Regular opens in constructive topology and a representation theorem for overlap algebras

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to Giovanni Sambin on the occasion of his 60th birthday

Abstract

Giovanni Sambin has recently introduced the notion of an overlap algebra in order to give a constructive counterpart to a complete Boolean algebra. We propose a new notion of regular open subset within the framework of intuitionistic, predicative Topology and we use it to give a representation theorem for (set-based) overlap algebras. In particular we show that there exists a duality between the category of set-based overlap algebras and a particular category of topologies in which all open subsets are regular.

Introduction

The content of this paper can be summarized as follows: we link overlap algebras with constructive topology via the notion of regular open subset.

The definition of an overlap algebra, as given by Sambin in [12], is an intuitionistic description of the power-collection of a set. It axiomatizes not only the relation of inclusion and the operations of union and intersection, but also the binary relation, called *overlap*, which says that two subsets have an element in common. With classical logic, overlap algebras are precisely complete Boolean algebras and hence the notion of an overlap algebra is strictly stronger than that of a complete Heyting algebra.

Constructive Topology is, roughly speaking, Topology within an intuitionistic and predicative framework. A formal topology (see [10]) is a predicative version of an overt (or open) locale. A *positive* topology is a formal topology in which the unary positivity predicate is replaced by a positivity relation, a new notion introduced in [12].

In this paper, we propose a new definition of regular formal open subset which works both for a positive and a formal topology. We show that the collection of regular open subsets has a structure of overlap algebra and that, moreover, every (set-based) overlap algebra can be represented in this way.

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Finally, we give a characterization of the category of set-based overlap algebra in terms of the opposite of a particular category of positive topologies.

Sections 1 and 2 are quite of an introductory nature; they are written with the aim of making the paper as self-contained as possible. The former deals with the category of overlap algebras and its subcategory of set-based overlap algebras. The latter recalls from [12] the definitions of the categories of basic and positive topologies, as well as other related notions.

Section 3 contains our new definition of regular formal open subset. This is employed in the construction of a notable class of overlap algebras.

Section 4 contains, among other things, the proof that every set-based overlap algebra can be represented as the overlap algebra of regular open subsets of a suitable positive (and also formal) topology.

In section 5 the correspondence between overlap algebras and positive topologies is extended to morphisms. In particular, it is shown that the category of set-based overlap algebras is dual to a suitable category of positive topologies.

Unless otherwise stated, all definitions and proofs of this paper are meant to be meaningful from the point of view of a *minimalist* foundational theory introduced by Maietti and Sambin in [9]. Since this theory, roughly speaking, lacks both the axiom of (unique) choice and the powerset axiom, as well as the law of excluded middle, the mathematics developed on it is valid in virtually all foundational theories such as Martin-Löf type theory, Topos theory, Aczel's CZF and so on (as well as in classical mathematics).

We shall use the word "predicative" for a statement which does not require the powerset axiom (and "impredicative" for its opposite), while "constructive" will means both predicative and intuitionistic. For the sake of predicativity, we distinguish "sets", whose elements are generated by rules, as in Martin-Löf theory, and which admits some kind of induction principle, from "collections". The standard example of a collection is given by all subsets of a given set. Here a subset is essentially a predicative propositional function over a set, up to equivalence of propositions (see [13] for a constructive theory of subsets). It should be clear that a definition which uses a quantification over a collection cannot have a predicative justification. Other remarks about foundations are going to be given within the text.

Throughout this paper, X, Y, S and T will always denote sets. Moreover, we shall use x, y, z, a, b, c for elements of those sets; D, E, U, V, W will denote subsets. We write $\{a \in S \mid \varphi(a)\}$ for the subset of S corresponding to the propositional function φ. The collection of all subset of S is written $\mathcal{P}(S)$. For $U, V \subseteq S$, we put:

$$U \between V \quad \stackrel{def}{\Longleftrightarrow} \quad U \cap V \text{ is inhabited.} \tag{1}$$

Given a function $\mathcal{F}$ on $\mathcal{P}(\mathcal{S})$ and an element $a \in S$, we shall very often write $\mathcal{F}(a)$ or simply $\mathcal{F}a$ instead of $\mathcal{F}(\{a\})$. The symbols P and Q will be reserved to collections with objects p, q, r. We keep the usual symbol $\in$ for membership in a set; on the contrary, membership in a subset and in a collection will be denoted by ϵ and : (colon), respectively.

Finally, a bibliographic remark: the main source for the notions we are going to use is the still unpublished [12] (of which the author possesses a draft). However, all the basic ideas and definitions (although sometimes with different names) can be found also in other papers such as [11], [5] and [2].

1 Overlap algebras

The notion of an overlap algebra has been recently introduced in [12] by Giovanni Sambin. It is an algebraic version of the power-collection of a set in which also the notion of "overlap", the $\between$ in equation (1), is axiomatized. The algebraic version of $\between$ is written $\mathrel{>\!\!\!<}$.

Definition 1.1 *An* overlap algebra *is a triple* $\mathcal{P} = (P, \leq, \mathrel{>\!\!\!<})$ *where* $(P, \leq)$ *is a complete lattice and* $\mathrel{>\!\!\!<}$ *is a binary relation on* P *satisfying the following conditions:*

symmetry: $$p \mathrel{>\!\!\!<} q \Longrightarrow q \mathrel{>\!\!\!<} p$$

transferring of meets: $$(p \wedge r) \mathrel{>\!\!\!<} q \Longrightarrow p \mathrel{>\!\!\!<} (r \wedge q)$$

splitting of joins: $$p \mathrel{>\!\!\!<} (\textstyle\bigvee_{i \in I} q_i) \Longleftrightarrow (\exists i \in I)(p \mathrel{>\!\!\!<} q_i)$$

density: $$(\forall r : P)(r \mathrel{>\!\!\!<} p \Longrightarrow r \mathrel{>\!\!\!<} q) \Longrightarrow p \leq q$$

for every $p, q : P$ *and every set-indexed family* $\{q_i : P \mid i \in I\}$ *(for* I *a set).*

For every set S, the structure $(\mathcal{P}(S), \subseteq, \between)$ is an overlap algebra. As an example, we check that $(\forall W \subseteq S)(W \between U \Rightarrow W \between V) \Longrightarrow U \subseteq V$ (density) holds. This is easy: the antecedent gives, in particular, $(\forall a \in S)(\{a\} \between U \Rightarrow \{a\} \between V)$, that is $(\forall a \in S)(a \,\epsilon\, U \Rightarrow a \,\epsilon\, V)$; so $U \subseteq V$.

A foundational remark is needed here. We use the adjective "complete", when referred to a lattice, to mean the existence of all *set-indexed* joins and meets. This is more convenient predicatively, though coincides with usual completeness (existence of *all* joins and meets) when working within an impredicative framework.

We write 0 and 1 for the bottom and top elements of an overlap algebra, respectively. They always exists since they are the join and meet of the empty family, respectively. Sometimes, it will be convenient to assume $1 \mathrel{>\!\!\!<} 1$. This will give $0 \neq 1$ as a consequence. In fact $0 \mathrel{>\!\!\!<} 0$ is always false since 0 is the join of the empty family and $\mathrel{>\!\!\!<}$ has to split joins. By the way, this same argument shows that $r \mathrel{>\!\!\!<} 0$ is always false, whatever r is.

The next proposition characterizes overlap algebras within a classical framework. This perhaps justifies the name "algebra" for a structure which has been defined via a relational symbol.

Proposition 1.2 *Assuming the Principle of Excluded Middle, if* $(P, \leq)$ *is a complete Boolean algebra, then* $(P, \leq, \mathrel{>\!\!\!<})$ *is an overlap algebra, where* $p \mathrel{>\!\!\!<} q$ *is* $p \wedge q \neq 0$.

Assuming the Principle of Excluded Middle and the Powerset Axiom, if $(P, \leq, \between)$ *is an overlap algebra (with* $1 \between 1$*), then* $(P, \leq)$ *is a complete Boolean algebra (with* $0 \neq 1$*).*

Proof See [5], Proposition 5.1. q.e.d.

Both statements in the previous proposition fails intuitionistically. In fact, the example of the power-collection shows that an overlap algebra is not a Boolean algebra, in general. Vice versa, there exist some properties of overlap algebras which are not provable, intuitionistically, for complete Boolean algebras. An example is the statement $\neg\neg(p = 0) \Rightarrow (p = 0)$ which can be proved in every overlap algebra as follows. We want to check that $p \leq 0$ follows from the assumption $\neg\neg(p = 0)$. The proof is by density. So we assume $r \between p$ and we claim $r \between 0$. Since $r \between 0$ is always false, as we have already noted, our task reduces to prove $\neg(r \between p)$ from the assumption $\neg\neg(p = 0)$. By intuitionistic logic, this is equivalent to check the implication $(r \between p) \Rightarrow \neg(p = 0)$ which holds because $r \between 0$ is false.

The new primitive $\between$ increases the expressive power of the language of lattices and allows for developing a lot of topology in algebraic terms and, moreover, in a positive way (no negation or complement needed). See [12], for the beginning of this approach to topology.

Proposition 1.3 *In every overlap algebra the following hold:*

1. $(r \between p)\ \ \&\ \ (p \leq q) \implies (r \between q)$
2. $p \leq q$ *iff* $\forall r(r \between p \Rightarrow r \between q)$ *and* $p = q$ *iff* $\forall r(r \between p \Leftrightarrow r \between q)$
3. $p \wedge (\bigvee_{i \in I} q_i) = \bigvee_{i \in I}(p \wedge q_i)$
4. $(r \between p) \Longrightarrow (p \between p)$
5. $\big((p \between p) \Longrightarrow (p \leq q)\big) \Longrightarrow (p \leq q)$

for every $p, q, r : \mathcal{P}$ *and every set-indexed family* $q_i : P (i \in I)$*.*

Proof (1) From $p \leq q$ one has $p \vee q = q$; on the other hand, $r \between p$ yields $r \between p \vee q$ ($\between$ splits joins); thus $r \between q$. (2) By density and item 1. (3) By applying item 2: $r \between p \wedge (\bigvee_{i \in I} q_i)$ iff $r \wedge p \between \bigvee_{i \in I} q_i$ iff $r \wedge p \between q_i$ for some i iff $r \between p \wedge q_i$ for some i iff $r \between \bigvee_{i \in I}(p \wedge q_i)$. (4) Assume $r \between p$, that is $r \between p \wedge p$; so $r \wedge p \between p$ and hence $p \between p$ by 1 because $r \wedge p \leq p$. (5) We make the hypothesis $p \between p \Rightarrow p \leq q$ and we prove $p \leq q$ by density. So, we make the further assumption $r \between p$ and we claim $r \between q$. Since $r \between p$, then $p \between p$ by item 4 and hence $p \leq q$ by hypothesis. This, together with $r \between p$, gives $r \between q$ by item 1, as wished. q.e.d.

Recall that a *frame* is a partial order with finite meets and set-indexed joins satisfying item 3 of the previous proposition. Frame forms a category with maps preserving joins and finite meets. The opposite of the category of frame is the category of *locales*. A locale is called *open* or *overt* (see [8] and [15]) if there exists a unary predicate Pos such that

$$\begin{array}{rcl} \mathsf{Pos}(p)\ \&\ (p \le q) & \Longrightarrow & \mathsf{Pos}(q) \\ \mathsf{Pos}(\bigvee_{i \in I} q_i) & \Longrightarrow & (\exists\, i \in I)\mathsf{Pos}(q_i) \\ \big(\mathsf{Pos}(p) \Rightarrow (p \le q)\big) & \Longrightarrow & (p \le q) \end{array} \tag{2}$$

Item 3 in the previous proposition says that $(P, \le)$ is a locale for every overlap algebra $(P, \le, \gtrless)$, in fact an overt locale with $p \gtrless p$ as $\mathsf{Pos}(p)$ thanks to item 1, the fact that $\gtrless$ splits joins and item 5.

A classical definition of order theory says that an atom is a minimal non-zero element. This idea can be formalized in the language of overlap algebras in a more positive way. We say that x is an atom if $x \gtrless x$ and, moreover, $x \le p$ (actually $x = p$) holds whenever $p \gtrless p$ and $p \le x$. In view of the next proposition, we adopt the following

Definition 1.4 *An element x of an overlap algebra $(P, \le, \gtrless)$ is an* atom *if*

$$x \gtrless p \quad \Longleftrightarrow \quad x \le p \tag{3}$$

for every p in P. An overlap algebra is atomic *if each element is the join of the atoms below it.*

Proposition 1.5 *For every element x of an overlap algebra $\mathcal{P}$, the following are equivalent:*

1. *x is an atom, that is, $(\forall p : P)\ (x \gtrless p \Longleftrightarrow x \le p)$.*
2. *$x \gtrless x$ and $(\forall p : P)\ (p \gtrless p\ \&\ p \le x \Longrightarrow x \le p)$.*
3. *$x \gtrless x$ and $(\forall p, q : P)\ (x \gtrless p\ \ \&\ \ x \gtrless q \ \Longrightarrow\ x \gtrless p \wedge q)$.*

Proof See [12]. Just for an example, we prove that $3 \Rightarrow 1$. Since $x \gtrless x$, we have $x \le p \Rightarrow x \gtrless p$ for every p. It remains to be checked that $x \gtrless p \Rightarrow x \le p$. So assume that $x \gtrless p$. By density, $x \le p$ will be proved if $x \gtrless r \Rightarrow p \gtrless r$ holds for every r. From $x \gtrless p$ and $x \gtrless r$, we have $x \gtrless p \wedge r$ by 3. So $x \wedge p \gtrless r$ and hence $p \gtrless r$, as wished. q.e.d.

In definition 1.1, P is not assumed to be a set. However, in order to keep to a predicative attitude (only quantification over sets), we shall very often consider *set-based* overlap algebras.

Definition 1.6 *A* set-based *overlap algebra is a triple $(\mathcal{P}, S, g)$ where $\mathcal{P}$ is an overlap algebra, S is a set and $g : S \to \mathcal{P}$ gives an indexing of a base for $\mathcal{P}$, that is, $p = \bigvee\{g(a) \mid a \in S,\ g(a) \le p\}$, for every p in $\mathcal{P}$.*

For the sake of simplicity, we shall use the same notation for an element of S and its image under g. Thus, for instance, we write $\bigvee U$ (for $U \subseteq S$) instead of $\bigvee\{g(a) \mid a \ \epsilon\ U\}$; similarly, $a \le b$ will stand for $g(a) \le g(b)$ and so on. We reserve the letters a, b, c, ... for (images under g of) elements in S and the letters p, q, r, ... for elements in $\mathcal{P}$. In a set-based overlap algebra density can be equivalently rewritten in the following predicative way: $(\forall a \in S)(a \between p \Longrightarrow a \between q) \Longrightarrow p \leq q$.

