A class of imprimitive groups [*†]

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Abstract

We classify imprimitive groups inducing the alternating group $\mathbf{A}_4$ on the set of blocks, with the inertia subgroup satisfying some very natural geometrical conditions which force the group to operate linearly.

Recently in [1] C. Bartolone, S. Musumeci and K. Strambach studied imprimitive permutation groups which are highly transitive on blocks and satisfy conditions common in geometry (for instance in Laguerre and Minkowski geometries, see [2]). In particular, they classified all imprimitive permutation groups $\mathsf{G} = (G, \Omega, \bar{\Omega})$, where G denotes the group of permutations, Ω the set of points and $\bar{\Omega}$ the set of blocks, fulfilling the following conditions for some integer m such that $3 < m \leq |\bar{\Omega}|$:

i) the inertia subgroup N_{G}, i.e. the subgroup fixing every block, induces a sharply transitive action on every block;

ii) given two ordered m-tuples $(X_1, \dots, X_m), (Y_1, \dots, Y_m) \in \Omega^m$, X_i and Y_i lying in the same block Δ_i, there is just one element in N_{G} moving $(X_1, \dots, X_m)$ to $(Y_1, \dots, Y_m)$, provided $\Delta_1, \dots, \Delta_m$ are distinct blocks;

iii) the stabilizer in G of a block has a 2-transitive action on it;

iv) the factor group G/N_{G} is finite and acts m-transitively on the set $\bar{\Omega}$ of blocks.

Conditions *i*) and *iii*) force the inertia subgroup N_{G} to be elementary Abelian and, consequently, G to be an affine group, of course of finite order in view of conditions *iv*) and *iii*). So one could envisage a wider programme of classifying finite imprimitive groups having an elementary Abelian inertia subgroup satisfying *ii*) for some positive integer $m \leq |\bar{\Omega}|$, provided the size of blocks and the factor permutation group $\bar{\mathsf{G}} = (G/N_{\mathsf{G}}, \bar{\Omega})$ are assigned. As in this context a large amount of non-splitting group extensions are expected, the programme can be carried out only if the size of blocks is small. This article is a first step for the envisaged programme: we deal with the case where there are 4 blocks, each of size 16, and $\bar{\mathsf{G}}$ is isomorphic to $\mathbf{A_4}$ (the reasons of such a size come from [1] and [4]). It turns out that G can be represented as an affine group with the inertia group N_{G} as the group of translations. Moreover G splits over N_{G}, apart from some exceptional cases (43 for $m = 1$, 11 for $m = 2$ and 6 for $m = 3$) we list in section 4 (for $m = 2$) and section 5 (for $m = 1$ and $m = 3$). Although some parts of the paper could be accomplished on a computer, we have preferred to achieve any result by using combinatorial arguments.

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1. Let m be a positive integer ≤ 4 and let $\mathsf{G}_m = (G_m, \Omega, \bar{\Omega})$ be a transitive imprimitive permutation group such that:

1. the inertia subgroup N_{G_m} is elementary Abelian;
2. the factor group $\bar{\mathsf{G}}_m = (G_m/N_{\mathsf{G}_m}, \bar{\Omega})$ is isomorphic to $\mathbf{A_4}$ and $|\Delta| = 16$ for all $\Delta \in \bar{\Omega}$;
3. for any m distinct blocks $\Delta_1, \ldots, \Delta_m \in \bar{\Omega}$ and points $X_i, Y_i \in \Delta_i$, $i = 1, \ldots, m$, there is just one element $g \in N_{\mathsf{G}_m}$ such that $g(X_i) = Y_i$ for all i.

As a block contains 16 points and there are 4 blocks at all, we have $|\Omega| = 64$ and we may regard G_m as a subgroup of the symmetric group $\mathbf{S_{64}}$ preserving $\bar{\Omega}$. Let $\mathsf{F} = (F, \Omega, \bar{\Omega})$ be the full subgroup of $\mathbf{S_{64}}$ preserving $\bar{\Omega}$ and let N_{F} be the corresponding inertia subgroup. Of course (N_{F}, Ω) is isomorphic to the direct product of 4 copies of $\mathbf{S_{16}}$. In view of Condition 3, N_{G_m} induces on every block $\Delta_l \in \bar{\Omega}$ a sharply transitive permutation group; also, $\forall X_l \in \Delta_l$, N_{F} has an elementary Abelian subgroup U_l of order 16 acting on Δ_l as $N_{\mathsf{G}_m}/(N_{\mathsf{G}_m})_{X_l}$ and leaving any other block point-wise fixed. Clearly N_{G_m} is contained in the direct sum

$$V = U_1 \oplus U_2 \oplus U_3 \oplus U_4 \subset N_{\mathsf{F}}, \tag{1}$$

that we may regard as a 16-dimensional vector space over the prime field $\mathbf{GF}(2)$.

Let $V_l = U_i \oplus U_j \oplus U_k$ with $\{i, j, k, l\} = \{1, 2, 3, 4\}$, then $V/V_l \simeq U_l$ acts sharply transitively on Δ_l and we may identify Δ_l with $V/V_l \simeq \mathbf{GF}(2)^4$, so

$$\Omega = \bigsqcup_{l=1}^{4} V/V_l = \Big\{ u + V_l : u \in U_l,\ l = 1, 2, 3, 4 \Big\}$$

with $\bar{\Omega} = \{V/V_1, V/V_2, V/V_3, V/V_4\}$. Thus we can let $G_m/N_{\mathsf{G}_m} \simeq \mathbf{A_4}$ act as a subgroup of $\mathbf{GL}(V)$ normalizing N_{G_m} and there are four equivalent linear representations $(G_m)_{\Delta_l}/N_{\mathsf{G}_m} \to \mathbf{GL}(V/V_l)$ that we may identify with a linear representation $\alpha : \mathbf{A_3} \to \mathbf{GL}(U)$ with $U \simeq \mathbf{GF}(2)^4$. This allows a double embedding

