leq c = cc$
(need not distribute over finite joins)

interior operator:

$$i(x) \leq y \quad \textit{iff} \quad i(x) \leq i(y)$$

that is, monotone and $ii = i \leq 1$
(need not distribute over finite meets)

The classical closure-complement problem

Let $(X, \leq, -)$ be a poset
with an antitone involution (complement)

let c be a closure operator on X

put $\boxed{i = -c-}$ (the corresponding interior operator)

then

there are at most $\boxed{14}$ possible combinations of $c, i, -$

What happens if we replace the complement with a pseudocomplement?

From complement to pseudocomplement: why?

MOTIVATIONS

- Greater generality
- Meaningful for constructive (intuitionistic, topos-valid) mathematics
- Application to pointfree topology / locale theory (the sublocales form a co-frame...)

What happens CONSTRUCTIVELY (no LEM)

$i = -c-$ does not make sense

($c = 1$ gives $i = --$ which is an interior operator iff LEM)

so we keep i as primitive

and we have 3 possibilities for c

- treat c as primitive too

(no pseudocomplement, no link between i and c)

- define c in terms of adherent points

(no pseudocomplement, need extra structure on the poset)

- put $c = -i-$

(need pseudocomplement)

The interior-closure problem (Sambin,?)

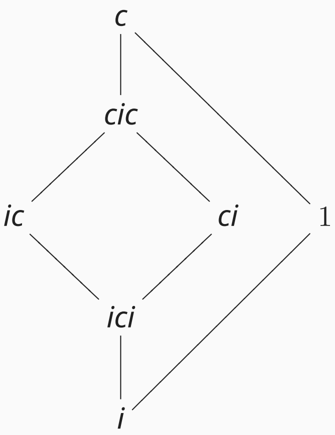
For an interior i
and a closure c

on an arbitrary poset:

at most $\boxed{7}$ combinations.

Constrcutive and
very general:
no link between i and c

Relevant for $\Box$ and $\Diamond$ in a general class of (non classical) modal logics...



The present framework is a bit too general for studying Kuratowski's monoids…

There are 26 additional inequalities which could be imposed:

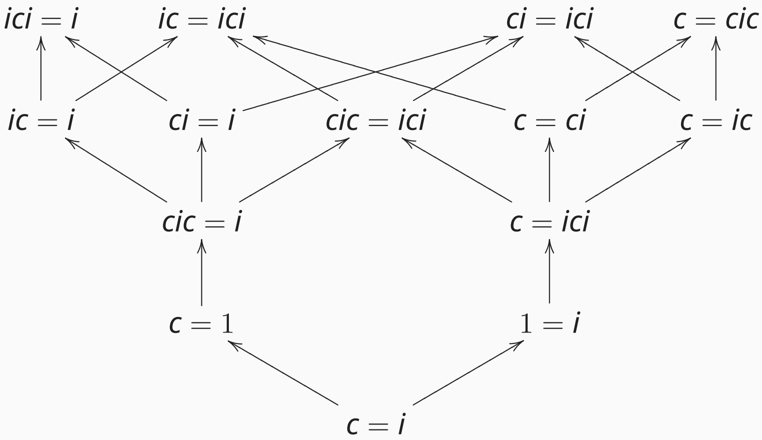
$c = i$, $c = ici$, $c = ic$, $c = ci$, $c = cic$, $c = 1$,
$cic = i$, $cic = ici$, $cic = ic$, $cic = ci$, $cic \leq 1$, $1 \leq cic$,
$ic = i$, $ic = ici$, $ic \leq ci$, $ic \leq 1$, $1 \leq ic$,
$ci = i$, $ci = ici$, $ci \leq ic$, $ci \leq 1$, $1 \leq ci$,
$ici = i$, $ici \leq 1$, $1 \leq ici$,
$1 = i$.

Although items in the same row are equivalent...