Note that, if $\mathcal{P}$ is set-based on S, then each item in proposition 1.5 is equivalent to that obtained by replacing each impredicative quantification over $\mathcal{P}$ with the corresponding quantification over the set S.

Clearly, for every set X, the structure $(\mathcal{P}(X), \subseteq, \between)$ is an example of a set-based overlap algebra with $S = X$ and $g(x) = \{x\}$ for every $x \in X$. Moreover, $\mathcal{P}(X)$ is atomic and the family of its atoms, that are the singletons, can be identified with a set, namely X itself.

1.1 Operators on overlap algebras

An *operator* between two overlap algebras $\mathcal{P}$ and $\mathcal{Q}$ is a mapping $\mathcal{F}$ between the corresponding carriers; $\mathcal{F}$ is *monotone* (or *order-preserving*) if $\mathcal{F}p \leq \mathcal{F}q$ whenever $p \leq q$.

Let $\mathcal{F}$ be an operator on an overlap algebra $\mathcal{P}$. We say that $\mathcal{F}$ is *idempotent* if $\mathcal{F}\mathcal{F}p = \mathcal{F}p$, for every p in $\mathcal{P}$. The collection $Fix(\mathcal{F}) = \{p \in \mathcal{P} \mid p = \mathcal{F}p\} = \{\mathcal{F}p \mid p \in \mathcal{P}\}$ of all fixed points of a monotone and idempotent operator is a complete lattice with respect to the following operations:

$$\bigvee_{i\in\mathcal{I}}^{\mathcal{F}} \mathcal{F}p_i \stackrel{def}{=} \mathcal{F}(\bigvee_{i\in\mathcal{I}} \mathcal{F}p_i) \qquad \text{and} \qquad \bigwedge_{i\in\mathcal{I}}^{\mathcal{F}} \mathcal{F}p_i \stackrel{def}{=} \mathcal{F}(\bigwedge_{i\in\mathcal{I}} \mathcal{F}p_i) \tag{4}$$

where $\bigvee$ and $\bigwedge$ denotes joins and meets in $\mathcal{P}$. In particular, the order in $Fix(\mathcal{F})$, which can be defined as usual by $\mathcal{F}p \leq^{\mathcal{F}} \mathcal{F}q$ iff $\mathcal{F}p \wedge^{\mathcal{F}} \mathcal{F}q = \mathcal{F}p$, is just that induced from $\mathcal{P}$, that is: $\mathcal{F}p \leq^{\mathcal{F}} \mathcal{F}q$ iff $\mathcal{F}p \leq \mathcal{F}q$.[1]

Definition 1.7 *An operator $\mathcal{F}$ on an overlap algebra $\mathcal{P}$ is called* expansive *if $p \leq \mathcal{F}p$ for every p in $\mathcal{P}$. It is called* reductive *if $\mathcal{F}p \leq p$ for every p.*

A monotone, idempotent and expansive operator is called a saturation. *A monotone, idempotent and reductive operator is called a* reduction.[2]

Provided that $\mathcal{A}$ is a saturation and $\mathcal{J}$ a reduction (as always in this paper), equations (4) simplify to

$$\bigvee_{i\in\mathcal{I}}^{\mathcal{A}} \mathcal{A}p_i = \mathcal{A}(\bigvee p_i) \qquad \text{and} \qquad \bigwedge_{i\in\mathcal{I}}^{\mathcal{A}} \mathcal{A}p_i = \bigwedge_{i\in\mathcal{I}} \mathcal{A}p_i \tag{5}$$

$$\bigvee_{i\in\mathcal{I}}^{\mathcal{J}} \mathcal{J}p_i = \bigvee \mathcal{J}p_i \qquad \text{and} \qquad \bigwedge_{i\in\mathcal{I}}^{\mathcal{J}} \mathcal{J}p_i = \mathcal{J}(\bigwedge_{i\in\mathcal{I}} p_i) \tag{6}$$

[1] In order to prove this, first note that $\mathcal{F}(\mathcal{F}p \wedge \mathcal{F}q) \leq \mathcal{F}p \wedge \mathcal{F}q$; then: $\mathcal{F}p \leq^{\mathcal{F}} \mathcal{F}q$ iff $\mathcal{F}p = \mathcal{F}p \wedge^{\mathcal{F}} \mathcal{F}q$ iff $\mathcal{F}p = \mathcal{F}(\mathcal{F}p \wedge \mathcal{F}q)$ iff $\mathcal{F}p \leq \mathcal{F}(\mathcal{F}p \wedge \mathcal{F}q)$ iff $\mathcal{F}p \leq \mathcal{F}p \wedge \mathcal{F}q$ iff $\mathcal{F}p \leq \mathcal{F}q$.

[2] A saturation (reduction) which preserves finite unions (intersections) is usually called a closure (interior) operator.

respectively. As an example, we check the first one, that is, $\mathcal{A}\bigvee_{i\in I}\mathcal{A}p_i = \mathcal{A}\bigvee_{i\in I}p_i$ (see [12] or [5] for the other proofs). The inequality $\mathcal{A}\bigvee_{i\in I}p_i \leq \mathcal{A}\bigvee_{i\in I}\mathcal{A}p_i$ holds because $p_i \leq \mathcal{A}p_i$ for all i ($\mathcal{A}$ is expansive) and $\mathcal{A}$ is monotone. We now prove the converse. Since $p_i \leq \bigvee_{i\in I}p_i$ for all i, then $\mathcal{A}p_i \leq \mathcal{A}\bigvee_{i\in I}p_i$ for all i, because $\mathcal{A}$ is monotone. Hence $\bigvee_{i\in I}\mathcal{A}p_i \leq \mathcal{A}\bigvee_{i\in I}p_i$ by the definition of join. The claim follows by applying $\mathcal{A}$ on both sides of this inequality and by recalling that $\mathcal{A}$ is idempotent.

Definition 1.8 *Let $\mathcal{A}$ and $\mathcal{J}$ be a saturation and a reduction, respectively, on the same overlap algebra $\mathcal{P}$. We say that $\mathcal{A}$ and $\mathcal{J}$ are* compatible *(or that they satisfy compatibility) if*

$$\mathcal{A}p \between \mathcal{J}q \Longrightarrow p \between \mathcal{J}q \tag{7}$$

(in fact $\mathcal{A}p \between \mathcal{J}q \Longleftrightarrow p \between \mathcal{J}q$) for every p and q in $\mathcal{P}$.[3]

Given two operators $\mathcal{F} : \mathcal{P} \to \mathcal{Q}$ and $\mathcal{F}^* : \mathcal{Q} \to \mathcal{P}$, one says that $\mathcal{F}^*$ is right adjoint to $\mathcal{F}$ and $\mathcal{F}$ is left adjoint to $\mathcal{F}^*$, written $\mathcal{F} \dashv \mathcal{F}^*$, if $\mathcal{F}p \leq q \Longleftrightarrow p \leq \mathcal{F}^*q$ holds for every p and q. Several simple facts hold for any adjunction $\mathcal{F} \dashv \mathcal{F}^*$; here we list some, without proof.

1. $\mathcal{F}$ and $\mathcal{F}^*$ are monotone;
2. $\mathcal{F}$ preserves joins and $\mathcal{F}^*$ preserves meets;
3. $\mathcal{F}\mathcal{F}^*\mathcal{F} = \mathcal{F}$ and $\mathcal{F}^*\mathcal{F}\mathcal{F}^* = \mathcal{F}^*$;
4. $\mathcal{F}^*q = \bigvee\{p \in \mathcal{P} \mid \mathcal{F}p \leq q\}$;
5. $\mathcal{F}^*\mathcal{F}$ is a saturation on $\mathcal{P}$ and $\mathcal{F}\mathcal{F}^*$ is a reduction on $\mathcal{Q}$;
6. $\mathcal{F}$ is an isomorphism of complete lattices from $Fix(\mathcal{F}^*\mathcal{F}) = \{\mathcal{F}^*q \mid q : \mathcal{Q}\}$ to $Fix(\mathcal{F}\mathcal{F}^*) = \{\mathcal{F}p \mid p : \mathcal{P}\}$, whose inverse is $\mathcal{F}^*$.

Definition 1.9 *Let $\mathcal{F} : \mathcal{P} \to \mathcal{Q}$ and $\mathcal{F}^- : \mathcal{Q} \to \mathcal{P}$ be two operators on overlap algebras. We say that $\mathcal{F}$ and $\mathcal{F}^-$ are* symmetric*, written $\mathcal{F} \cdot\!|\!\cdot \mathcal{F}^-$, if*

$$\mathcal{F}p \between q \Longleftrightarrow p \between \mathcal{F}^-q \tag{8}$$

for every $p \in \mathcal{P}$ and $q \in \mathcal{Q}$.[4]

[3]Classically, $\mathcal{A}$ and $\mathcal{J}$ are compatible precisely when $\mathcal{A} \leq -\mathcal{J}-$ or, equivalently, $\mathcal{J} \leq -\mathcal{A}-$, where $-$ is the complement in $\mathcal{P}$. Also, in a classical framework, the operators $-\mathcal{J}-$ and $-\mathcal{A}-$ are a saturation and a reduction, respectively. So, in the case in which $\mathcal{A}$ and $\mathcal{J}$ are a closure and an interior operator, compatibility expresses inclusion between two different topologies on $\mathcal{P}$: the one whose closed subsets are defined via the closure operator $\mathcal{A}$ and the other one whose open subsets are defined via the interior operator $\mathcal{J}$. Since $\mathcal{J}p = p$ (that is, $p \leq \mathcal{J}p$) yields $-\mathcal{A}-p = p$ (that is, $p \leq -\mathcal{A}-p$), every open subsets of the latter topology is open also with respect to the former. In other words the intuitive content of "compatibility" is that the topology defined via $\mathcal{A}$ is finer than that defined via $\mathcal{J}$.

[4]The term "symmetric" operators is taken from [12]; essentially the same notion, though in a classical setting, was studied in [7] where the term "conjugate" is used instead.

Classically, $\mathcal{F}p \between q \Leftrightarrow p \between \mathcal{F}^- q$ is tantamount to $p \leq F^* - q \Leftrightarrow p \leq -\mathcal{F}^- q$ (where $-$ is the complement) because $p \between q$ is classically equivalent to $p \wedge q \neq 0$. So $\mathcal{F}^* - = -\mathcal{F}^-$ holds and hence $\mathcal{F}^* = -\mathcal{F}^- -$ and $\mathcal{F}^- = -\mathcal{F}^* -$.

Proposition 1.10 *Each operator F from $\mathcal{P}$ to $\mathcal{Q}$ has at most one symmetric operator $\mathcal{F}^-$. When F^- exists, then it is*

$$\mathcal{F}^- q \;=\; \bigvee \{p \in \mathcal{P} \mid (\forall r \in \mathcal{P})(r \between p \;\Rightarrow\; \mathcal{F}r \between q)\} \tag{9}$$

and, in this case, $\mathcal{F}$ (and hence also $\mathcal{F}^-$) preserves joins.

Proof Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be such that $\mathcal{F} \cdot|\cdot \mathcal{F}_1$ and $\mathcal{F} \cdot|\cdot \mathcal{F}_2$. For every q in $\mathcal{Q}$ and p in $\mathcal{P}$, $p \between \mathcal{F}_1 q$ if and only if $\mathcal{F}p \between q$ if and only if $p \between \mathcal{F}_2 q$. By a two-fold application of density (in $\mathcal{P}$), we get $\mathcal{F}_1 q = \mathcal{F}_2 q$ for every q in $\mathcal{Q}$.

Now, assume that $\mathcal{F}^-$ is the symmetric of $\mathcal{F}$. We have: $p \leq \mathcal{F}^- q$ iff (by density in $\mathcal{P}$) $(\forall r \in \mathcal{P})$ $\big(r \between p \Rightarrow r \between \mathcal{F}^- q\big)$ iff $(\forall r \in \mathcal{P})$ $\big(r \between p \Rightarrow \mathcal{F}r \between q\big)$. So that (9) follows.

Finally, for arbitrary q in $\mathcal{Q}$, we have: $(\mathcal{F} \bigvee_{i \in I} p_i) \between q$ iff $\bigvee_{i \in I} p_i \between \mathcal{F}^- q$ iff $(\exists i \in I)$ $\big(p_i \between \mathcal{F}^- q\big)$ iff $(\exists i \in I)$ $\big(\mathcal{F}p_i \between q_i\big)$ iff $(\bigvee_{i \in I} \mathcal{F}p_i) \between q$; so $\mathcal{F}$ preserves joins (by a twofold application of density). q.e.d.

For every overlap algebra $\mathcal{P}$ and every $p : \mathcal{P}$, the map $x \mapsto p \wedge x$ defines a self-symmetric operator on $\mathcal{P}$ (by transferring of meets); in particular, $\wedge$ preserves joins, which proves again that every overlap algebra is a frame.

It is well known that $\mathcal{F}$ has a right adjoint $\mathcal{F}^*$ (defined as in 4. above) if and only if $\mathcal{F}$ preserves joins. In proposition 5.1 we shall investigate under which conditions an operator $\mathcal{F}$ admits a symmetric.[5] Classically, this is clear: since $\mathcal{F}^-$ exists if and only if $\mathcal{F}^*$ exists (thanks to the classically valid equations $\mathcal{F}^- = -\mathcal{F}^* -$ and $\mathcal{F}^* = -\mathcal{F}^- -$), then $\mathcal{F}$ admits a symmetric precisely when it preserves joins. Intuitionistically, the situation is more complex (actually, having a symmetric is a stronger property than having a right adjoint). The answer we shall give involves topological notions and makes essential use of the binary positivity relation introduced by Sambin (see [12] and [11]).

1.2 Overlap-relations

Several notions of morphism appear natural in dependence of the several structures (sup-lattices, locales, etc.) overlap algebras can be thought of as a special case. Following [12] we shall use an only apparently artificial notion of morphism. The idea is to think of the category of overlap algebras as a generalization of the category **Rel** of sets and binary relations. This is possible by identifying each set with the corresponding power-collection and each relation with the operator defined below.

[5]In fact, given an arbitrary $\mathcal{F}$, the operator defined by the right-hand side of equation (9) fails, in general, to be symmetric of $\mathcal{F}$.