$$\mathsf{G}_m \hookrightarrow \mathcal{G}_m \hookrightarrow \mathbf{AGL}(V),$$

where $\mathcal{G}_m$ denotes the twisted wreath product $U \,\mathrm{wr}_\alpha\, \mathbf{A_4}$, and we may regard $\mathcal{G}_m$ as an imprimitive permutation group with the same point set and block set as G_m (see [5] p. 86, [1] §2.1 and §2.2). In this context the inertia subgroup N_{G_m} corresponds to the group of translations determined by a subspace W of V which is *m-transversal* with respect to the decomposition (1) of V, which means that the projection $W \longrightarrow \bigoplus_{r=i_1}^{i_m} U_r$ is an isomorphism for any m-subset $\{i_1, \ldots, i_m\}$ of $\{1, 2, 3, 4\}$ (hence $\dim W = 4m$) (see [1], §2.1.2 for details). Manifestly we have $\mathsf{G}_4 = \mathcal{G}_4$, so we may assume $m < 3$ from now on.

2. In order to represent the twisted wreath product $\mathcal{G}_m$, we need a set S of representatives of $\mathbf{A_4}/\mathbf{A_3}$. In view of [5], p. 86, the structure of the group is independent of the choice of S, so we take

$$S = \{\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4\}$$

with $\vartheta_k \equiv (ij)(k4)$, where $\{i, j, k\} = \{1, 2, 3\}$ for $k = 1, 2, 3$ and $\vartheta_4 = 1$. Fix now a basis $\{\mathbf{e}_{rs}\}_{r,s=1,2,3,4}$ of V with $U_r = \langle \mathbf{e}_{r1}, \mathbf{e}_{r2}, \mathbf{e}_{r3}, \mathbf{e}_{r4} \rangle$ and, for $l = 1, 2, 3, 4$, let ϑ_l act on V as the linear map moving $\mathbf{e}_{rs}$ to $\mathbf{e}_{\vartheta_l(r)s}$. For any $\sigma \in \mathbf{A_4}$ each permutation $\vartheta_{\sigma(l)}\sigma\vartheta_l$ fixes 4 and may be identified with the corresponding element in $\mathbf{A_3}$. So we have the linear mappings $\alpha(\vartheta_{\sigma(l)}\sigma\vartheta_l) \in \mathbf{GL}(U)$ and we can let σ linearly act on V by putting $\sigma(\mathbf{e}_{rs}) = \sum_{l=1}^{4} a_{ls}\, \mathbf{e}_{\sigma(r)l}$, where

$$A(\sigma)_r := (a_{ls})$$

is the matrix defining $\alpha(\vartheta_{\sigma(r)}\sigma\vartheta_r)$ with respect to the fixed basis. Clearly $A(\vartheta_l)_r$ is the identity and, putting $\zeta = (123)$, we have $\vartheta_{\zeta(r)}\zeta\vartheta_r = \zeta$ for all $r = 1, 2, 3, 4$, i.e. $A(\zeta)_r$ is, up to similarity, one of the following:

$$1.\ \begin{pmatrix} I & O \\ O & I \end{pmatrix}, \qquad 2.\ \begin{pmatrix} I & O \\ O & E \end{pmatrix}, \qquad 3.\ \begin{pmatrix} E & O \\ O & E \end{pmatrix}, \tag{2}$$

where

$$E = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad O = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Throughout the paper we use the symbol G_m^i (resp. $\mathcal{G}_m^i$) instead of G_m (resp. $\mathcal{G}_m$) to specify which of the above cases occurs. Besides we indicate by

$$\varphi_M \text{ and } \phi_{\tau,\, M}$$

to indicate, respectively, the automorphism of U defined through the matrix $M \in \mathbf{GL}(4, 2)$ and the automorphism of $\mathbf{GL}(V)$ obtained from the 4×4 permutation matrix corresponding to $\tau \in \mathbf{S_4}$ by placing M at the place $(\tau(r), r)$. Thus the above argument says that the permutations in $\mathcal{G}_m^i$ are the affine mappings

$$g_{\gamma,\, \mathbf{v}} : \mathsf{x} \mapsto \gamma(\mathsf{x}) + \mathbf{v}$$

with $\mathbf{v} \in V$ and $\gamma \in H_i := \langle \phi_{\zeta,\, A_i}, \phi_{\vartheta_3,\, I} \rangle$ ($\simeq \mathbf{A_4}$), where A_i is the matrix (2.i).

3. Let G_m^i be a subgroup of $\mathcal{G}_m^i$ fulfilling conditions $1-3$ in §1. Then the group of permutations in G_m^i is

$$G_m^i = \langle g_{\gamma,\, \mathbf{u}_\gamma + \mathbf{w}} : \gamma \in H_i, \mathbf{w} \in W \rangle, \tag{3}$$

where W is a H_i-invariant subspace of V, m-transversal with respect to the decomposition (1), and $\mathbf{u}_\gamma$ is a vector depending on γ, only. The invariance of W under $\phi_{\zeta,\, A_i}$ and $\phi_{\vartheta_3,\, I}$ yields