$cic \leq 1$	$cic = i$	
$1 \leq cic$	$c = cic$	
$ic \leq ci$	$cic = ci$	$ic = ici$
$ic \leq 1$	$ic = i$	
$1 \leq ic$	$c = ic$	
$ci \leq ic$	$cic = ic$	$ci = ici$
$ci \leq 1$	$ci = i$	
$1 \leq ci$	$c = ci$	
$ici \leq 1$	$ici = i$	
$1 \leq ici$	$c = ici$	

(like axioms B and 5 over $S4$)

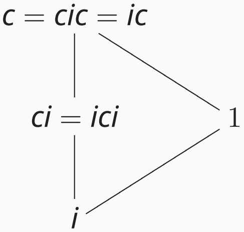
These are the implications which hold in general:



(some counterexample still missing...).

Example:

if $\boxed{c = ic}$ (every closed is open), then



Closure via adherent points

A point is in the closure of a set
if every open neighbourhood of the point
intersects the set.

For a constructively-sound algebraic version of this,
we need a poset with an overlap relation…

(with LEM, overlap = non empty intersection)

Overlap relations

Poset with overlap $(X, \leq, \between)$
= poset $(X, \leq)$ + binary relation $\between$ on X s.t.

$x \between y \Rightarrow y \between x$ (symmetry)

$(x \between y)$ & $(y \leq z) \Rightarrow (x \between z)$ (monotonicity)

$\forall z\ (x \between z \Rightarrow y \between z) \Rightarrow x \leq y$ (density)

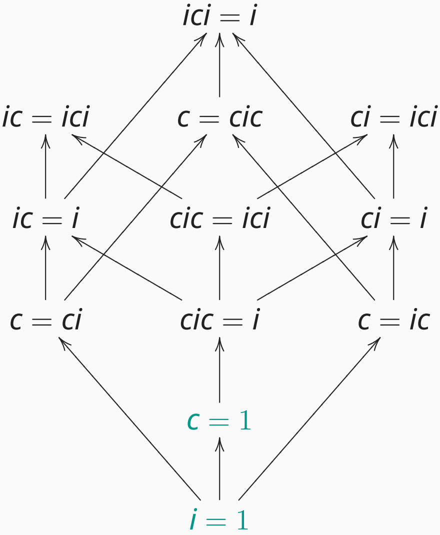
A "positive" link between interior and closure

Compatibility:

$$ix \between cy \;\Rightarrow\; ix \between y$$

In a poset with $\between$
if i and c are compatible then:

- $i = 1 \;\Rightarrow\; c = 1$
- ...
- $c = ic \;\Rightarrow\; ci = i$
- $c = ci \;\Rightarrow\; ic = i$
- ....



TFAE:

- LEM
- $c = 1 \Rightarrow i = 1$
- $ci = i \Rightarrow c = ic$
- ...

Example (a version of Booleanness):

if $\boxed{c = ic}$ (every closed is open), then

$$
\begin{array}{c}
c = cic = ic \\
| \\
1 \\
| \\
ci = ici = i
\end{array}
$$

(as in the classical case).

[More to appear in my paper...]

The interior-pseudocomplement problem

Pseudo-complement operator $-$ on a poset X:

$$x \leq -y \iff y \leq -x$$

($-$ defines an antitone Galois connection on X)

Properties:

$x \leq --x$ $\quad (1 \leq --)$

$x \leq y \Rightarrow -y \leq -x$ $\quad$ ($-$ is antitone)

$---x = -x$ $\quad (--- = -)$

$x \leq --y \Leftrightarrow --x \leq --y$ $\quad$ ($--$ is a closure operator)