Let r be a binary relation between X and S. Let $r : \mathcal{P}(X) \to \mathcal{P}(S)$ be the operator, called the *direct image* of the relation r, defined by

$$rD \stackrel{def}{=} \{a \in S \mid (\exists x \,\epsilon\, D)(x\, r\, a)\} \tag{10}$$

for every $D \subseteq X$. Similarly, we can consider an operator r^- corresponding to the inverse relation of r:

$$r^- U \stackrel{def}{=} \{x \in X \mid (\exists a \,\epsilon\, U)(x\, r\, a)\} \tag{11}$$

for $U \subseteq S$. It is easy to verify that the two operators r and r^- are symmetric, that is:

$$rD \between U \quad \Longleftrightarrow \quad D \between r^- U \tag{12}$$

for all $D \subseteq X$ and $U \subseteq S$. In fact, this says precisely that there exists $a \,\epsilon\, U$ such that $x\, r\, a$ for some $x \,\epsilon\, D$ if and only if there exists $x \,\epsilon\, D$ such that $x\, r\, a$ for some $a \,\epsilon\, U$. It follows by proposition 1.10, and it is easy to check directly, that both r and r^- preserves unions; hence they admit right adjoint operators, r^* and r^{-*}, respectively.

Definition 1.11 *An* overlap-relation *from the overlap algebra $\mathcal{P}$ to the overlap algebra $\mathcal{Q}$ is an operator $\mathcal{F} : \mathcal{P} \to \mathcal{Q}$ which admits a symmetric operator.*[6]

Proposition 1.12 *$\mathcal{F}$ is an overlap-relation between $\mathcal{P}(X)$ and $\mathcal{P}(S)$ if and only if there exists a binary relation r between X and S such that $\mathcal{F} = r$ (the direct image of the relation r).*

Proof Let $\mathcal{F}$ be an overlap-relation from $\mathcal{P}(X)$ to $\mathcal{P}(S)$; then $\mathcal{F}$ preserves joins in $\mathcal{P}(X)$, that is unions, by proposition 1.10. So $\mathcal{F}D = \bigcup_{x \epsilon D} \mathcal{F}\{x\}$, that is, $\mathcal{F}$ is uniquely determined by its behaviour on singletons. Thus we put: $x\, r\, a \Leftrightarrow a \,\epsilon\, \mathcal{F}\{x\}$, that is, $r\{x\} = \mathcal{F}\{x\}$ for every $x \in X$. Also the operator r preserves unions; thus $\mathcal{F}D = \bigcup_{x \epsilon D} \mathcal{F}\{x\} = \bigcup_{x \epsilon D} r\{x\} = rD$, that is, $\mathcal{F} = r$. q.e.d.

Note that the identity map on an overlap algebra is a self-symmetric operators. Note also that $(\mathcal{F}_1\mathcal{F}_2)^-$ exists and is equal to $\mathcal{F}_2^-\mathcal{F}_1^-$ provided that $\mathcal{F}_1^-$ and $\mathcal{F}_2^-$ exist (and $\mathcal{F}_1\mathcal{F}_2$ makes sense). In fact: $\mathcal{F}_1\mathcal{F}_2 x \gtrless y$ iff $\mathcal{F}_2 x \gtrless \mathcal{F}_1^- y$ iff $x \gtrless \mathcal{F}_2^-\mathcal{F}_1^- y$. So overlap-relations are closed under usual composition of maps.

Let **OA** be the category of overlap algebras and overlap-relations with usual composition and identities. The mappings $S \mapsto \big(\mathcal{P}(S), \subseteq, \between\big)$ and r (as a relation) $\mapsto r$ (as an operator on subsets) define a full embedding (= full, faithful and injective on objects) of **Rel** into **OA**. Moreover, since an overlap algebra is atomic if and only if it is isomorphic to $(\mathcal{P}(S), \subseteq, \between)$, where S is the set of all its atoms (see [12]), the above embedding is also dense (= essentially surjective) on atomic overlap algebras (see [14] for details).

The following proposition, whose proof is essentially taken from [12], shows that isomorphisms in **OA** are precisely invertible $\gtrless$-preserving maps.

[6]This definition is slightly different from that required in [12]. Sambin asks an overlap-relation to be an $\mathcal{F}$ for which $\mathcal{F}^-$, $\mathcal{F}^*$ and $\mathcal{F}^{-*}$ exist. Though this definition is redundant impredicatively, it is not so predicatively. However, his and our notion coincide in the set-based case, which is the only relevant for this paper.

Proposition 1.13 *Two overlap algebras $\mathcal{P}$ and $\mathcal{Q}$ are isomorphic in* **OA** *if and only if there exists a bijective operator $\mathcal{F} : \mathcal{P} \to \mathcal{Q}$ with inverse $\mathcal{F}^{-1}$ such that*

$$p_1 \between p_2 \Longleftrightarrow \mathcal{F}p_1 \between \mathcal{F}p_2 \tag{13}$$

for every p_1 and p_2 in $\mathcal{P}$. In that case $\mathcal{F}^- = \mathcal{F}^{-1}$, that is the symmetric of $\mathcal{F}$ is its inverse map.

Proof Let $\mathcal{F}$ be an isomorphism in **OA**, with inverse $\mathcal{F}^{-1}$, from $\mathcal{P}$ to $\mathcal{Q}$.[7] Also, let $1_\mathcal{P}$ and $1_\mathcal{Q}$ be the top elements of $\mathcal{P}$ and $\mathcal{Q}$, respectively. We first claim that $\mathcal{F}^- 1_\mathcal{Q} = 1_\mathcal{P}$: for every p, $p \between 1_\mathcal{P} \Leftrightarrow \mathcal{F}^{-1}\mathcal{F}p \between 1_\mathcal{P} \Leftrightarrow \mathcal{F}p \between (\mathcal{F}^{-1})^- 1_\mathcal{P} \Rightarrow \mathcal{F}p \between 1_\mathcal{Q} \Leftrightarrow p \between \mathcal{F}^- 1_\mathcal{Q}$; so $1_\mathcal{P} \leq \mathcal{F}^- 1_\mathcal{Q}$. Also, as $\mathcal{F}$ is monotone and invertible, we have $\mathcal{F}(p_1 \wedge p_2) = \mathcal{F}p_1 \wedge \mathcal{F}p_2$. Now $p_1 \between p_2$ iff $p_1 \wedge p_2 \between 1_\mathcal{P}$ iff $p_1 \wedge p_2 \between \mathcal{F}^- 1_\mathcal{Q}$ iff $\mathcal{F}(p_1 \wedge p_2) \between 1_\mathcal{Q}$ iff $\mathcal{F}p_1 \wedge \mathcal{F}p_2 \between 1_\mathcal{Q}$ iff $\mathcal{F}p_1 \between \mathcal{F}p_2$, that is (13). Consequently, $p \between \mathcal{F}^{-1}q$ iff $\mathcal{F}p \between \mathcal{F}\mathcal{F}^{-1}q$ iff $\mathcal{F}p \between q$ iff $p \between \mathcal{F}^- q$; hence $\mathcal{F}^{-1} = \mathcal{F}^-$.

Vice versa, let $\mathcal{F}$ be a bijection satisfying the above equivalence. We have to check only that $\mathcal{F} \cdot|\cdot \mathcal{F}^{-1}$, so that both $\mathcal{F}$ and $\mathcal{F}^{-1}$ are overlap-relations. This is easy: $\mathcal{F}p \between q$ iff $\mathcal{F}p \between \mathcal{F}\mathcal{F}^{-1}q$ iff $p \between \mathcal{F}^{-1}q$. q.e.d.

2 Constructive Topology

The beginning of the Basic Picture (see [12]) stands in realizing that much of topology can be developed on the basis of an *arbitrary* relation between two sets. Let $\Vdash$ be a binary relation between the set X (which is thought of as, but not necessarily is, the set of points of a topology) and the set S (which is thought of as, but not necessarily is, a (set of labels for a) base of the topology). For $x \in X$ and $a \in S$, the intended meaning of $x \Vdash a$ is "the point x lies in the basic neighbourhood (whose label is) a". In this context, we say that $\mathcal{X} = (X, \Vdash, S)$ is a *basic pair*. Following [12], we define the following four operators between subsets

$$\Diamond \;= \Vdash \qquad \mathsf{ext} \;= \Vdash^- \qquad \mathsf{rest} \;= \Vdash^* \qquad \Box \;= \Vdash^{-*} \tag{14}$$

which satisfy the symmetry condition $\Diamond \cdot|\cdot \mathsf{ext}$ and the adjunctions $\Diamond \dashv \mathsf{rest}$ and $\mathsf{ext} \dashv \Box$. According to the intended meaning, $\mathsf{ext}\{a\}$ is the basic open subset whose name is a and $\Diamond\{x\}$ is the system of basic open neighbourhoods of x. We have that a point x belongs to $\mathsf{int}\, D$, the interior of a subset D, iff $(\exists a \in \Diamond\{x\})(\mathsf{ext}\{a\} \subseteq D)$ (there exists a neighbourhood of x which is contained in D) iff $(\exists a \in \Diamond\{x\})(a \in \Box D)$ iff $\Diamond\{x\} \between \Box D$ iff $x \in \mathsf{ext}\Box D$; in other words, $\mathsf{int} = \mathsf{ext}\Box$. Dually $x \in \mathsf{cl}\, D$ (the closure of D) iff $(\forall a \in \Diamond\{x\})(\mathsf{ext}\{a\} \between D)$ (every neighbourhood of x meets D) iff $(\forall a \in \Diamond\{x\})(a \in \Diamond D)$ iff $\Diamond\{x\} \subseteq \Diamond D$ iff

[7] Since the identity morphisms of **OA** are just identity maps, the inverse of an overlap relation, when it exists, is given by its inverse map. Of course, this does not imply that every invertible map between overlap algebras is an isomorphism in **OA**.

$x \,\epsilon\, \mathsf{rest}\,\Diamond D$; so $\mathsf{cl} = \mathsf{rest}\,\Diamond$. Symmetrically, we can define two operators $\mathcal{J}$ and $\mathcal{A}$ on $\mathcal{P}(S)$:

$$a \,\epsilon\, \mathcal{J}U \quad \text{iff} \quad (\exists x \,\epsilon\, \mathsf{ext}\,\{a\})(\Diamond\{x\} \subseteq U) \text{ iff } \mathsf{ext}\,\{a\} \between \mathsf{rest}\,U \text{ iff } a \,\epsilon\, \Diamond\,\mathsf{rest}\,U$$
$$a \,\epsilon\, \mathcal{A}U \quad \text{iff} \quad (\forall x \,\epsilon\, \mathsf{ext}\,\{a\})(\Diamond\{x\} \between U) \text{ iff } \mathsf{ext}\,\{a\} \subseteq \mathsf{ext}\,U \text{ iff } a \,\epsilon\, \Box\,\mathsf{ext} \tag{15}$$

for $a \in S$ and $U \subseteq S$.

It follows from $\Diamond \cdot|\cdot\ \mathsf{ext}$, $\Diamond \dashv \mathsf{rest}$ and $\mathsf{ext} \dashv \Box$ that $\mathcal{A}$ is a saturation, $\mathcal{J}$ is a reduction and that they satisfy compatibility (recall definition 1.8).

Definition 2.1 *A* basic topology *is a triple $\mathcal{S} = (S, \mathcal{A}, \mathcal{J})$ made of a set S and two operators of saturation and reduction on $\mathcal{P}(S)$ which are compatible.*

For every basic pair $\mathcal{X} = (X, \Vdash, \mathcal{S})$, the structure $\mathcal{S}_{\mathcal{X}} = (S, \Box\,\mathsf{ext}, \Diamond\,\mathsf{rest})$ is a basic topology which is said to be *represented* by $\mathcal{X}$. The operator ext defines an isomorphism (whose inverse is $\Box$) of complete lattices between $Sat(\mathcal{A}) = Fix(\mathcal{A}) = \{\mathcal{A}U : U \subseteq S\} = \{\Box D : D \subseteq X\}$ (the collection of *formal open subsets*) and $Red(\mathsf{int}) = Fix(\mathsf{int}) = \{\mathsf{int}\,D : D \subseteq X\} = \{\mathsf{ext}\,U : U \subseteq S\}$ (the collection of (concrete) open subsets). Similarly, $Red(\mathcal{J}) = Fix(\mathcal{J}) = \{\mathcal{J}U : U \subseteq S\} = \{\Diamond D : D \subseteq X\}$, the complete lattice of *formal closed subsets*, is isomorphic (via rest and $\Diamond$) to the collection $Sat(cl) = Fix(\mathsf{cl}) = \{\mathsf{cl}\,D : D \subseteq X\} = \{\mathsf{rest}\,U : U \subseteq S\}$ of (concrete) closed subsets.

The operators $\mathcal{A}$ and $\mathcal{J}$ of a basic topology are usually presented via two infinitary relations $\lhd$ and $\ltimes$, respectively called *cover* and *binary positivity*, such that $a \lhd U$ iff $a \,\epsilon\, \mathcal{A}U$ and $a \ltimes U$ iff $a \,\epsilon\, \mathcal{J}U$, for every $a \in S$ and $U \subseteq S$.

A basic pair $\mathcal{X}$ is a *concrete space* if $Red(\mathsf{int})$ is a topology on X. This happens if and only if $Red(\mathsf{int})$ is closed under finite intersections. In that case,[8] the lattice $Sat(\mathcal{A})$ is a frame.

Lemma 2.2 *Let $\mathcal{A}$ be a saturation on $\mathcal{P}(S)$; then the following are equivalent:*

1. *$Sat(\mathcal{A})$ is a frame, that is, $\mathcal{A}U \cap \mathcal{A}\bigcup_{i\in I} V_i = \mathcal{A}\bigcup_{i\in I}(\mathcal{A}U \cap \mathcal{A}V_i)$, for every $U \subseteq S$ and every family $\{V_i \subseteq S\}_{i\in I}$;*

2. *$\mathcal{A}$ is* convergent*, that is, $\mathcal{A}U \cap \mathcal{A}V = \mathcal{A}(U{\downarrow}V)$ for every $U, V \subseteq S$, where $U{\downarrow}V = \big\{c \in S \mid c \,\epsilon\, \mathcal{A}\{u\} \cap \mathcal{A}\{v\}$, for some $u \,\epsilon\, U$ and $v \,\epsilon\, V\big\}$.*

Proof If $Sat(\mathcal{A})$ is a frame, then $\mathcal{A}U \cap \mathcal{A}V = (\mathcal{A}\bigcup_{u\epsilon U}\mathcal{A}\{u\}) \cap (\mathcal{A}\bigcup_{v\epsilon V}\mathcal{A}\{v\}) = \mathcal{A}\bigcup_{u\epsilon U}\bigcup_{v\epsilon V}(\mathcal{A}\{u\} \cap \mathcal{A}\{v\}) = \mathcal{A}(U{\downarrow}V)$. Vice versa, if item 2 holds, then $\mathcal{A}U \cap \mathcal{A}\bigcup_{i\in I} V_i = \mathcal{A}(U{\downarrow}\bigcup_{i\in I} V_i) = \mathcal{A}(\bigcup_{i\in I}(U{\downarrow}V_i)) = \mathcal{A}(\bigcup_{i\in I}\mathcal{A}(U{\downarrow}V_i)) = \mathcal{A}(\bigcup_{i\in I}(\mathcal{A}U \cap \mathcal{A}V_i))$. q.e.d.