Proposition 1. *W is one of the following subspaces of V:*

$$\left\langle \left(\mathsf{x}, \omega(\mathsf{x}),\, \varphi_{A_i^2}\omega\varphi_{A_i}(\mathsf{x}),\, \varphi_{A_i}\omega\varphi_{A_i^2}(\mathsf{x})\right) : \mathsf{x} \in U \right\rangle \qquad (m = 1),$$

$$\left\langle \left(\mathsf{x}_1, \mathsf{x}_2, \omega(\mathsf{x}_1) + \varphi_{A_i^2}\omega^{-1}\varphi_{A_i}(\mathsf{x}_2), \varphi_{A_i^2}\omega^{-1}\varphi_{A_i}(\mathsf{x}_1) + \omega(\mathsf{x}_2)\right) : \mathsf{x}_h \in U \right\rangle \quad (m = 2),$$

$$\left\langle \left(\mathsf{x}_1, \mathsf{x}_2, \mathsf{x}_3, \omega(\mathsf{x}_1) + \varphi_{A_i}\omega\varphi_{A_i^2}(\mathsf{x}_2) + \varphi_{A_i^2}\omega\varphi_{A_i}(\mathsf{x}_3)\right) : \mathsf{x}_k \in U \right\rangle \qquad (m = 3),$$

for a suitable automorphism ω of U such that $(\omega\varphi_{A_i})^3 = (\omega\varphi_{A_i^2})^3 = 1$ with $\omega + \omega^{-1} = \varphi_{A_i}\omega\varphi_{A_i}\omega^{-1}\varphi_{A_i}$ if $m = 2$ and $\omega^2 = 1$ if $m = 1$, or $m = 3$. □

Proposition 2. *Let G_m^j be a subgroup of $\mathcal{G}_m^i$ fulfilling the same conditions* 1−3 *of* §1 *satisfied by G_m^i. There is an isomorphism $\mathsf{G}_m^i \to \mathsf{G}_m^j$ if and only if $i = j$ and there exist $M \in \mathcal{N}_{\mathbf{GL}(4,2)}\langle A_i \rangle$, $\tau \in \mathbf{S_4}$ and $\mathbf{v} \in V$ such that*

$$G_m^j = \langle g_{\gamma', \mathbf{u}'_{\gamma'} + \mathbf{w}'} : \gamma' \in H_i,\, \mathbf{w}' \in \phi_{\tau, M}(W) \rangle,$$

where $\mathbf{u}'_{\gamma'} = \phi_{\tau, M}(\mathbf{u}_\gamma + \mathbf{v} + \gamma(\mathbf{v}))$ with $\phi_{\tau, M}\gamma = \gamma' \phi_{\tau, M}$ and $M \in \mathcal{C}_{\mathbf{GL}(4,2)}\langle A_i \rangle$ if $\tau \in \mathbf{A_4}$ or $i = 1$, $M \notin \mathcal{C}_{\mathbf{GL}(4,2)}\langle A_i \rangle$ if $\tau \notin \mathbf{A_4}$ and $i \neq 1$. In such a case an isomorphism $\psi = (\psi', \psi'')$: $\mathsf{G}_m^i \to \mathsf{G}_m^j$ is given by

$$\begin{aligned} \psi' \;&:\; g_{\gamma, \mathbf{u}_\gamma + \mathbf{w}} \longmapsto g_{\gamma', \mathbf{u}'_{\gamma'} + \phi_{\tau, M}(\mathbf{w})}, \\ \psi'' \;&:\; \mathsf{u} + V_l \quad \longmapsto \phi_{\tau, M}(\mathsf{u} + \mathbf{v}) + V_{\tau(l)}. \end{aligned}$$

Proof. Assume there exists an isomorphism $\psi = (\psi', \psi'') : \mathsf{G}_m^i \to \mathsf{G}_m^j$. Then there exist $\tau \in \mathbf{S_4}$ and bijective mappings $\psi_l : U_l \to U_{\tau(l)}$ $(l = 1, 2, 3, 4)$ such that

$$\psi''(\mathsf{u} + V_l) = \psi_l(\mathsf{u}) + V_{\tau(l)} \quad \forall \mathsf{u} \in U_l. \tag{4}$$

Furthermore ψ' restricts to an isomorphism $\lambda : N_{\mathsf{G}_m^i} \to N_{\mathsf{G}_m^j}$ of the corresponding inertia subgroups mapping the point-wise stabilizer $(N_{\mathsf{G}_m^i})_{[\Delta_l]}$ in $N_{\mathsf{G}_m^i}$ of the block Δ_l onto the point-wise stabilizer $(N_{\mathsf{G}_m^j})_{[\Delta_{\tau(l)}]}$ in $N_{\mathsf{G}_m^j}$ of the block $\Delta_{\tau(l)}$. Thus λ induces isomorphisms $N_{\mathsf{G}_m^i}/(N_{\mathsf{G}_m^i})_{[\Delta_l]} \to N_{\mathsf{G}_m^j}/(N_{\mathsf{G}_m^j})_{[\Delta_{\tau(l)}]}$, that we may regards as linear isomorphisms

$$\lambda_l : U_l \to U_{\tau(l)},$$

and, by identifying $N_{\mathsf{G}_m^i}$ and W, we have $\lambda = \bigoplus_{r=1}^4 \lambda_r \pi_r$, where π_r denotes the natural projection $W \to U_r$. Moreover,

$$\pi'_{\tau(l)} \lambda = \lambda_l \pi_l, \quad (l = 1, 2, 3, 4) \tag{5}$$

using the other projections $\pi'_r : W' \to U_r$, if W' denotes the subspace of V underlying $N_{\mathsf{G}_m^j}$. Now for all $\mathbf{w} \in W$ we have $\psi''(\mathbf{w} + V_l) = \lambda(\mathbf{w}) + \psi''(V_l)$ that, in view of (4), yields

$$\psi_l(\pi_l(\mathbf{w})) + V_{\tau(l)} = \pi'_{\tau(l)}(\lambda(\mathbf{w})) + \psi_l(0_{U_l}) + V_{\tau(l)},$$

i.e.

$$\psi_l \pi_l(\mathbf{w}) = \lambda_l \pi_l(\mathbf{w}) + \psi_l(0_{U_l}), \tag{6}$$

thanks to (5). As W is transversal, each projection π_l is surjective, so the identities (6) say that ψ_l is an affinity $U_l \to U_{\tau(l)}$.