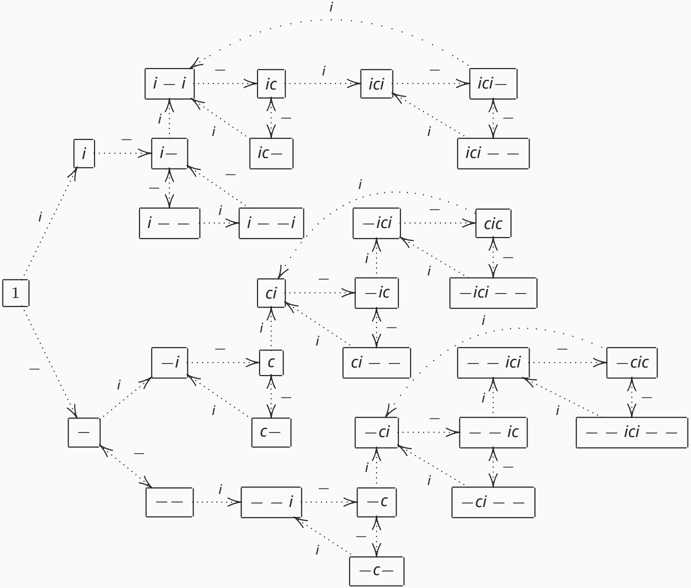
Let $\boxed{i}$ be an interior operator on X and put $\boxed{c = -i-}$.

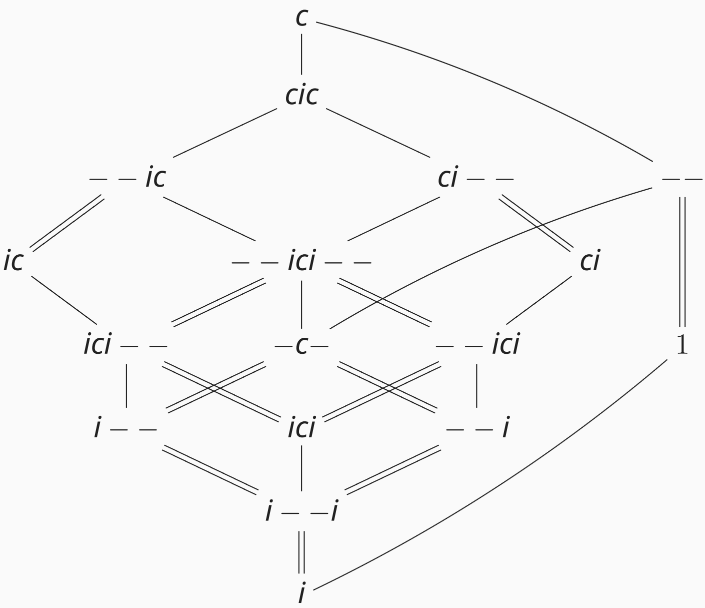
31 (?) possible combinations

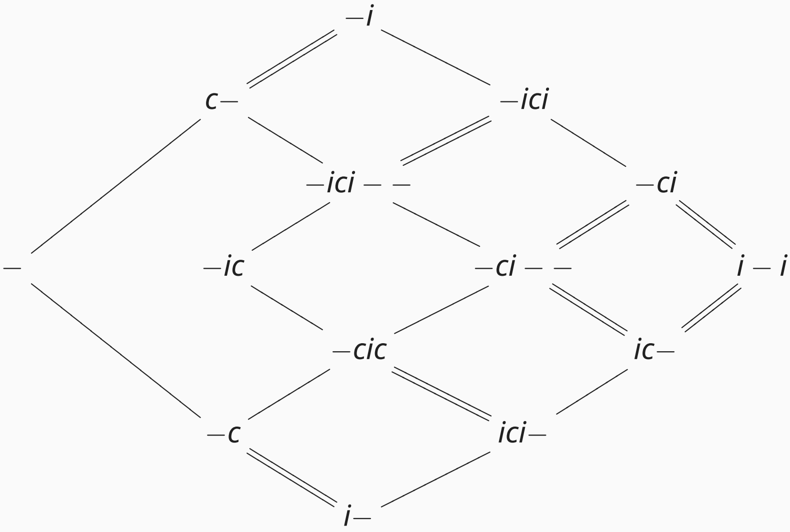
monotone
1
$--$
i
$i--$
$--i$
$-c-$
$i--i$
c
ic
$--ic$
ci
$ci--$
ici
$ici--$
$--ici$
$--ici--$
cic
17

antitone
$-$
$i-$
$-c$
$-i$
$c-$
$i-i$
$ic-$
$-ci$
$-ci--$
$-ic$
$ici-$
$-cic$
$-ici$
$-ici--$
14