Definition 2.3 *A* positive topology *is a basic topology $(S, \mathcal{A}, \mathcal{J})$ such that $\mathcal{A}$ is convergent, that is*

$$\mathcal{A}U \cap \mathcal{A}V \;=\; \mathcal{A}(U{\downarrow}V) \qquad \textit{with} \quad U{\downarrow}V = \bigcup_{u\epsilon U,\, v\epsilon V} \big(\mathcal{A}\{u\} \cap \mathcal{A}\{v\}\big) \tag{16}$$

[8]But not only in that case, as we shall see.

for every $U, V \subseteq S$. In other words, a positive topology is a basic topology such that $Sat(\mathcal{A})$ is a frame.

A strictly related, but not equivalent, notion is that of a *formal topology* as introduced in [10]. Essentially, a formal topology is a presentation of an overt locale, namely it is a triple $(S, \mathcal{A}, \mathsf{Pos})$ such that $\mathcal{A}$ is a convergent saturation on S and Pos is a *unary positivity predicate* satisfying:

$$\big(\mathsf{Pos}(a) \quad \& \quad a \,\epsilon\, \mathcal{A}U\big) \quad \Longrightarrow \quad \mathsf{Pos}(U) \qquad \text{(monotonicity)} \tag{17}$$

$$\big(\mathsf{Pos}(a) \quad \Longrightarrow \quad a \,\epsilon\, \mathcal{A}U\big) \quad \Longrightarrow \quad a \,\epsilon\, \mathcal{A}U \qquad \text{(positivity axiom)} \tag{18}$$

where $\mathsf{Pos}(U)$ abbreviates $(\exists a \,\epsilon\, U)\ \mathsf{Pos}(a)$. These conditions, as it is easy to check, imply that $\mathsf{Pos}(\mathcal{A}U)$ behaves as a positivity predicate for the locale $Sat(\mathcal{A})$.

The intended meaning of $\mathsf{Pos}(a)$ is exactly the same as that of $a \,\epsilon\, \mathcal{J}S$ in a positive topology, namely that the basic open corresponding to a is inhabited. However, even though (17) holds with respect to $a \,\epsilon\, \mathcal{J}S$ (thanks to compatibility between $\mathcal{A}$ and $\mathcal{J}$), this is not the case for (18), in general. Nevertheless, when working with positive topologies, we shall write $\mathsf{Pos}(a)$ for $a \,\epsilon\, \mathcal{J}S$ (and hence $\mathsf{Pos}(U)$ for $U \between \mathcal{J}S$). Let us note that almost all the definitions and results we are going to give hold for both positive topologies and formal topologies. In fact, with regards to our aims, it would be sufficient to deal with structures of the kind $(S, \mathcal{A}, \mathsf{Pos})$ where $\mathcal{A}$ is a convergent saturation and Pos is a unary predicate satisfying (17), but not necessarily (18).

2.1 Continuous relations

Following [12], we are now going to introduce morphisms between basic pairs, between basic topologies and between positive topologies. First, some notation: in what follows, if $\mathcal{X}$ and $\mathcal{Y}$ are two basic pairs, we assume that $\mathcal{X} = (X, \Vdash, S)$ and $\mathcal{Y} = (Y, \Vdash, T)$ (and use the same symbol $\Vdash$ for two different relations); also we write $\mathcal{S} = (S, \mathcal{A}, J)$ and $\mathcal{T} = (T, \mathcal{B}, \mathcal{K})$ for the corresponding basic topologies.

Let us assume for a moment that X and Y are two concrete spaces with bases S and T, respectively, and that $f : X \to Y$ is a continuous function. Then f^{-1} restricts to a map (in fact a frame homomorphism) between open subsets of Y and open subsets of X. This is equivalent to say that $f^{-1}(\mathsf{ext}\, b)$ is open (in X) for every $b \in T$, that is: $\mathsf{int}\, f^{-1}\, \mathsf{ext}\, b = f^{-1}\, \mathsf{ext}\, b$. The latter equality, hence continuity of f, can be expressed by the commutativity of the following diagram of relations[9]:

$$\begin{array}{ccc} X & \xrightarrow{\Vdash} & S \\ \big\downarrow f & & \big\downarrow s \\ Y & \xrightarrow{\Vdash} & T \end{array} \qquad \text{where } a\,s\,b \text{ is } \mathsf{ext}\,\{a\} \subseteq f^{-1}\,\mathsf{ext}\, b$$

[9]Given two binary relations r, between X and Y, and s, between Y and Z, the composition $s \circ r$ is the binary relation between X and Z defined by $x\,(s \circ r)\,z$ iff there exists $y \in Y$ such that $x\,r\,y$ and $y\,s\,z$.

where f is thought as a relation. This fact suggests the following.

Definition 2.4 *Let $\mathcal{X}$ and $\mathcal{Y}$ be two basic pairs. A morphism from $\mathcal{X}$ to $\mathcal{Y}$, called a* relation pair*, is given by a pair of binary relations (r, s) such that the diagram*

$$\begin{array}{ccc} X & \stackrel{\Vdash}{\longrightarrow} & S \\ \downarrow{\scriptstyle r} & & \downarrow{\scriptstyle s} \\ Y & \stackrel{\Vdash}{\longrightarrow} & T \end{array}$$

commutes, that is, $\Vdash \circ r = s \circ \Vdash$.[10] *Two relation pairs (r_1, s_1) and (r_2, s_2) are equal if $s_1 \circ \Vdash \; = s_2 \circ \Vdash$ or, equivalently, $\Vdash \circ r_1 = s_1 \circ \Vdash \; = s_2 \circ \Vdash \; = \; \Vdash \circ r_2$.*

Thus, properly speaking, a relation pair is an equivalence class. This notion of equality between relation pairs is justified by topological reasons (see [12]) and is what makes the category of basic pairs and relation pairs differ from **Rel**.

It is possible to prove that the right component of any relation pair gives rise to two morphisms of sup-lattices and inf-lattices, respectively, defined by

$$\begin{array}{ccc} Sat(\mathcal{B}) & \longrightarrow & Sat(\mathcal{A}) \\ \mathcal{B}U & \longmapsto & \mathcal{A}s^{-}U \end{array} \quad \text{and} \quad \begin{array}{ccc} Red(\mathcal{K}) & \longrightarrow & Red(\mathcal{J}) \\ \mathcal{K}U & \longmapsto & \mathcal{J}s^{*}U \end{array} \tag{19}$$

for $U \subseteq T$. Vice versa, if a relation s between S and T is such that the maps above are well-defined (in that case, they automatically become morphisms of sup-lattices and inf-lattices, respectively), then s is the right component of a relation pair (r, s), where $x\,r\,y$ (for $x \in X$ and $y \in Y$) can be taken to be $\Diamond y \subseteq s\Diamond\{x\}$. This justifies the following definition.

Definition 2.5 *A* continuous relation *from the basic topology $\mathcal{S} = (S, \mathcal{A}, \mathcal{J})$ to the basic topology $\mathcal{T} = (T, \mathcal{B}, \mathcal{K})$ is a binary relation s between S and T such that the two maps in (19) are well-defined morphisms of sup-lattices and inf-lattices, respectively.*

We identify two continuous relations if they give rise to the same maps.

In the case in which $\mathcal{S}$ and $\mathcal{T}$ are represented by $\mathcal{X}$ and $\mathcal{Y}$ respectively, the discussion above says that the continuous relations from $\mathcal{S}$ to $\mathcal{T}$ are precisely the right components of relation-pairs from $\mathcal{X}$ to $\mathcal{Y}$.

Note that the identity relation on S is continuous on $(S, \mathcal{A}, \mathcal{J})$ for every $\mathcal{A}$ and $\mathcal{J}$. Moreover, the composition of two continuous relation is continuous.

Two continuous relations s_1 and s_2 are equal if they corresponds to the same maps in (19). We claim that this happens if and only if

$$\mathcal{A}s_1^{-}b \; = \; \mathcal{A}s_2^{-}b \tag{20}$$

for every $b \in T$. For, by assuming the latter, we have: $\mathcal{A}s_1^{-}U = \mathcal{A}\bigcup_{b\epsilon U} s_1^{-}b = \mathcal{A}\bigcup_{b\epsilon U} \mathcal{A}s_1^{-}b = \mathcal{A}\bigcup_{b\epsilon U} \mathcal{A}s_2^{-}b = \mathcal{A}\bigcup_{b\epsilon U} s_2^{-}b = \mathcal{A}s_2^{-}U$. We want to check also

[10] From the point of view of the operators in (14) this means that $\Diamond r = s\Diamond$, $r^{-}\,\mathsf{ext} = \mathsf{ext}\,s^{-}$, $r^{*}\,\mathsf{rest} = \mathsf{rest}\,s^{*}$ and $\Box r^{-*} = s^{-*}\Box$.

that $\mathcal{J}s_1^*U = \mathcal{J}s_2^*U$, for every $U \subseteq T$. Since $\mathcal{J}$ is a reduction, it is enough to show that $\mathcal{J}s_1^*U \subseteq s_2^*U$, which is equivalent to $s_2\mathcal{J}s_1^*U \subseteq U$. So let $b \,\epsilon\, s_2\mathcal{J}s_1^*U$; then $s_2^- b \between \mathcal{J}s_1^*U$. Thanks to compatibility between $\mathcal{A}$ and $\mathcal{J}$ and by equality between s_1 and s_2, we get $s_1^- b \between \mathcal{J}s_1^*U$, which is equivalent to $b \,\epsilon\, s_1\mathcal{J}s_1^*U$. Since both $\mathcal{J}$ and $s_1 s_1^*$ are reduction operators, we can conclude $b \,\epsilon\, U$.

Finally, a morphism from the positive topology $\mathcal{S} = (S, \mathcal{A}, \mathcal{J})$ to the positive topology $\mathcal{T} = (T, \mathcal{B}, \mathcal{K})$ is given by a continuous relation s from $\mathcal{S}$ to $\mathcal{T}$ such that the map

$$\begin{array}{ccc} Sat(\mathcal{B}) & \longrightarrow & Sat(\mathcal{A}) \\ \mathcal{B}U & \longmapsto & \mathcal{A}s^-U \end{array} \tag{21}$$

is a morphism of frames. Such a continuous relation is called a *total and convergent continuous relation* and can be characterized by the following two properties: $\mathcal{A}s^-T = \mathcal{A}S$ and $\mathcal{A}s^-(U{\downarrow}V) = \mathcal{A}(s^-U{\downarrow}s^-V)$, for every $U, V \subseteq T$.

We write **BP** for the category of basic pairs and relation pairs, **BTop** for that of basic topologies and continuous relations and **PTop** for that of positive topologies and total and convergent continuous relations.

We end this section with a lemma we shall need later on.

Lemma 2.6 *Let s be a continuous relation from $\mathcal{S}$ to $\mathcal{T}$; then:*[11]

- *the operator s is* formal closed *(or* reduced*), that is, it maps $Red(\mathcal{J})$ to $Red(\mathcal{K})$;*
- *the operator s^{-*} is* formal open *(or* saturated*), that is, it maps $Sat(\mathcal{A})$ to $Sat(\mathcal{B})$.*

Proof By the definition of continuous relation, the map $\mathcal{K}U \longmapsto \mathcal{J}s^*U$ is well-defined on $Red(\mathcal{K})$. This gives, in particular, $\mathcal{J}s^*\mathcal{K}U = \mathcal{J}s^*U$ for all U because $\mathcal{K}\mathcal{K}U = \mathcal{K}U$. Since $\mathcal{K}$ is reductive, this is equivalent to $\mathcal{J}s^* \subseteq \mathcal{J}s^*\mathcal{K}$ and hence to $\mathcal{J}s^* \subseteq s^*\mathcal{K}$ because $\mathcal{J}$ is a reduction. In its turn, this is precisely $s\mathcal{J}s^* \subseteq \mathcal{K}$ (because $s \dashv s^*$). In particular, $s\mathcal{J}s^*s\mathcal{J} \subseteq \mathcal{K}s\mathcal{J}$; hence $s\mathcal{J} \subseteq s\mathcal{J}\mathcal{J} \subseteq s\mathcal{J}s^*s\mathcal{J} \subseteq \mathcal{K}s\mathcal{J}$, because s^*s is expansive and $\mathcal{J}$ is idempotent. So $s\mathcal{J} = \mathcal{K}s\mathcal{J}$ because $\mathcal{K}$ is a reduction. In other words, $\mathcal{K}(s\mathcal{J}U) = s\mathcal{J}U$ for all U, that is, the image of $Red(\mathcal{J})$ under s is contained in $Red(\mathcal{K})$. The second part has a dual proof. q.e.d.

3 The overlap algebra of regular opens

Before starting this section, let us fix some notation: when dealing with singletons such as $\{a\}$ we shall almost always suppress brackets; so $\mathcal{A}a$, $b \lhd a$, $U{\downarrow}a$, s^-a and so on will stand for $\mathcal{A}\{a\}$, $b \lhd \{a\}$, $U{\downarrow}\{a\}$ and $s^-\{a\}$ respectively.

[11] Actually, these two items characterize a continuous relation.