It follows from paragraph 2 that $G_m^i/N_{\mathsf{G}_m^i}$ and $G_m^j/N_{\mathsf{G}_m^j}$ share the same subgroup

$$\bar{K} := \langle \phi_{\vartheta_k, I} : k = 1, 2, 3 \rangle$$

of $\mathbf{GL}(V)$. So ψ' induces an isomorphism $\rho : G_m^i/N_{\mathsf{G}_m^i} \to G_m^j/N_{\mathsf{G}_m^j}$ mapping $\bar{K}$ onto itself. More precisely we have

$$\rho(\phi_{\vartheta_k, I}) = \phi_{\vartheta_{k'}, I},\; k' = \tau \vartheta_k \tau^{-1}(4) \quad (k = 1, 2, 3). \tag{7}$$

Let M_l be the 4×4 matrix defining the affinity ψ_l with respect to the above fixed bases of U_l and $U_{\tau(l)}$. Then (7) gives $I = M_{\tau^{-1}(k')}M^{-1}_{\tau^{-1}(4)} = M_{\vartheta_k\tau^{-1}(4)}M^{-1}_{\tau^{-1}(4)}$ for $k = 1,2,3$, i.e. M_1, M_2, M_3, M_4 are the same matrix M and we have $\psi''(\mathsf{u}+V_l) = \phi_{\tau,M}(\mathsf{u}+\mathbf{v}) + V_{\tau(l)}$ with

$$\mathbf{v} = \phi^{-1}_{1,M}\Big(\psi_1(0_{U_1}), \psi_2(0_{U_2}), \psi_3(0_{U_3}), \psi_4(0_{U_4})\Big).$$

Moreover $\rho(\phi_{\zeta,A_i}) \in \langle\phi_{\zeta,A_j}\rangle\bar{K}$ gives $MA_iM^{-1} \in \langle A_j\rangle$, and this occurs just if $j = i$. Furthermore $\phi_{\tau,M} \in \mathcal{N}_{\mathbf{GL}(4,2)}(\bar{G}^i_m)$ yields $M \in \mathcal{C}_{\mathbf{GL}(4,2)}\langle A_i\rangle$ if $\tau \in \mathbf{A_4}$ or $i = 1$, $M \notin \mathcal{C}_{\mathbf{GL}(4,2)}\langle A_i\rangle$ if $\tau \notin \mathbf{A_4}$ and $i \neq 1$.

Finally $\psi'(g_{\gamma,\mathbf{u}_\gamma+\mathbf{w}})$, as an element of $\mathbf{Sym}(\Omega)$, is $\psi'' g_{\gamma,\mathbf{u}_\gamma+\mathbf{w}}\psi''^{-1}$ which moves the point $\mathsf{u}+V_l$ to $\gamma'(\mathsf{u})+\phi_{\tau,M}(\mathbf{v}+\gamma(\mathbf{v})+\mathbf{u}_\gamma)+\phi_{\tau,M}(\mathbf{w})+\gamma'(V_l)$ with $\gamma' = \phi_{\tau,M}\gamma\phi^{-1}_{\tau,M}$, or $\psi'(g_{\gamma,\mathbf{u}_\gamma+\mathbf{w}}) = g_{\gamma',\mathbf{u}'_{\gamma'}+\phi_{\tau,M}(\mathbf{w})}$ as claimed. □

Remark 1. Proposition 2 says that a subspace W' of V defines the inertia subgroup of a group G'_m isomorphic to G^i_m if, and only if, one of the following occurs:

$$W' = \begin{cases} \phi_{1_{\mathbf{S_4}},N}(W), & \text{for some } N \in \mathcal{C}_{\mathbf{GL}(4,2)}\langle A_i\rangle,\\ \phi_{\chi,N}(W), & \text{for some } N \in \mathcal{N}_{\mathbf{GL}(4,2)}\langle A_i\rangle \setminus \mathcal{C}_{\mathbf{GL}(4,2)}\langle A_i\rangle, \text{ if } i\neq 1,\\ \phi_{\chi,N}(W), & \text{for some } N \in \mathbf{GL}(4,2), \text{ if } i = 1,\end{cases}$$

where χ denotes the transposition (12) of $\mathbf{S_4}$. In fact, $\tau = \zeta^j\vartheta_l$, or $\tau = \chi\zeta^j\vartheta_l$, for suitable $j \in \{1,2,3\}$ and $l \in \{1,2,3,4\}$, according as whether $\tau \in \mathbf{A_4}$ or not. Hence $W' = \phi_{\tau,M}(W) = \phi_{1_{\mathbf{S_4}},MA_i^{-j}}(W)$, or $W' = \phi_{\chi,MA_i^{-j}}(W)$, in view of the fact that W is H_i-invariant. This means that we obtain W' from W by replacing the linear mapping ω, defining W as claimed in Proposition 1, by $\varphi_N\omega\varphi_N^{-1}$ if $N \in \mathcal{C}_{\mathbf{GL}(4,2)}\langle A_i\rangle$, or by $\varphi_N\omega^{-1}\varphi_N^{-1}$ if either $N \in \mathcal{N}_{\mathbf{GL}(4,2)}\langle A_i\rangle \setminus \mathcal{C}_{\mathbf{GL}(4,2)}\langle A_i\rangle$ or $i = 1$.