Cayley graph of the monoid generated by *i* and —




c
cic
– – ic
ci – –
– –
ic
– – ici – –
ci
ici – –
–c–
– – ici
1
i – –
ici
– – i
i – –i
i


−i
c−
−ici
−ici − −
−ci
−
−ic
−ci − −
i − i
−cic
ic−
−c
ici−
i−

Question

What happens when X is a cHa and
i and c preserve finite $\wedge$ and $\vee$ respectively?

An application:
the pointfree Kuratowski's problem

The Kuratowski's problem for locales

X = co-frame of sublocales of a given locale L

$\leq$ is the sub-locale relation
(opposite of the pointwise order on nuclei)

$-$ is the supplement (co-pseudocomplement) in X
(that is, $-x \leq y \;\Leftrightarrow\; -y \leq x$)
[for j a nucleus, $-L_j$ is given by the nucleus $x \mapsto \bigwedge_{x \leq y}(jy \to y)$]

i is the interior in the sense of sublocales
(it is a closure operator on nuclei)

c is the closure in the sense of sublocales
(it is an interior operator on nuclei)

Some facts about open and closed sublocales

Open and closed sublocales are complemented.

Consequences:

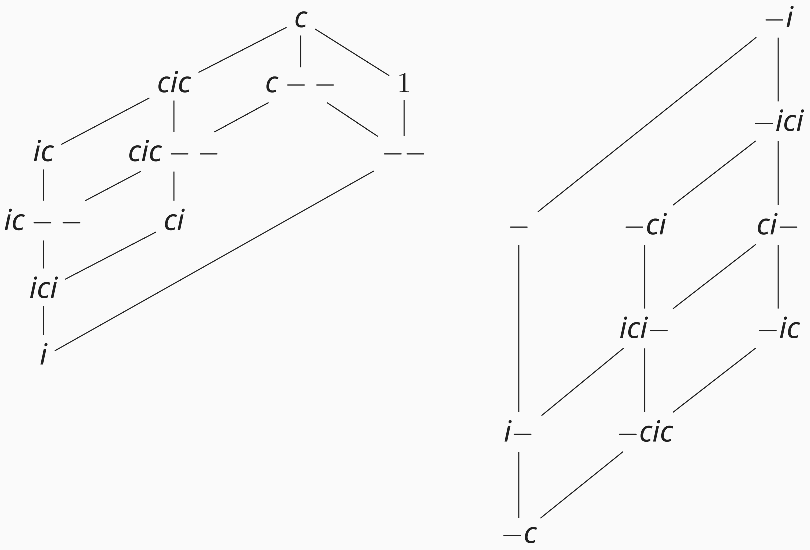
$$- - i = i = -c- = i - - \leq -- \leq c - - = -i- \leq c = - - c$$

$$-c \leq i- \leq - \leq c- = -i$$

Formally, the Kuratowski's problem for locales is a special case of the interior-pseudocomplement problem above because $i = -c-$

Warning: since now $-$ is a co-pseudocomplement, one has to apply the previous result with respect to the opposite order, and so the roles of the interior and closure operators are switched.

Solution: (i) take the previous diagrams, (ii) reverse them, (iii) write c in place of i and vice versa (simultaneously), and (v) simplify based on the previous facts...



11+10=21 (?)

$c \neq -i-$

L = the opposite of $\omega + 1$

Z = its smallest dense sublocale
(defined by the double negation nucleus)
so $cZ = L$.

But also $-Z = L$ and hence
$-i - Z = \emptyset$ (the smallest sublocale of L).

References

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Gardner & MJackson, The Kuratowski closure-complement theorem, New Zealand J. Math. 38 (2008)

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Related works:

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He & Zhang, Interior and boundary in a locale, Advances in Mathematics (China) 29 (2000)

Other references?

Gracias!
Gràcies!