Definition 3.1 *For every positive topology $\mathcal{S}$, let $\mathcal{R} : \mathcal{P}(S) \to \mathcal{P}(S)$ be the operator on subsets defined by:*

$$a \,\epsilon\, \mathcal{R}U \quad \stackrel{def}{\Longleftrightarrow} \quad (\forall b \in S)\big(\mathsf{Pos}(a{\downarrow}b) \Rightarrow \mathsf{Pos}(U{\downarrow}b)\big) \tag{22}$$

for $a \in S$ and $U \subseteq S$. We say that U is (formal) regular *if $U = \mathcal{R}U$.*

Proposition 3.2 *For every positive topology $\mathcal{S} = (S, \mathcal{A}, \mathcal{J})$, the operator $\mathcal{R}$ defined in (22) is a saturation on S.*

Moreover $\mathcal{A} \subseteq \mathcal{R}$, that is, $\mathcal{A}U \subseteq \mathcal{R}U$ for every $U \subseteq S$. Hence $\mathcal{A}\mathcal{R} = \mathcal{R}\mathcal{A} = \mathcal{R}$ and $Sat(\mathcal{R}) \subseteq Sat(\mathcal{A})$, that is, each regular subset is a formal open.

Proof Let us consider the basic pair $(S, \Vdash, S)$ where $x \Vdash a \stackrel{def}{\Longleftrightarrow} \mathsf{Pos}(x{\downarrow}a)$. Then the operator $\Box\,\mathsf{ext}$ is a saturation on S. We claim that $\Box\,\mathsf{ext} = \mathcal{R}$: $a \,\epsilon\, \Box\,\mathsf{ext}\, U$ iff $\mathsf{ext}\,\{a\} \subseteq \mathsf{ext}\, U$ iff $(\forall x \in S)$ $(x \Vdash a \Rightarrow (\exists u \,\epsilon\, U)(x \Vdash u))$ iff $(\forall x \in S)$ $(\mathsf{Pos}(x{\downarrow}a) \Rightarrow (\exists u \,\epsilon\, U)\mathsf{Pos}(x{\downarrow}u))$ iff $(\forall x \in S)$ $(\mathsf{Pos}(x{\downarrow}a) \Rightarrow \mathsf{Pos}(x{\downarrow}U))$ iff $a \,\epsilon\, \mathcal{R}U$.

$\mathcal{A} \subseteq \mathcal{R}$: if $a \,\epsilon\, \mathcal{A}U$, that is $\mathcal{A}a \subseteq \mathcal{A}U$, then $a{\downarrow}b = \mathcal{A}a \cap \mathcal{A}b \subseteq \mathcal{A}U \cap \mathcal{A}b = \mathcal{A}(U{\downarrow}b)$ (because $\mathcal{A}$ is convergent); if $\mathsf{Pos}(a{\downarrow}b)$ holds, then $\mathsf{Pos}\big(\mathcal{A}(U{\downarrow}b)\big)$ by condition (17); hence $\mathsf{Pos}(U{\downarrow}b)$ by compatibility; so $a \,\epsilon\, \mathcal{R}U$.

From $\mathcal{A} \subseteq \mathcal{R}$ one gets both $\mathcal{A}\mathcal{R} \subseteq \mathcal{R}\mathcal{R} = \mathcal{R}$ and $\mathcal{R}\mathcal{A} \subseteq \mathcal{R}\mathcal{R} = \mathcal{R}$. Conversely $\mathcal{R} \subseteq \mathcal{A}\mathcal{R}$ and $\mathcal{R} \subseteq \mathcal{R}\mathcal{A}$ because $\mathcal{A}$ is expansive and $\mathcal{R}$ is monotone.

Finally, if $U = \mathcal{R}U$, then $\mathcal{A}U = \mathcal{A}\mathcal{R}U = \mathcal{R}U = U$. q.e.d.

A standard definition in topology says that an open subset is regular when it equals the interior of its closure. To justify our definition of regular formal open subset, we analyze the case of a positive topology which is represented by a concrete space $(X, \Vdash, S)$.[12] For every formal open $\mathcal{A}U$, we are going to show that $\mathsf{ext}\, U$, the concrete open corresponding to $\mathcal{A}U$, is regular (in the usual sense) if and only if $\mathcal{A}U$ is formal regular (according to definition 3.1), namely $\mathcal{A}U = \mathcal{R}U$. First we note that: $\mathsf{Pos}(U{\downarrow}V)$ iff $U{\downarrow}V \between \mathcal{J}S$ iff $U{\downarrow}V \between \Diamond\,\mathsf{rest}\, S$ iff $\mathsf{ext}\,(U{\downarrow}V) \between \mathsf{rest}\, S$ iff [13] $(\mathsf{ext}\, U \cap \mathsf{ext}\, V) \between X$ iff $\mathsf{ext}\, U \between \mathsf{ext}\, V$. Hence $a \,\epsilon\, \mathcal{R}U$ iff $(\forall b \in S)$ $(\mathsf{ext}\,\{a\} \between \mathsf{ext}\,\{b\} \to \mathsf{ext}\, U \between \mathsf{ext}\,\{b\})$ iff $\mathsf{ext}\,\{a\} \subseteq \mathsf{cl}\ \mathsf{ext}\, U$ iff $a \,\epsilon\, \Box\,\mathsf{cl}\ \mathsf{ext}\, U$. In other words, $\mathcal{R} = \Box\,\mathsf{cl}\ \mathsf{ext}$ and $\mathsf{ext}\,\mathcal{R} = \mathsf{int}\ \mathsf{cl}\ \mathsf{ext}$. So $\mathcal{A}U = \mathcal{R}U$ iff $\mathcal{A}U = \mathcal{A}\mathcal{R}U$ iff $\mathsf{ext}\, U = \mathsf{ext}\,\mathcal{R}U$ iff $\mathsf{ext}\, U = \mathsf{int}\ \mathsf{cl}\ \mathsf{ext}\, U$ iff $\mathsf{ext}\, U$ is regular.[14]

It is worth noting that definition 3.1 differs, at lest intuitionistically, from the usual definition. A regular element of a frame is an x such that $x = - - x$ (recall that, at least impredicatively, every frame has a pseudocomplement). In the case of a frame of the form $Sat(\mathcal{A})$, one can define the pseudocomplement of $\mathcal{A}U$ as $\{a \in S \mid a{\downarrow}U \subseteq \mathcal{A}\emptyset\}$ (see [12]). This would bring to a weaker definition

[12] This essentially amounts to require the locale $Sat(\mathcal{A})$ to be *spatial* (see [6]).

[13] $\mathsf{ext}\,(U{\downarrow}V) = \mathsf{ext}\,\Box\,\mathsf{ext}\,(U{\downarrow}V) = \mathsf{ext}\,\mathcal{A}(U{\downarrow}V) = \mathsf{ext}\,(\mathcal{A}U \cap \mathcal{A}V) = \mathsf{ext}\,(\Box\,\mathsf{ext}\, U \cap \Box\,\mathsf{ext}\, V) = \mathsf{ext}\,\Box(\mathsf{ext}\, U \cap \mathsf{ext}\, V) = \mathsf{int}\,(\mathsf{ext}\, U \cap \mathsf{ext}\, V) = \mathsf{ext}\, U \cap \mathsf{ext}\, V$

[14] The basic pair $(S, \mathsf{Pos}(\_\_ {\downarrow} \_\_), S)$, besides the saturation $\mathcal{R}$, induces also a reduction operator whose fixed points can be shown to correspond to "regular" closed subsets (those which are equal to the closure of their interior). See section 4.1 for details.

of regular. In fact, in the case of a concrete space, one can show that $\mathcal{A}U$ would be regular in this sense precisely when $\mathsf{ext}\,U = \mathsf{int} - \mathsf{int} - \mathsf{ext}\,U$.

Recall from [6] that a *nucleus* on a locale is a map j on the underlying frame such that: (1) $j(x \wedge y) = j(x) \wedge j(y)$; (2) $x \leq j(x)$; (3) $j(j(x)) \leq j(x)$. In other words, a nucleus is a saturation which preserves binary meets. Nuclei are identified with sublocales: by definition, the collection of all fixed points of a nucleus form the underlying frame of a sublocale.

Lemma 3.3 *For every positive topology $\mathcal{S}$, the operator $\mathcal{R}$ satisfies*

$$\mathcal{R}U \cap \mathcal{R}V \quad = \quad \mathcal{R}(U{\downarrow}V) \tag{23}$$

for all $U, V \subseteq S$.

Proof The inclusion $\mathcal{R}(U{\downarrow}V) \subseteq \mathcal{R}U \cap \mathcal{R}V$ follows from $U{\downarrow}V \subseteq \mathcal{A}U$ and $U{\downarrow}V \subseteq \mathcal{A}V$ because $\mathcal{R}\mathcal{A} = \mathcal{R}$. Before proving the converse, le us observe that $\mathcal{A}(U{\downarrow}U) = \mathcal{A}U$, $\mathcal{A}(U{\downarrow}V) = \mathcal{A}(V{\downarrow}U)$ and $\mathcal{A}((U{\downarrow}V){\downarrow}W) = \mathcal{A}(U{\downarrow}(V{\downarrow}W))$ for all $U, V, W \subseteq S$. This implies that $\mathsf{Pos}(U{\downarrow}U)$ is equivalent to $\mathsf{Pos}(U)$, that $\mathsf{Pos}(U{\downarrow}V)$ is equivalent to $\mathsf{Pos}(V{\downarrow}U)$ and that $\mathsf{Pos}((U{\downarrow}V){\downarrow}W)$ is equivalent to $\mathsf{Pos}(U{\downarrow}(V{\downarrow}W))$. We now check that $\mathcal{R}U \cap \mathcal{R}V \subseteq \mathcal{R}(U{\downarrow}V)$ holds. Take an a in S such that $a \,\epsilon\, \mathcal{R}U$ and $a \,\epsilon\, \mathcal{R}V$. For every $b \in S$ such that $\mathsf{Pos}(a{\downarrow}b)$, we must prove that $\mathsf{Pos}((U{\downarrow}V){\downarrow}b)$ holds. From $\mathsf{Pos}(a{\downarrow}b)$ one gets $\mathsf{Pos}(a{\downarrow}(a{\downarrow}b))$ and hence $\mathsf{Pos}(U{\downarrow}(a{\downarrow}b))$ because $a \,\epsilon\, \mathcal{R}U$. This is tantamount to $\mathsf{Pos}(a{\downarrow}(U{\downarrow}b))$ which, together with $a \,\epsilon\, \mathcal{R}V$, gives $\mathsf{Pos}(V{\downarrow}(U{\downarrow}b))$. This is equivalent to $\mathsf{Pos}((U{\downarrow}V){\downarrow}b)$. q.e.d.

Proposition 3.4 *For every positive topology, $Sat(\mathcal{R})$ is a sublocale of $Sat(\mathcal{A})$. Moreover, $Sat(\mathcal{R})$ is overt with respect to* Pos*, that is $(S, \mathcal{R}, \mathsf{Pos})$ is a formal topology (in the sense of [10]).*

Proof We have to prove that $\mathcal{R}$ is a nucleus on $Sat(\mathcal{A})$. We already know that $\mathcal{R}$ can be seen as an operator on $Sat(\mathcal{A})$ since $\mathcal{R}\mathcal{A}U = \mathcal{A}\mathcal{R}U$ for every U. Moreover, $\mathcal{R}$ is a saturation (not only on $\mathcal{P}(S)$ but also) on $Sat(\mathcal{A})$. Finally $\mathcal{R}(\mathcal{A}U \wedge^{\mathcal{A}} \mathcal{A}V) = \mathcal{R}(\mathcal{A}U \cap \mathcal{A}V) =$ ($\mathcal{A}$ convergent) $\mathcal{R}\mathcal{A}(U{\downarrow}V) = \mathcal{R}(U{\downarrow}V) =$ (lemma) $\mathcal{R}U \cap \mathcal{R}V = \mathcal{R}\mathcal{A}U \cap \mathcal{R}\mathcal{A}V = \mathcal{R}\mathcal{A}U \wedge^{\mathcal{A}} \mathcal{R}\mathcal{A}V$.

Finally, we prove that Pos is a positivity predicate for $Sat(\mathcal{R})$. Monotonicity: since $\mathsf{Pos}(a)$ is equivalent to $\mathsf{Pos}(a{\downarrow}a)$, from $a \,\epsilon\, \mathcal{R}U$ and $\mathsf{Pos}(a)$ one gets $\mathsf{Pos}(U{\downarrow}a)$ which, in turn, entails $\mathsf{Pos}(U)$ (because $U{\downarrow}a \subseteq \mathcal{A}U$). Positivity axiom: assume that $\mathsf{Pos}(a) \Rightarrow (a \,\epsilon\, \mathcal{R}U)$; for every $b \in S$, if $\mathsf{Pos}(a{\downarrow}b)$, then $\mathsf{Pos}(a)$ (because $a{\downarrow}b \subseteq \mathcal{A}a$); so $a \,\epsilon\, \mathcal{R}U$ by the assumption; this together with $\mathsf{Pos}(a{\downarrow}b)$ gives $\mathsf{Pos}(U{\downarrow}b)$; hence $a \,\epsilon\, \mathcal{R}U$ by the definition of $\mathcal{R}$. q.e.d.

The lattice $Sat(\mathcal{R})$ has also a natural structure of overlap algebra.[15] Let us define

$$\mathcal{R}U \between \mathcal{R}V \quad \stackrel{def}{\Longleftrightarrow} \quad \mathsf{Pos}(U{\downarrow}V) \tag{24}$$

[15]This corresponds to the classical fact that the regular open sets of a topological space form a complete Boolean algebra.

for every $U, V \subseteq S$. This is well-defined. In fact, since Pos is a positivity predicate for $Sat(\mathcal{R})$, then one has: $\mathsf{Pos}(U{\downarrow}V)$ iff $\mathsf{Pos}(\mathcal{R}(U{\downarrow}V))$ iff $\mathsf{Pos}(\mathcal{R}U \cap \mathcal{R}V)$. Easily, $\between$ is an overlap relation in the sense of definition 1.1. For instance, $\between$ satisfies density by the very definition of $\mathcal{R}$.

Definition 3.5 *For every positive topology $\mathcal{S}$, we put $Reg(\mathcal{S}) = (Sat(\mathcal{R}), \subseteq, \between)$ (with $\mathcal{R}$ and $\between$ defined in (22) and (24), respectively) and we refer to it as "the overlap algebra of regular opens of $\mathcal{S}$".*

Note that $Reg(\mathcal{S})$ is always a set-based overlap algebra via the set S itself and the map $a \mapsto \mathcal{R}\{a\}$ because $\mathcal{R}U = \mathcal{R}\mathcal{R}U = \mathcal{R}\bigcup_{a \epsilon \mathcal{R}U}\{a\} = \bigvee^{\mathcal{R}}_{\mathcal{R}\{a\} \leq^{\mathcal{R}} \mathcal{R}U} \mathcal{R}\{a\}$. In the next section, we shall prove that each set-based overlap algebra can be represented as the overlap algebra of regular opens of some positive topology.