Remark 2. The Abelian group U has a natural structure of free module of rank 2 over the centralizer k of the matrix E. Since the minimal polynomial x^2+x+1 of E is irreducible over $\mathbf{GF}(2)$, Schur's Lemma guaranties that k is a field, of course isomorphic to $\mathbf{GF}(4)$. Thus we may regard U as a 2-dimensional k-vector space and we have

$$\mathcal{C}_{\mathbf{GL}(4,2)}\langle A_i\rangle = \begin{cases}\mathbf{GL}(2,2)\times\mathsf{k}^*, & \text{if } i=2,\\ \mathbf{GL}(2,\mathsf{k}), & \text{if } i=3,\end{cases}$$

$$\mathcal{N}_{\mathbf{GL}(4,2)}\langle A_i\rangle = \begin{cases}\mathbf{GL}(2,2)\times\mathbf{\Gamma L}(1,\mathsf{k}), & \text{if } i=2,\\ \mathbf{\Gamma L}(2,\mathsf{k}), & \text{if } i=3,\end{cases}$$

and $\mathcal{C}_{\mathbf{GL}(4,2)}\langle A_1\rangle = \mathcal{N}_{\mathbf{GL}(4,2)}\langle A_1\rangle = \mathbf{GL}(4,2)$.

Put $\gamma_i = \phi_{\zeta,A_i}$, $\delta = \phi_{\vartheta_3,I}$, $\varphi_i = \varphi_{A_i}$ and $\psi_i = \varphi_i^2+\varphi_i+\mathrm{id}_U$. We have

Proposition 3. Up to isomorphism of permutation groups

$$G^i_m = \langle g_{\gamma_i,0_V}, g_{\delta,\mathbf{u}}, g_{1,\mathbf{w}} : \mathbf{w}\in W\rangle$$

with $\mathbf{u} = (\mathsf{u}_1, \mathsf{u}_2, \mathsf{u}_3, 0_U)$, and

$$\begin{array}{ll} \mathsf{u}_1+\mathsf{u}_2 = \varphi_i\omega\varphi_i^2(\mathsf{u}_3) \in \ker(\omega+\mathrm{id}_U),\ (\varphi_i^2+\mathrm{id}_U)(\mathsf{u}_1) = (\varphi_i+\omega)(\mathsf{u}_2), & \text{if } m=1;\\ \mathsf{u}_1 = \mathsf{u}_2 \in \ker(\varphi_i + \omega + \varphi_i^2\omega^{-1}\psi_i),\ \mathsf{u}_3 = 0_U, & \text{if } m=2;\\ \mathsf{u}_1 \in \ker(\varphi_i^2\omega + \omega\varphi_i^2),\ \psi_i(\mathsf{u}_1) = \mathsf{u}_2 = \mathsf{u}_3 = 0_U, & \text{if } m=3. \end{array}$$

Moreover, for any such a triple $(\mathsf{u}_1, \mathsf{u}_2, \mathsf{u}_3)$, one extension does exist and G_m^i splits over N_{G_m} if there exist $\mathsf{x} \in U$ such that

$$\begin{array}{ll} \mathsf{u}_1 + \mathsf{u}_2 = (\omega + \mathrm{id}_U)(\mathsf{x}),\ \psi_i(\mathsf{u}_1) = \psi_i(\mathsf{x}), & \text{if } m=1;\\ \mathsf{u}_1 = \big(\mathrm{id}_U + (\varphi_i + \mathrm{id}_U)(\varphi_i\omega + \omega^{-1}\varphi_i)\big)(\mathsf{x}), & \text{if } m=2;\\ \mathsf{u}_1 = \big(\omega + \varphi_i + \varphi_i^2 + \omega\varphi_i\omega + \varphi_i^2\omega + \varphi_i\omega\varphi_i^2\big)(\mathsf{x}), & \text{if } m=3. \end{array}$$

Proof. We may assume

$$G_m^i = \langle g_{\gamma_i,\mathbf{z}'}, g_{\delta,\mathbf{z}''}, g_{1,\mathbf{w}} : w \in W \rangle$$

with $\mathbf{z}' \in \ker(\gamma_i^2 + \gamma_i + \mathrm{id}_V)$ because we may take $g_{\gamma_i,\mathbf{z}'}$ of order 3 (see [1], Lemma 4.3.3). Besides, thanks to Proposition 2, we may replace $\mathbf{z}'$ and $\mathbf{z}''$ by $\mathbf{z}' + (\gamma_i + \mathrm{id}_V)(\mathbf{v})$ and $\mathbf{z}'' + (\delta + \mathrm{id}_V)(\mathbf{v})$ for any $\mathbf{v} \in V$. As $\ker\psi_i = (\varphi_i + \mathrm{id}_U)(U)$, this allows one to take $\mathbf{z}' = 0_V$, as well as $\mathbf{z}'' = (\mathsf{u}_1, \mathsf{u}_2, \mathsf{u}_3, 0_U)$ for suitable $\mathsf{u}_1, \mathsf{u}_2, \mathsf{u}_3 \in U$. Thus the vector $\mathbf{z}''$ must satisfy the conditions

$$\begin{array}{ll} a) & (\delta + \mathrm{id}_V)(\mathbf{z}'') \in W,\\ b) & ((\delta\gamma_i)^2 + \delta\gamma_i + \mathrm{id}_V)(\mathbf{z}'') \in W, \end{array} \tag{8}$$

as a consequence of the fact that $G_m^i/N_{\mathsf{G}_m^i} \simeq \mathbf{A}_4$.

Let $m = 1$. Then (8.a) gives $\mathsf{u}_1+\mathsf{u}_2 \in \ker(\omega+\mathrm{id}_U)$ and $\mathsf{u}_3 = \varphi_i^2\omega\varphi_i(\mathsf{u}_1+\mathsf{u}_2)$ ($= \varphi_i\omega\varphi_i^2(\mathsf{u}_1+\mathsf{u}_2)$ since $(\varphi_i\omega)^3 = \mathrm{id}_U$), whereas (8.b) yields $\omega\psi_i(\mathsf{u}_1) = \mathsf{u}_2 + \varphi_i\mathsf{u}_3$ which is equivalent to $(\varphi_i^2+\mathrm{id}_U)(\mathsf{u}_1) = (\varphi_i+\omega)(\mathsf{u}_2)$. Assume there exists $\mathsf{x} \in U$ such that $\mathsf{u}_1 + \mathsf{u}_2 = (\omega + \mathrm{id}_U)(\mathsf{x})$, then $(g_{\delta,\mathbf{z}''+\mathbf{x}})^2 = \mathrm{id}_V$ with $\mathbf{x} = (\mathsf{x}, \omega(\mathsf{x}), \varphi_i^2\omega\varphi_i(\mathsf{x}), \varphi_i\omega\varphi_i^2(\mathsf{x})) \in W$. Assume also $\psi_i(\mathsf{x}) = \psi_i(\mathsf{u}_1)$, then