4 Topological representation of overlap algebras

Let $(\mathcal{P}, S, g)$ be a set-based overlap algebra. With these data at our disposal, it is natural to consider the basic pair $(S, \between, S)$ (in fact, we should use the symbol $\between_{|g(S)\times g(S)}$ instead of $\between$, but, for the sake of simplicity, we shall not). And given this, the following natural step is to construct the basic topology represented by it.

Definition 4.1 *Given an overlap algebra $\mathcal{P}$ set-based on S, we write $Top(\mathcal{P})$ ("the topology associated to $\mathcal{P}$") for the basic topology represented by the basic pair $(S, \between, S)$.*

Of course, since $\between$ is symmetric we have $\mathsf{ext} = \between^{-} = \between = \Diamond$, $\mathsf{rest} = \Box$, $\mathcal{A} = \mathsf{cl}$, $\mathcal{J} = \mathsf{int}$ and so on. According to the general definitions, for $a \in S$ and $U \subseteq S$, we have: $a \lhd U$ iff $a \,\epsilon\, \mathcal{A}U$ iff $a \,\epsilon\, \Box\,\mathsf{ext}\,U$ iff $\mathsf{ext}\{a\} \subseteq \mathsf{ext}\,U$ iff $\between^{-} a \subseteq \between^{-} U$ iff $(\forall x \in S)\ \big(x \between a \Rightarrow (\exists u \,\epsilon\, U)\ (x \between u)\big)$ iff $(\forall x \in S)\ (x \between a \Rightarrow x \between \bigvee U)$ (because $\between$ splits joins) iff $a \leq \bigvee U$ (by density). For p in $\mathcal{P}$ we put ${\downarrow}p = \{a \in S : a \leq p\}$, so that $\mathcal{A}U = {\downarrow}\bigvee U$ hold for all $U \subseteq S$. One can easily check that the following are all isomorphisms of complete lattices:

$$\mathcal{P} \ \underset{\bigvee}{\overset{\downarrow}{\rightleftarrows}}\ Sat(\mathcal{A}) = Sat(cl) \ \underset{\mathsf{rest}}{\overset{\Diamond}{\rightleftarrows}}\ Red(\mathcal{J}) = Red(\mathsf{int}) \tag{25}$$

(since ${\downarrow}\bigvee = \mathcal{A}$ and $\bigvee{\downarrow} = id_{\mathcal{P}}$, it is enough to check that $\bigvee \dashv {\downarrow}$, that is, $\bigvee U \leq p \Leftrightarrow U \subseteq {\downarrow}p$, and then apply the general results on page 7). In particular, U is a formal closed subset ($U = \mathcal{J}U$) if and only if there exists p in $\mathcal{P}$ (in fact, $p = \bigvee \mathsf{rest}\,U$) such that $U = \Diamond{\downarrow}p = \{a \in S : (\exists b \in S)(a \between b \ \&\ b \leq p)\} = \{a \in S : a \between p\}$. Also, $a \ltimes U$ ($a \,\epsilon\, \mathcal{J}U$) if and only if $a \between \bigvee \mathsf{rest}\,U$, where $\mathsf{rest}\,U = \{x \in S \mid (\forall b \in S)(x \between b \Rightarrow b \,\epsilon\, U\}$. In particular, $\mathsf{Pos}(a)$, that is $a \,\epsilon\, \mathcal{J}S$, if and only if $(\exists x \in S)\ (a \between x)$. More generally, note that: $\mathsf{Pos}(U{\downarrow}V)$ iff $\mathsf{Pos}\big(\mathcal{A}(U{\downarrow}V)\big)$ iff $\mathsf{Pos}(\mathcal{A}U \cap \mathcal{A}V)$ iff $(\exists a \in S)\ \big((a \lhd U)\ \&\ (a \lhd V)\ \&\ (a \between a)\big)$ iff $(\exists a \in S)\ \big((a \leq \bigvee U)\ \&\ (a \leq \bigvee V)\ \&\ (a \between a)\big)$ iff $(\bigvee U) \between (\bigvee V)$.[16]

[16] To justify the latter equivalence, note that $p \between q$ is equivalent to $p \wedge q \between p \wedge q$. Since $p \wedge q = \bigvee \{a \in S \mid a \leq p \wedge q\}$, this holds if and only if there exists $a \leq p \wedge q$ such that $a \between a$.

Summing up, we have the following properties of $Top(\mathcal{P})$:

- $a \lhd U$ iff $a \leq \bigvee U$, that is, $\mathcal{A} = \downarrow \bigvee$;
- $\mathsf{Pos}(a)$ iff $a \between a$, hence $\mathsf{Pos}(U)$ iff $\bigvee U \between \bigvee U$;
- $\mathsf{Pos}(U{\downarrow}V)$ iff $\bigvee U \between \bigvee V$;
- formal closed subsets are precisely the subsets of the form $\{a \in S \mid a \between p\}$, for some p in $\mathcal{P}$;
- $a \ltimes U$, that is $a \,\epsilon\, \mathcal{J}U$, iff $a \between \bigvee\{x \in S \mid (\forall\, b \in S)(x \between b \Rightarrow b \,\epsilon\, U)\}$.

Even if $(S, \between, S)$ is not a concrete space in general (see proposition 4.5), nevertheless $Top(\mathcal{P})$ is always a positive topology ($Sat(\mathcal{A})$ is a frame), because $Sat(\mathcal{A}) \cong \mathcal{P}$ as complete lattices and the latter is a frame.

According to definition (22), it is possible to define an operator $\mathcal{R}$ on $\mathcal{P}(S)$ whose fixed points are the regular formal open subsets of the positive topology $Top(\mathcal{P})$. We have: $a \,\epsilon\, \mathcal{R}U$ iff $(\forall b \in S)$ $\big(\mathsf{Pos}(a{\downarrow}b) \Rightarrow \mathsf{Pos}(U{\downarrow}b)\big)$ iff $(\forall b \in S)$ $\big((a \between b) \Rightarrow (\bigvee U \between b)\big)$ iff $(a \leq \bigvee U)$ iff $a \,\epsilon\, \mathcal{A}U$. Hence $\mathcal{R} = \mathcal{A}$ and $Sat(\mathcal{R})$ $= Sat(\mathcal{A})$. In other words, every open subset of $Top(\mathcal{P})$ is regular. Following definition (24), we can endow $Sat(\mathcal{R})$ with an overlap algebra structure by defining: $\mathcal{R}U \between \mathcal{R}V$ iff $\mathsf{Pos}(U{\downarrow}V)$ iff $\bigvee U \between \bigvee V$.

By putting all these facts together and by remembering the isomorphisms showed in (25), one probably expects $Reg\big(Top(\mathcal{P})\big)$ (*the overlap algebra of regular opens of the positive topology associated to a set-based overlap algebra*) to be isomorphic to $\mathcal{P}$ itself; and in fact it is so.

Proposition 4.2 *Let $\mathcal{P}$ be a set-based overlap algebra. Then $Reg\big(Top(\mathcal{P})\big)$ is isomorphic to $\mathcal{P}$ via the maps:*

$$\begin{array}{ccc} Reg\big(Top(\mathcal{P})\big) & \longrightarrow & \mathcal{P} \\ \mathcal{R}U & \longmapsto & \bigvee U \end{array} \quad \textit{and} \quad \begin{array}{ccc} \mathcal{P} & \longrightarrow & Reg\big(Top(\mathcal{P})\big) \\ p & \longmapsto & \downarrow p \end{array}.$$

Proof The map $\bigvee$ is well-defined since $\bigvee \mathcal{R}U = \bigvee \mathcal{A}U = \bigvee\{a \in S \mid a \leq \bigvee U\}$ $= \bigvee U$. Also $\bigvee$ is bijective and $\downarrow$ is its inverse. Moreover, $\mathcal{R}U \between \mathcal{R}V$ iff $(\bigvee U) \between (\bigvee V)$; so $\bigvee$ satisfies the hypotheses of proposition 1.13 and hence it is an isomorphism in **OA**. q.e.d.

Let us denote by $\eta_{\mathcal{P}}$ the isomorphism from $Reg(Top(\mathcal{P}))$ to $\mathcal{P}$; so:

$$\eta_{\mathcal{P}}(\mathcal{R}U) \quad = \quad \bigvee U \tag{26}$$

for every $U \subseteq S$.

Corollary 4.3 *Every set-based overlap algebra is isomorphic to the overlap algebra of regular open subsets of a positive topology.*

4.1 The topology of regular subsets

Being at this point, the question naturally arises of what the link is between a positive topology $\mathcal{S}$ and the positive topology $Top(Reg(\mathcal{S}))$. The latter is the topology represented by the basic pair $(S, \Vdash, S)$ where $x \Vdash a$ is $\mathsf{Pos}(x{\downarrow}a)$. As we know (see the proof of proposition 3.2), the corresponding saturation is just $\mathcal{R}$ whose fixed points are the regular formal open subsets of $\mathcal{S}$. We now consider also the reduction induced by $\Vdash$, call it $\mathcal{J}_{\mathcal{R}}$. Thus, from now on, we can write $Top(Reg(\mathcal{S})) = (S, \mathcal{R}, \mathcal{J}_{\mathcal{R}})$. We have:

$$a \,\epsilon\, \mathcal{J}_{\mathcal{R}} U \quad \stackrel{def}{\Longleftrightarrow} \quad (\exists x \in S)\Big(\mathsf{Pos}(x{\downarrow}a) \;\;\&\;\; (\forall b \in S)\big(\mathsf{Pos}(x{\downarrow}b) \Rightarrow b \,\epsilon\, U\big)\Big)$$

(remember the general construction in (15); by the way, note that $a \,\epsilon\, \mathcal{J}_{\mathcal{R}} S$ is just $\mathsf{Pos}(a)$, that is, $\mathcal{J}_{\mathcal{R}} S = \mathcal{J} S$). By an argument dual to that used for $\mathcal{R}$, it is possible to show that $Red(\mathcal{J}_{\mathcal{R}})$ can be rightfully called "the collection of regular formal closed subsets of $\mathcal{S}$" since, in the case of a topological space, it is isomorphic to the lattice of regular closed subset (where a closed subset is regular if it equals the closure of its interior).

However, the collection $Red(\mathcal{J}_{\mathcal{R}})$ is not in general a subcollection of $Red(\mathcal{J})$. This implies that $Top(Reg(\mathcal{S})) = (S, \mathcal{R}, \mathcal{J}_{\mathcal{R}})$ is not, in general, a subobject of $\mathcal{S}$ $= (S, \mathcal{A}, \mathcal{J})$ even though $Sat(\mathcal{R})$ is a sublocale of $Sat(\mathcal{A})$. To avoid this problem, we require $\mathcal{J}_{\mathcal{R}} \subseteq \mathcal{J}$; this is equivalent to assume that each regular formal closed subset is a formal closed subset. In this case, and only in this case, the identity relation on S is a total and convergent continuous relation from $Top(Reg(\mathcal{S}))$ to $\mathcal{S}$ which is, in fact, a monomorphism in **PTop** (hence in **BTop**). To avoid confusion with the identity morphism on $\mathcal{S}$, we write

$$\varepsilon_{\mathcal{S}} \;:\; Top(Reg(\mathcal{S})) \longrightarrow \mathcal{S} \tag{27}$$

for this morphism. Of course, we have $\varepsilon_{\mathcal{S}}^{-} b = \{b\}$ for all $b \in S$. We end this paragraph with a proposition we shall need later.

Proposition 4.4 *For every set-based overlap algebra $\mathcal{P}$ and every positive topology $\mathcal{S}$ we have:*

$$Top(Reg(Top(\mathcal{P}))) \;=\; Top(\mathcal{P}) \quad \textit{and} \quad Reg(Top(Reg(\mathcal{S}))) \;=\; Reg(\mathcal{S})\ .$$

Proof First note that $Top(Reg(\mathcal{S})) = \mathcal{S}$ if and only if $\mathcal{A} = \mathcal{R}$ and $\mathcal{J} = \mathcal{J}_{\mathcal{R}}$. From discussions in the previous paragraph, we already know that $\mathcal{A} = \mathcal{R}$ in every positive topology of the form $Top(\mathcal{P})$. In this case, moreover, $a \,\epsilon\, \mathcal{J}_{\mathcal{R}} U$ iff $(\exists x \in S)$ $(\mathsf{Pos}(x{\downarrow}a)$ & $(\forall b \in S)$ $(\mathsf{Pos}(x{\downarrow}b) \Rightarrow b \,\epsilon\, U))$ iff $(\exists x \in S)$ $(x \gtrless a$ & $(\forall b \in S)$ $(b \gtrless x \Rightarrow b \,\epsilon\, U))$ iff $(\exists x \in S)$ $(a \gtrless x$ & $\Diamond\{x\} \subseteq U)$ iff $(\exists x \in S)$ $(a \gtrless x$ & $x \,\epsilon\, \mathsf{rest}\, U)$ iff $a \gtrless \bigvee \mathsf{rest}\, U$ iff $a \,\epsilon\, \mathcal{J} U$. Thus the first equality is proved.

Since all formal open subsets of $Top(Reg(\mathcal{S})) = (S, \mathcal{R}, \mathcal{J}_{\mathcal{R}})$ are regular and, moreover, they coincides with the regular formal open subsets of $\mathcal{S}$, the overlap algebra $Reg(Top(Reg(\mathcal{S})))$ is precisely the overlap algebra $Reg(\mathcal{S})$. q.e.d.

4.2 Ideal points and atoms

We are now going to discuss some links between the notion of ideal point in positive topology (see definition below) and that of atom in an overlap algebra. Even if these results have little consequence for the rest of this paper, we think they may have some interest on their own.

Proposition 4.5 *Let $\mathcal{P}$ be an overlap algebra set-based on S. The basic pair $(S, \gtrless, S)$ is a concrete space if and only if $\mathcal{P}$ is atomic with S as its set of atoms.*

Proof We want to show that the collection $Red(\mathsf{int})$ of all concrete open subsets of the basic pair $(S, \gtrless, S)$ is closed under finite intersections if and only if every $x \in S$ is an atom. More explicitly, we prove: (1) S is open if and only if $x \gtrless x$ holds for every $x \in S$; (2) the intersection of two basic open subsets is open if and only if $x \gtrless a$ & $x \gtrless b \Rightarrow x \gtrless a \wedge b$ for every $x, a, b \in S$ (recall proposition 1.5). Of course, S is open if and only if every $x \in S$ has an open neighbourhood. This means literally that $(\forall x \in S)$ $(\exists a \in S)$ $(x \gtrless a)$, which is equivalent to $(\forall x \in S)$ $(x \gtrless x)$. The intersection of two basic opens is open if and only if, for every $a, b \in S$ and every $x \,\epsilon\, \mathsf{ext}\,\{a\} \cap \mathsf{ext}\,\{b\}$, there exists $c \in S$ such that $x \,\epsilon\, \mathsf{ext}\,\{c\} \subseteq \mathsf{ext}\,\{a\} \cap \mathsf{ext}\, b$. Now $\mathsf{ext}\,\{c\} \subseteq \mathsf{ext}\,\{a\}$ is equivalent to $c \lhd \{a\}$, that is, $c \leq a$; similarly for b. So we have reached the following: for every $x, a, b \in S$, if $x \gtrless a$ and $x \gtrless b$, then there exists $c \leq a \wedge b$ such that $x \gtrless c$. However, $x \gtrless c$ for some $c \leq a \wedge b$ is tantamount to $x \gtrless a \wedge b$. q.e.d.

In [10], a formal point of a formal topology $(S, \lhd, \mathsf{Pos})$ is defined as a subset $\alpha \subseteq S$ satisfying the following: $\alpha \between S$ (α is inhabited); $a \,\epsilon\, \alpha$ & $b \,\epsilon\, \alpha \Rightarrow a{\downarrow}b \between \alpha$ (α is *convergent*); $a \,\epsilon\, \alpha \Rightarrow \mathsf{Pos}(a)$ (α is *positive*); $a \lhd U$ & $a \,\epsilon\, \alpha \Rightarrow U \between \alpha$ (α *splits* $\lhd$). In [12], the notion of an *ideal point* was introduced by replacing the requirement "α is positive" with "α is formal closed", that is, $\alpha = \mathcal{J}\alpha$. This is, in general, more restrictive: $a \,\epsilon\, \alpha \Leftrightarrow a \,\epsilon\, \mathcal{J}\alpha \Rightarrow a \,\epsilon\, \mathcal{J}S \Leftrightarrow \mathsf{Pos}(a)$. Moreover, it is easy to check that if α is formal closed, then it automatically splits $\lhd$ (this is just "compatibility" between $\mathcal{A}$ and $\mathcal{J}$).

Here we want to prove that the notion of atom for a set-based overlap algebra $\mathcal{P}$ essentially coincides with the notion of an ideal point for the positive topology $Top(\mathcal{P})$.

Proposition 4.6 *Let $\mathcal{P}$ be an overlap algebra set-based on S. Then the maps:*

$$\begin{array}{ccc} Atoms(\mathcal{P}) & \longrightarrow & iPt\big(Top(\mathcal{P})\big) \\ x & \longmapsto & \Diamond{\downarrow}x \end{array} \quad \textit{and} \quad \begin{array}{ccc} iPt\big(Top(\mathcal{P})\big) & \longrightarrow & Atoms(\mathcal{P}) \\ \alpha & \longmapsto & \bigvee \mathsf{rest}\,\alpha \end{array}$$

define a bijective correspondence between atoms of $\mathcal{P}$ and ideal points of the positive topology $Top(\mathcal{P})$.

Proof Thanks to (25), $\Diamond{\downarrow}x = \{a \in S \mid a \gtrless x\}$ is a formal closed subset for every $x \in P$. Assume now that x is an atom. In particular, $x \gtrless x$ and hence $\Diamond{\downarrow}x$ is inhabited. Also, thanks to proposition 1.5, $x \gtrless a$ & $x \gtrless b \Rightarrow x \gtrless a \wedge b$;

therefore $\Diamond{\downarrow}x$ is convergent (note that $a \wedge b = \bigvee a{\downarrow}b$). Summing up, $\Diamond{\downarrow}x$ is an ideal point.

Vice versa, suppose that α is an ideal point. Then $a \,\epsilon\, \alpha$ for some $a \in S$ (α is inhabited) and $a \,\epsilon\, \mathcal{J}\alpha$ (α is formal closed), that is, $a \gtrless \bigvee \mathsf{rest}\,\alpha$; hence $\bigvee \mathsf{rest}\,\alpha \gtrless \bigvee \mathsf{rest}\,\alpha$. Moreover, $a \gtrless \bigvee \mathsf{rest}\,\alpha$ & $b \gtrless \bigvee \mathsf{rest}\,\alpha \Leftrightarrow a \,\epsilon\, \mathcal{J}\alpha$ & $b \,\epsilon\, \mathcal{J}\alpha \Leftrightarrow a \,\epsilon\, \alpha$ & $b \,\epsilon\, \alpha \Rightarrow a{\downarrow}b \between \alpha \Leftrightarrow a{\downarrow}b \between \mathcal{J}\alpha \Leftrightarrow \bigvee(a{\downarrow}b) \gtrless \bigvee \mathsf{rest}\,\alpha \Leftrightarrow (a \wedge b) \gtrless \bigvee \mathsf{rest}\,\alpha$. Summing up, $\bigvee \mathsf{rest}\,\alpha$ is an atom by proposition 1.5.

Finally, $\bigvee \mathsf{rest}\,\Diamond{\downarrow}x = x$ and $\Diamond{\downarrow}\bigvee \mathsf{rest}\,\alpha = \alpha$ by (25). q.e.d.

Following [12] and in analogy with what is done in Local Theory, we say that a positive topology is *spatial* when $a \lhd U$ holds if and only if $a \,\epsilon\, \alpha \Rightarrow \alpha \between U$ for every point α. It is easy to check that the positive topology represented by a concrete space is always spatial.

Proposition 4.7 *The positive topology $Top(\mathcal{P})$ is spatial if and only if the overlap algebra $\mathcal{P}$ is atomic.*

Proof If $\mathcal{P}$ is atomic, then $(S, \gtrless, S)$ is a concrete space, hence $Top(\mathcal{P})$ is spatial. Vice versa, it is sufficient to check that each $a \in S$ is the join of the atoms below it. So, for every $a \in S$, we must prove that $a \leq \bigvee\{x \in Atoms(\mathcal{P}) \mid x \leq a\}$, that is, $a \lhd \{x \in Atoms(\mathcal{P}) \mid x \leq a\}$. By spatiality of $Top(\mathcal{P})$, this is equivalent to $a \,\epsilon\, \alpha \Rightarrow \alpha \between \{x \in Atoms(\mathcal{P}) \mid x \leq a\}$ for every point α. By the previous proposition, the claim becomes $z \gtrless a \Rightarrow \big(\exists\, x \in Atoms(\mathcal{P})\big)(z \gtrless x$ & $x \leq a)$ for every atom z (because $a \,\epsilon\, \alpha$ is $a \,\epsilon\, \Diamond{\downarrow}z$, where z is the atom $\bigvee \mathsf{rest}\,\alpha$; this is equivalent to $z \gtrless a$). This last claim is trivial: if $z \gtrless a$, then take $x = z$ and use the fact that z is an atom. q.e.d.

5 Overlap-relations topologically

Since each overlap algebra can be seen as a positive topology, it is natural to look for a topological reading of overlap-relations. We start from investigating the notion of overlap-relation in the case of set-based overlap algebras. Much of this material is joint work with Maria Emilia Maietti and Paola Toto; some of the ideas can be found in [14]; a more exhaustive paper on this subject is in preparation.

From now on let $\mathcal{P}$ and $\mathcal{Q}$ be two overlap algebras set-based on S and T, respectively, and let $\mathcal{F}$ be an overlap-relation from $\mathcal{P}$ to $\mathcal{Q}$. We shall show that each overlap-relation from $\mathcal{P}$ to $\mathcal{Q}$ is essentially a continuous relation, that is a morphism of basic topologies, between the associated positive topologies.

In the case of set-based overlap algebras, equation (9) can be replaced with the following predicative version:

$$\mathcal{F}^{-}q \;=\; \bigvee\{a \in S \mid (\forall x \in S)(x \gtrless a \;\Rightarrow\; \mathcal{F}x \gtrless q)\} \tag{28}$$

for every q in Q. Similarly: $\mathcal{F}^{*}q = \bigvee\{a \in S \mid \mathcal{F}a \leq q\}$ and $\mathcal{F}^{-*}p = \bigvee\{b \in T \mid \mathcal{F}^{-}b \leq p\}$. Our aim is to investigate under which conditions equation (28) indeed defines the symmetric operator of $\mathcal{F}$.

Let us start with some notation. To any $\mathcal{F} : \mathcal{P} \to \mathcal{Q}$, we associate a binary relation, say $\downarrow\mathcal{F}$, between S and T defined by:

$$a\,(\downarrow\mathcal{F})\,b \quad \stackrel{def}{\Longleftrightarrow} \quad b \leq \mathcal{F}a \tag{29}$$

for every $a \in S$ and $b \in T$. If, as usual, we write $(\downarrow\mathcal{F}) : \mathcal{P}(X) \to \mathcal{P}(S)$ also for the direct image of the relation $(\downarrow\mathcal{F})$, then we have: $b \,\epsilon\, (\downarrow\mathcal{F})a$ if and only if $b \leq \mathcal{F}a$ if and only if $b \,\epsilon\, \downarrow\mathcal{F}a$ (which probably justifies the notation). As usual we write $(\downarrow\mathcal{F})^-$ both for the inverse relation of $(\downarrow\mathcal{F})$ and for the symmetric of $(\downarrow\mathcal{F})$ as an operator.

Proposition 5.1 *Let $\mathcal{F}$ be an operator from $\mathcal{P}$ to $\mathcal{Q}$ which preserves joins and let $\mathcal{G}$ be the operator defined via equation (28). Then the following conditions are equivalent:*

1. $\mathcal{F} \cdot\!|\!\cdot \mathcal{G}$*, that is,* $G = \mathcal{F}^-$*;*
2. $\big((\downarrow\mathcal{G}), (\downarrow\mathcal{F})^-\big)$ *is a relation pair from* $(T, \gtrless, T)$ *to* $(S, \gtrless, S)$*;*
3. $(\downarrow\mathcal{F})^-$ *is a continuous relation from* $Top(\mathcal{Q})$ *to* $Top(\mathcal{P})$*;*
4. $(\downarrow\mathcal{F})^-$ *is a* formal closed map*, that is, it maps formal closed subsets to formal closed subsets;*
5. *for every* q *in* $\mathcal{Q}$*,* $\{a \in S : \mathcal{F}a \gtrless q\}$ *is a formal closed subset of* $Top(\mathcal{P})$*.*

Proof $(1 \Rightarrow 2)$ For every $x \in S$ and $y \in T$, $\mathcal{F}x \gtrless y \Leftrightarrow x \gtrless \mathcal{G}y$ holds; this yields $(\exists b \in T)\ \big(y \gtrless b\ \&\ b \leq \mathcal{F}x\big) \Leftrightarrow (\exists a \in S)\ \big(x \gtrless a\ \&\ a \leq \mathcal{G}y\big)$ which can be read as: $(\exists b \in T)\ \big(y \gtrless b\ \&\ b(\downarrow\mathcal{F})^- x\big) \Leftrightarrow (\exists a \in S)\ \big(y(\downarrow\mathcal{G})a\ \&\ a \gtrless x\big)$; finally, by the definition of composition between binary relations, we can rewrite things as: $(\downarrow\mathcal{F})^- \circ \gtrless\, = \,\gtrless \circ (\downarrow\mathcal{G})$.

$(2 \Rightarrow 3)$ Because the right component of a relation-pair is a continuous relation between the represented basic topologies.

$(3 \Rightarrow 4)$ By lemma 2.6.

$(4 \Rightarrow 5)$ For every q in $\mathcal{Q}$, $\{b \in T \mid b \gtrless q\}$ is a formal closed subset of $Top(\mathcal{Q})$; therefore $(\downarrow\mathcal{F})^-\{b \in T \mid b \gtrless q\}$ is a closed subset of $Top(\mathcal{P})$. But, for every $a \in S$, we have: $a \,\epsilon\, (\downarrow\mathcal{F})^-\{b \in T \mid b \gtrless q\}$ iff $(\downarrow\mathcal{F})a \between \{b \in T \mid b \gtrless q\}$ iff $(\exists b \in T)\ (b \leq \mathcal{F}a\ \&\ b \gtrless q)$ iff $\mathcal{F}a \gtrless q$.

$(5 \Rightarrow 1)$ We first show that $p \gtrless \mathcal{G}q \Rightarrow \mathcal{F}p \gtrless q$ (this holds for any $\mathcal{F}$ which is monotone). Assume that $p \gtrless \mathcal{G}q$; by definition of $\mathcal{G}$ (use also the fact that $\gtrless$ splits joins) there exists $a \in S$ such that $p \gtrless a$ and $(\forall x \in S)\ \big(x \gtrless a \Rightarrow \mathcal{F}x \gtrless q\big)$; since $p \gtrless a$, there exists $x \leq p$ such that $x \gtrless a$; hence $\mathcal{F}x \gtrless q$ and $\mathcal{F}p \gtrless q$ (because $\mathcal{F}$ is monotone). Now we are going to prove the other direction: $\mathcal{F}p \gtrless q \Rightarrow p \gtrless \mathcal{G}q$. Let $U = \{x \in S \mid \mathcal{F}x \gtrless q\}$; then $\mathcal{G}q = \bigvee\{a \in S \mid (\forall x \in S)\ (x \gtrless a \Rightarrow x \,\epsilon\, U)\} = \bigvee\{a \in S \mid \Diamond a \subseteq U)\} = \bigvee \mathsf{rest}\, U$. Assume that $\mathcal{F}p \gtrless q$; since $\mathcal{F}$ preserves joins, $\mathcal{F}p = \bigvee_{a \leq p} \mathcal{F}a$; so there exists $a \leq p$ such that $\mathcal{F}a \gtrless q$, that is, $a \,\epsilon\, U$. Since U is a formal closed subset of $Top(\mathcal{P})$, we get $a \,\epsilon\, \mathcal{J}U$, that is, $a \gtrless \bigvee \mathsf{rest}\, U$. So $a \gtrless \mathcal{G}q$, hence $p \gtrless \mathcal{G}q$. q.e.d.

5.1 A category-theoretic summing-up

In this section we want to read Top and Reg as two functors between set-based overlap algebras and positive topologies. It is clear that we cannot consider the standard category of positive topologies because, as we have just seen, overlap relations corresponds to continuous relation which are not necessarily total and convergent. On the other hand, the category of basic topologies is not suitable because the construction Reg work for positive topologies only. Thus we must consider a category which is halfway between **PTop** and **BTop**, namely the category of positive topologies and continuous relations. This is however not enough. If we want to extend Reg to a functor, we must be able to restrict every continuous relation to regular subsets (see proposition 5.4 below). In other words, given $s : \mathcal{S} \longrightarrow \mathcal{T}$, we will need to induce a continuous relation between $Top(Reg(\mathcal{S}))$ and $Top(Reg(\mathcal{T}))$.