$$\begin{aligned} \omega\psi_i(\mathsf{x}) &= \omega(\varphi_i^2 + \mathrm{id}_U)(\mathsf{u}_1) + \omega\varphi_i(\mathsf{u}_1) = \omega(\varphi_i + \omega)(\mathsf{u}_2) + \omega\varphi_i(\mathsf{u}_1) = \omega\varphi_i(\mathsf{u}_1 + \mathsf{u}_2) + \mathsf{u}_2\\ &= \varphi_i(\mathsf{u}_3) + \mathsf{u}_2 \end{aligned}$$

and the order of $g_{\delta,\mathbf{z}''+\mathbf{x}}\, g_{\gamma_i,0_V}$ is 3, i.e. we have a split extension over the inertia subgroup.

Let $m = 2$. Then $(\delta + \mathrm{id}_V)(V) \cap W = (\delta + \mathrm{id}_V)(W)$ and we may assume $g_{\delta,\mathbf{z}''}$ of order 2, which means $\mathsf{u}_1 = \mathsf{u}_2$ and $\mathsf{u}_3 = 0_U$. Thus (8.b) gives $\mathsf{u}_1 \in \ker(\omega\psi_i + \varphi_i^2\omega^{-1}\varphi_i + \varphi_i^2) = \ker(\varphi_i^2\omega^{-1}\psi_i + \varphi_i + \omega)$ because $\omega\varphi_i\omega = \varphi_i^2\omega^{-1}\varphi_i^2$ and $\varphi_i^2\psi_i = \psi_i$. Assume there exists $\mathsf{x} \in U$ such that $\mathsf{u}_1 = (\mathrm{id}_U + (\varphi_i + \mathrm{id}_U)(\varphi_i\omega + \omega^{-1}\varphi_i))(\mathsf{x})$, then $\psi_i(\mathsf{u}_1) = \psi(\mathsf{x})$ and this gives in turn $(g_{\delta,\mathbf{z}''+\mathbf{x}})^2 = (g_{\delta,\mathbf{z}''+\mathbf{x}}g_{\gamma_i,0_V})^3 = \mathrm{id}_V$ with $\mathbf{x} = (\mathsf{x}, \mathsf{x}, (\omega + \varphi_i^2\omega^{-1}\varphi_i)(\mathsf{x}), (\omega + \varphi_i^2\omega^{-1}\varphi_i)(\mathsf{x})) \in W$.

Let $m = 3$. Up to multiplying by a translation in the inertia group we may assume $\mathsf{u}_2 = \mathsf{u}_3 = 0_U$. Then (8.a) yields $(\varphi_i^2\omega + \omega\varphi_i^2)(\mathsf{u}_1) = 0_U$, whereas (8.b) gives $\psi_i(\mathsf{u}_1) = 0_U$. Assume there exists $\mathsf{x} \in U$ such that $\mathsf{u}_1 = (\omega + \varphi_i + \varphi_i^2 + \omega\varphi_i\omega + \varphi_i^2\omega + \varphi_i\omega\varphi_i^2)(\mathsf{x})$. Then $(\delta+\mathrm{id}_V)(\mathbf{x}) = (\delta+\mathrm{id}_V)(\mathbf{z}'')$ and $((\delta\varphi_i)^2+\delta\varphi_i+\mathrm{id}_V)(\mathbf{x}) = 0_V = ((\delta\varphi_i)^2+\delta\varphi_i+\mathrm{id}_V)(\mathbf{z}'')$ with

$$\mathbf{x} = (\mathsf{u}_1 + \varphi_i(\mathrm{id}_U + \varphi_i)\mathsf{x}, \varphi_i(\mathrm{id}_U + \varphi_i)\mathsf{x}, \mathsf{x}, \mathsf{x}) \in W$$

and we may assume the order of $g_{\delta,\mathbf{z}''}$ and $g_{\delta,\mathbf{z}''}g_{\gamma_i,0_V}$ to be 2 and 3, respectively.

Finally $(g_{\delta,\mathbf{z}''})^2$ and $(g_{\delta,\mathbf{z}''}\, g_{\gamma_i,0_V})^3$ are translations $g_{1,a}$ and $g_{1,b}$ with $a = (\mathsf{u}_1 + \mathsf{u}_2, \mathsf{u}_1 + \mathsf{u}_2, \mathsf{u}_3, \mathsf{u}_3)$ and $b = (\psi_i(\mathsf{u}_1), \mathsf{u}_2 + \varphi_i(\mathsf{u}_3), \varphi_i^2(\mathsf{u}_2) + \mathsf{u}_3, \varphi_i(\mathsf{u}_2) + \varphi_i^2(\mathsf{u}_3))$ vectors of W satisfying the conditions $a \in \ker(\delta + \mathrm{id}_V)$, $b \in \ker(\delta\gamma_i + \mathrm{id}_V)$ and $(\gamma_i^2 + \gamma_i + \mathrm{id}_V + \delta\gamma_i^2)(b) = (\gamma_i^2 + \gamma_i + \mathrm{id}_V)(\mathrm{id}_V + \delta\gamma_i)(a)$, so Theorem 1 in [3] guaranties that

$$G_m^i = \langle g_{\gamma_i,0_V}, g_{\delta,\mathbf{z}''}, g_{1,\mathbf{w}} : w \in W \rangle$$

with $\mathbf{z}'' = (\mathsf{u}_1, \mathsf{u}_2, \mathsf{u}_3, 0_U)$ is an extension of W by H_i for any triple $(\mathsf{u}_1, \mathsf{u}_2, \mathsf{u}_3)$ of vectors of U fulfilling the asked conditions. □