Definition 5.2 *Let s be a continuous relation between two positive topologies $\mathcal{S}$ and $\mathcal{T}$. We say that s* preserves regular subsets *(or is* regular-preserving*) if it is also a continuous relation from $Top(Reg(\mathcal{S}))$ to $Top(Reg(\mathcal{T}))$.*[17]

To avoid confusion, we write $Top(Reg(s))$ for s read as a continuous relation from $Top(Reg(\mathcal{S}))$ to $Top(Reg(\mathcal{T}))$. Note that regular-preserving continuous relations are closed under composition and identities. Moreover, every continuous relation between topologies of the form $Top(\mathcal{P})$ is regular-preserving because of proposition 4.4.

This almost completes the definition of the category of topologies we need. We only add the requirement $\mathcal{J}_\mathcal{R} \subseteq \mathcal{J}$ in order to make $\varepsilon_\mathcal{S}$ of equation (27) a continuous relation. Note that every $\varepsilon_\mathcal{S}$ preserves regular subsets since it is induced by the identity relation.

Definition 5.3 *We write $\mathbf{PTop_{reg}}$ for the category whose objects are positive topologies satisfying $\mathcal{J}_\mathcal{R} \subseteq \mathcal{J}$ and whose morphisms are continuous relations preserving regular subsets.*

Let $\mathbf{OA_{sb}}$ be the category of set-based overlap algebras and overlap-relations. We define two (cotravariant) functors

$$\mathbf{OA_{sb}}^{op} \underset{Reg}{\overset{Top}{\rightleftarrows}} \mathbf{PTop_{reg}} \tag{30}$$

in the following way.

For any $\mathcal{P}$ in $\mathbf{OA_{sb}}$ we have already defined an object in $\mathbf{PTop_{reg}}$ namely $Top(\mathcal{P})$. Now, for any morphisms $\mathcal{F} : \mathcal{P} \longrightarrow \mathcal{Q}$ in $\mathbf{OA_{sb}}$ we put

$$Top(\mathcal{F}) \overset{def}{=} ({\downarrow}\mathcal{F})^- \ : \ Top(\mathcal{Q}) \ \longrightarrow \ Top(\mathcal{P})$$

(remember equation (29) and proposition 5.1). The continuous relation $Top(\mathcal{F})$ obviously preserves regular subsets since all formal open subsets in $Top(\mathcal{Q})$ and

[17] This notion is analogous to that of weakly open frame homomorphism in [1].

$Top(\mathcal{P})$ are regular. If $id_{\mathcal{P}}$ is the identity map on $\mathcal{P}$, then $a \,\epsilon\, \mathcal{A}(Top(Id_{\mathcal{P}}))^{-}b$ iff $a \,\epsilon\, \mathcal{A}(\downarrow Id_{\mathcal{P}})b$ iff $a \leq \bigvee \downarrow b$ iff $a \leq b$ iff $a \,\epsilon\, \mathcal{A}b$; so $Top(id_{\mathcal{P}})$ is equal (as a continuous relation) to the identity on $\mathcal{P}$. Moreover, $a\, Top(\mathcal{F}_1 \circ \mathcal{F}_2)\, b$ iff $a\, (\downarrow(\mathcal{F}_1 \circ \mathcal{F}_2))^{-}\, b$ iff $a \leq (\mathcal{F}_1 \circ \mathcal{F}_2)b$ iff $a \leq \mathcal{F}_1\mathcal{F}_2 b$ iff $(\exists c \leq \mathcal{F}_2 b)(a \leq \mathcal{F}_1 c)$ iff $(\exists c \,\epsilon\, \downarrow\mathcal{F}_2 b)(a \,\epsilon\, \downarrow\mathcal{F}_1 c)$ iff $a \,\epsilon\, (\downarrow\mathcal{F}_1)(\downarrow\mathcal{F}_2)b$ iff $a \,\epsilon\, (Top(\mathcal{F}_1))^{-}(Top(\mathcal{F}_2))^{-}b$ iff $a \,\epsilon\, (Top(\mathcal{F}_2) \circ Top(\mathcal{F}_1))^{-}b$ iff $a\,(Top(\mathcal{F}_2) \circ Top(\mathcal{F}_1))\, b$.

We now came to the case of Reg. For every morphism $s : \mathcal{S} \longrightarrow \mathcal{T}$ in $\mathbf{PTop_{reg}}$, we put:

$$\begin{array}{rcrcl} Reg(s) & : & Reg(\mathcal{T}) & \longrightarrow & Reg(\mathcal{S}) \\ & & \mathcal{R}V & \longmapsto & \mathcal{R}s^{-}V \end{array} \tag{31}$$

for every $V \subseteq T$. Note that this definition agrees with the notation $Top(Reg(s))$ proposed after definition 5.2.

In fact, if we compute $Top(Reg(s))$ according to the definitions of Top and Reg, then we have: $aTop(Reg(s))b$ iff $a(\downarrow Reg(s))^{-}b$ iff $b(\downarrow Reg(s))a$ iff $a \leq Reg(s)b$. This inequality must be read in the overlap algebras $Reg(\mathcal{S})$ and $Reg(\mathcal{T})$, which are set based on S and T with respect to the maps $a \mapsto \mathcal{R}a$ and $b \mapsto \mathcal{R}'b$. So $a \leq Reg(s)b$ is an abbreviation for $\mathcal{R}a \subseteq Reg(s)\mathcal{R}'b$, that is, $\mathcal{R}a \subseteq \mathcal{R}s^{-}b$. Since $\mathcal{R}$ is a saturation, this says precisely that $a \,\epsilon\, \mathcal{R}s^{-}b$. Summing up, from the definitions of Top and Reg, we get

$$Top(Reg(s))^{-}b \;=\; \mathcal{R}s^{-}b \tag{32}$$

for all $b \in T$. This implies that $Top(Reg(s))$ is equal to s as a continuous relation (recall that the saturation of $Top(Reg(\mathcal{S}))$ is just $\mathcal{R}$) as wished.

Lemma 5.4 *Let s be a morphism from $\mathcal{S}$ to $\mathcal{T}$ in $\mathbf{PTop_{reg}}$; then the operator $Reg(s)$ defined in (31) is an overlap-relation.*

Proof Since $Reg(s)$ preserves joins, it is sufficient (by proposition 5.1) to prove that the the relation $Top(Reg(s))$ is continuous from $Top(Reg(\mathcal{S}))$ to $Top(Reg(\mathcal{T}))$. This is true since s is regular-preserving. q.e.d.

We want to show that Reg is in fact a (contravariant) functor. Clearly, $Reg(id_{\mathcal{S}}) = id_{Reg(\mathcal{S})}$ since $id_{\mathcal{S}}$ can be presented by the identity relation. Moreover, we have: $Reg(s_1 \circ s_2)\ \mathcal{R}U = \mathcal{R}(s_1 \circ s_2)^{-}U = \mathcal{R}s_2^{-}s_1^{-}U = Reg(s_2)\ \mathcal{R}s_1^{-}U$ $= Reg(s_2)Reg(s_1)\ \mathcal{R}U = (Reg(s_2) \circ Reg(s_1))\ \mathcal{R}U$.

Recall from proposition 4.4 that $Top(Reg(Top(\mathcal{P})))$ and $Reg(Top(Reg(\mathcal{S})))$ always coincides with $Top(\mathcal{P})$ and $Reg(\mathcal{S})$, respectively, for every set-based overlap algebra $\mathcal{P}$ and every positive topology $\mathcal{S}$ (in particular, neither Top nor Reg is injective on objects).

Lemma 5.5 *For every set-based overlap algebra $\mathcal{P}$ and every positive topology $\mathcal{S}$ with $\mathcal{J}_{\mathcal{R}} \subseteq \mathcal{J}$ the following hold:*

1. $Top(\eta_{\mathcal{P}}) = id_{Top(\mathcal{P})} = \varepsilon_{Top(\mathcal{P})}$
2. $Reg(\varepsilon_{\mathcal{S}}) = id_{Reg(\mathcal{S})} = \eta_{Reg(\mathcal{S})}$

(see equations (26) and (27) for the relevant definitions).

Proof For $a, b \in S$ we have: $a \ \epsilon\ (Top(\eta_{\mathcal{P}}))^{-}b$ iff $a \ \epsilon\ (\downarrow\eta_{\mathcal{P}})b$ iff $a \leq \bigvee\{b\}$ iff $a \leq b$ (to be read in $Reg(\mathcal{S})$, set-based on S with respect to the map $a \mapsto \mathcal{R}a$) iff $\mathcal{R}a \subseteq \mathcal{R}b$ iff $a \ \epsilon\ \mathcal{R}b$; hence $\mathcal{R}(Top(\eta_{\mathcal{P}}))^{-}b = \mathcal{R}b = \mathcal{R}(\varepsilon_{Top(\mathcal{P})})^{-}b$. For $U \subseteq S$ we have: $Reg(\varepsilon_{\mathcal{S}})\mathcal{R}U = \mathcal{R}(\varepsilon_{\mathcal{S}})^{-}U = \mathcal{R}U = \bigvee^{\mathcal{R}}\{\mathcal{R}a \mid a \ \epsilon\ \mathcal{R}U\} = \eta_{Reg(\mathcal{S})}\mathcal{R}U$. q.e.d.

Lemma 5.6 *The two maps $\mathcal{P} \mapsto \eta_{\mathcal{P}}$ and $\mathcal{S} \mapsto \varepsilon_{\mathcal{S}}$ define two natural transformations $\eta : 1_{\mathbf{OA_{sb}}^{op}} \to Reg\ Top$ and $\varepsilon : Top\ Reg \to 1_{\mathbf{PTop_{reg}}}$.*

Moreover η is a natural isomorphism and

$$Reg(Top(\mathcal{F})) \quad = \quad \eta_{\mathcal{Q}}^{-1} \circ \mathcal{F} \circ \eta_{\mathcal{P}} \tag{33}$$

for every overlap-relation $\mathcal{F} : \mathcal{P} \longrightarrow \mathcal{Q}$ in $\mathbf{OA_{sb}}$.

Proof We first prove (33). For every $U \subseteq S$ (S being the base of $\mathcal{P}$), $Reg(Top(\mathcal{F}))\ \mathcal{R}U = R\ (Top(\mathcal{F}))^{-}U = \mathcal{R}(\downarrow\mathcal{F})U = \mathcal{R}\bigcup_{a\epsilon U}\downarrow\mathcal{F}a = \mathcal{A}\bigcup_{a\epsilon U}\downarrow\mathcal{F}a$ $= \downarrow\bigvee\bigcup_{a\epsilon U}\downarrow\mathcal{F}a = \downarrow\bigvee_{a\epsilon U}\bigvee\downarrow\mathcal{F}a = \downarrow\bigvee_{a\epsilon U}\mathcal{F}a = \downarrow\mathcal{F}(\bigvee U) = \eta_{\mathcal{Q}}^{-1}\ \mathcal{F}(\bigvee U) =$ $(\eta_{\mathcal{Q}}^{-1} \circ \mathcal{F})\bigvee U = (\eta_{\mathcal{Q}}^{-1} \circ \mathcal{F} \circ \eta_{\mathcal{P}})\ \mathcal{R}U$. So η is a natural transformation and, in fact, a natural isomorphism (proposition 4.2).

We now want to prove that $\varepsilon_{\mathcal{T}} \circ Top(Reg(s)) = s \circ \varepsilon_{\mathcal{S}}$ for every $s : \mathcal{S} \to \mathcal{T}$. By the definition of equality between continuous relations, we must check that $\mathcal{R}(\varepsilon_{\mathcal{T}} \circ Top(Reg(s)))^{-}b = \mathcal{R}(s \circ \varepsilon_{\mathcal{S}})^{-}b$ for all $b \in T$. This follows from equation (33) and the fact that $\varepsilon_{\mathcal{S}}$ and $\varepsilon_{\mathcal{T}}$ are induced by the identity relation on S and T, respectively. q.e.d.

Note that, as a consequence of the previous two lemmas, $Top(Reg(Top(\mathcal{F}))) = Top(\mathcal{F})$ and $Reg(Top(Reg(s))) = Reg(s)$, for every $\mathcal{F}$ and every s.

Proposition 5.7 *The adjunction $Top \dashv Reg$ holds, with unit η and counit ε.*

Proof Once written down rightly, the triangular identities follow at once from lemma 5.5. q.e.d.

Proposition 5.8 *The functors Top and Reg are full; Top is also faithful and reflects isomorphisms.*

Proof Let s be a morphism from $Top(\mathcal{P})$ to $Top(\mathcal{Q})$. Then $s = Top(Reg(s))$ (which makes sense thanks to proposition 4.4) and so Top is full.

Let $\mathcal{F}$ be a morphism from $Reg(\mathcal{S})$ to $Reg(\mathcal{T})$. By equation (33), we have: $Reg(Top(\mathcal{F})) = \eta_{Reg(\mathcal{T})}^{-1} \circ \mathcal{F} \circ \eta_{Reg(\mathcal{S})} = \mathcal{F}$ (recall lemma 5.5). So Reg is full.

If $Top(\mathcal{F}_1) = Top(\mathcal{F}_2)$, then $Reg(Top(\mathcal{F}_1)) = Reg(Top(\mathcal{F}_2))$, that is, $\eta_{\mathcal{Q}}^{-1} \circ \mathcal{F}_1 \circ \eta_{\mathcal{P}} = \eta_{\mathcal{Q}}^{-1} \circ \mathcal{F}_2 \circ \eta_{\mathcal{P}}$. So $\mathcal{F}_1 = \mathcal{F}_2$ and Top is faithful.

Finally, If $Top(\mathcal{P}) \cong Top(\mathcal{Q})$, then $\mathcal{P} \cong Reg(Top(\mathcal{P})) \cong Reg(Top(\mathcal{Q})) \cong \mathcal{Q}$. q.e.d.

As a consequence, there exists a duality between $\mathbf{OA_{sb}}$ and its essential image along Top (the objects in this image are exactly the representable positive topologies whose formal open subsets are all regular). Moreover this image is a coreflective subcategory of $\mathbf{PTop_{reg}}$ with the composition $Top\ Reg$ as the coreflector.

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