4 $(m = 2)$**.** In case $m = 2$ the linear mapping ω acts fixed-point freely, $\omega + \omega^{-1}$ being invertible. Besides the order of ω is either 3 or 6, as immediately follows from the following

Lemma. *The minimal polynomial $\mu_\omega(x)$ of ω divides $x^4 + x^2 + 1$ and ω^2 centralizes φ_i. Moreover $\mu_\omega(x) = x^2 + x + 1$ precisely if ω centralizes φ_i.*

Proof. Proposition 1 yields $\omega + \omega^{-1} = \varphi_i\omega\varphi_i\omega^{-1}\varphi_i$ and $(\varphi_i\omega)^3 = 1$ and this gives in turn $\omega^{-2} + 1 = \varphi_i\omega^2\varphi_i^{-1}$, which means that ω and ω^2 have no fixed point and

$$\mu_\omega(\omega^{-2} + 1) = \varphi_i\mu_\omega(\omega^2)\varphi_i^{-1} = \varphi_i(\mu_\omega(\omega))^2\varphi_i^{-1} = 0. \tag{9}$$

Assume $\mu_\omega(x)$ does not divide $x^4 + x^2 + 1$, then $\mu_\omega(x)$ is either $x^4 + x + 1$, or $x^4 + x^3 + 1$, or $x^4 + x^3 + x^2 + x + 1$. The latter requires the order of ω to be 5, so $\omega^4 + \omega^2 + 1 = \mu_\omega(\omega^3 + 1) = 0$ by (9), a contradiction. In the remaining cases the order of ω is 15 with $\omega^{-2} + 1 = \omega^2 + \omega + 1$ and $\omega^{-2} + 1 = \omega^3 + \omega^2$ according as whether $\mu_\omega(x) = x^4 + x^3 + 1$, or $\mu_\omega(x) = x^4 + x + 1$. But in both cases a contradiction occurs. Therefore $\omega^2 = \omega^4 + 1 = \omega^{-2} + 1 = \varphi_i\omega^2\varphi_i^2$ and we see that ω^2 centralizes φ_i.

Let ω centralize φ_i, then $(\varphi_i\omega)^3 = 1$ says that the order of ω is 3, hence its minimal polynomial must be $x^2 + x + 1$. Let $\mu_\omega(x) = x^2 + x + 1$, then $\omega^{-1} = \omega + 1$ and the identity $\omega + \omega^{-1} = \varphi_i\omega\varphi_i\omega^{-1}\varphi_i$ forces ω to centralize φ_i. □

In case $i = 3$ the above lemma says that if the minimal polynomial of ω is $x^2 + x + 1$, then any matrix representing ω may be regarded as a 2×2-matrix over k. Thus, in view of Remark 1 and Remark 2 we may take to represent ω

$$\begin{cases} P^{11} = \begin{pmatrix} E & O \\ O & E \end{pmatrix}, & \text{if } i = 1; \\[2ex] P^{21} = \begin{pmatrix} E & O \\ O & E \end{pmatrix},\ P^{22} = \begin{pmatrix} E^2 & O \\ O & E^2 \end{pmatrix}, & \text{if } i = 2; \\[2ex] P^{31} = \begin{pmatrix} E & O \\ O & E \end{pmatrix},\ P^{32} = \begin{pmatrix} E^2 & O \\ O & E^2 \end{pmatrix},\ P^{33} = \begin{pmatrix} E^2 & O \\ O & E \end{pmatrix}, & \text{if } i = 3. \end{cases} \tag{10}$$

Let now the minimal polynomial of ω be $x^4 + x^2 + 1 = (x^2 + x + 1)^2$ (hence ω does not centralize φ_i and $i = 2$, or $i = 3$). Then ω acts reducibly leaving a 2-dimensional subspace Z

invariant with $Z = \langle \mathsf{u}, \omega(\mathsf{u})\rangle$ for a suitable vector $\mathsf{u} \in U$, ω fixing no point. Moreover, by the above lemma, ω^2 centralizes φ_i.

Since ω^2 satisfies conditions such as the ones satisfied by ω, the above argument says that ω^2 is k-linear, even k-scalar if $i = 2$. Thus ω^2 operates on the set $\mathcal{L} = \{l_1, \ldots, l_5\}$ of lines of the vector plane over k by stabilizing at least two of them, say l_4 and l_5, its order being 3. Of course we may assume $\varphi_i(l_4) = l_4$ and $\varphi_i(l_5) = l_5$, as well.

Assume $Z \notin \mathcal{L}$, then ω^2 cannot be k-scalar, hence $i = 3$. Up to an arrangement of indices, we may put $\mathsf{u} \in l_1$, $\omega^2(\mathsf{u}) \in l_2$ and $\omega^4(\mathsf{u}) = \omega(\mathsf{u}) \in l_3$. There are just three subspaces of dimension 3 over $\mathbf{GF}(2)$ meeting at Z we can indicate as

$$L_k := Z + l_{\mathsf{k}} \quad (k = 1, 2, 3).$$

Thus $\omega^2(L_k) = L_{k+1}$ and $\omega(L_k) = L_{k-1}$ (mod 3). Furthermore ω moves the pair of points of $l_k \setminus Z$ to the pair of points of L_{k-1} lying neither in Z nor in l_{k-1} (otherwise $\omega^2(l_k) = l_{k+1}$, $\omega^2(l_4) = l_4$ and $\omega^2(l_5) = l_5$ would force ω to operate as a permutation group of $\mathcal{L}$, hence to centralize the k-scalar mapping φ_3). As a consequence $\varphi_3(\mathsf{u}) \in (l_1 \setminus Z)$ gives $\omega\varphi_3(\mathsf{u}) \in L_3 \setminus (l_3 \cup Z) \subset l_4 \cup l_5$, hence

$$\varphi_3\omega\varphi_3(\mathsf{u}) \in (L_1 \setminus (l_1 \cup Z)) \cup (L_2 \setminus (l_2 \cup Z)) \Longrightarrow (\omega\varphi_3)^2(\mathsf{u}) \in (l_3 \cup l_1) \setminus Z.$$

Thus we infer $(\omega\varphi_3)^3(\mathsf{u}) \neq \mathsf{u}$, but $(\omega\varphi_3)^3 = 1$, so we have $Z \in \mathcal{L}$.

Let $i = 2$. The fact that ω centralizes the k-scalar mapping ω^2 forces ω to be k-linear. Assume the k-line Z fixed by ω is none of the lines l_4 and l_5 stabilized by φ_2, then we may assume $\omega(l_1) = l_1 = \varphi_2(l_2) = \varphi_2^2(l_3)$. This reduces matters to discuss two cases: either $\omega^2(l_2) = \omega(l_3) = l_2$ and $\omega^2(l_4) = \omega(l_5) = l_4$, or $\omega^2(l_2) = \omega(l_4) = l_2$ and $\omega^2(l_3) = \omega(l_5) = l_3$. But both possibilities lead to $(\omega\varphi_2)^3 \neq 1$. So we may represent ω by a matrix of the shape

$$\begin{pmatrix} E^h & X \\ O & E^h \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} E^h & O \\ X & E^h \end{pmatrix}$$

with $h = 1$, or $h = 2$ because ω acts fixed-point freely. Then X centralizes E and we may take $X = E^h$ in view of Remark 1 and Remark 2.

Let $i = 3$. If ω^2 were k-scalar ω should be k-linear, but we are assuming that ω does not centralize φ_3. So we can take P^{33} in (10) to represent ω^2 and, consequently,

$$\begin{pmatrix} E & J \\ O & E^2 \end{pmatrix}, \quad \text{or} \quad \begin{pmatrix} E & O \\ J & E^2 \end{pmatrix},$$

with J satisfying $EJ = JE^2$, to represent ω. Furthermore, up to conjugation by an element in $\mathbf{GL}(2, \mathsf{k})$, we may take

$$J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Summing up, in case ω does not centralize φ_i we obtain the representations

$$\begin{cases} P^{23} = \begin{pmatrix} E & E \\ O & E \end{pmatrix}, \; P^{24} = \begin{pmatrix} E^2 & E^2 \\ O & E^2 \end{pmatrix}, & \\ \qquad P^{25} = \begin{pmatrix} E & O \\ E & E \end{pmatrix}, \; P^{26} = \begin{pmatrix} E^2 & O \\ E^2 & E^2 \end{pmatrix}, & \text{if } i = 2; \\ P^{34} = \begin{pmatrix} E & J \\ O & E^2 \end{pmatrix}, \; P^{35} = \begin{pmatrix} E & O \\ J & E^2 \end{pmatrix}, & \text{if } i = 3. \end{cases} \tag{11}$$

Hence there are at most twelve non-equivalent instances for the inertia subgroup of a subgroup G_2^{ij} of $\mathcal{G}_2$ fulfilling conditions such as $1-3$ in §1. Each of them corresponds to a subspace, we shall denote by $\mathbf{W}_2^{ij}$, defined as specified in Proposition 1 through $\omega = \varphi_{P^{ij}}$ with P^{ij} given by (10) and (11). Looking at Remarks 1 and 2 we see that actually no two different such instances yield equivalent permutation groups and by Proposition 3 we have

$$G_2^{ij} = \left\langle g_{\gamma_i, 0_V}, g_{\delta, \mathbf{u}}, g_{1, \mathbf{w}} : \mathbf{w} \in \mathbf{W}_2^{ij} \right\rangle,$$

with $\mathbf{u} = (\mathsf{u}, \mathsf{u}, 0_U, 0_U)$ and $\mathsf{u} \in \ker(\varphi_i + \omega + \varphi_i^2 \omega^{-1} \psi_i)$. Moreover G_2^{ij} splits over $\mathbf{W}_2^{ij}$ if there exists $\mathsf{x} \in U$ such that $\mathsf{u} = (\mathrm{id}_U + (\varphi_i + \mathrm{id}_U)(\varphi_i \omega + \omega^{-1} \varphi_i))(\mathsf{x})$: this confines non-splitting extensions to the following cases (in terms of k-coordinates)

$$\mathsf{u} = \begin{cases} (X, Y), \quad Y \neq O, & \text{with } (i, j) = (2, 1), (2, 5); \\ (X, Y) \neq (O, O), & \text{with } (i, j) = (3, 1); \\ (O, Y), \quad Y \neq O, & \text{with } (i, j) = (3, 3); \\ (X, JX), X \neq O, & \text{with } (i, j) = (3, 5). \end{cases}$$

Thus, up to transforming by an automorphism of U centralizing both φ_i and ω, we can reduce matters to the cases

$$\mathsf{u} = \begin{cases} (O, I), & \text{with } (i, j) = (2, 1), (2, 5), (3, 1), (3, 3); \\ (I, I), & \text{with } (i, j) = (2, 1), (2, 5), (3, 1); \\ (E, I), & \text{with } (i, j) = (2, 5), (3, 5); \\ (E^2, I), & \text{with } (i, j) = (2, 5); \\ (I, O), & \text{with } (i, j) = (3, 1). \end{cases} \tag{12}$$

It is a straightforward calculation to verify that none of the above eleven extensions splits over the inertia subgroup. On the other hand Proposition 2 guaranties that actually no two of them are equivalent as permutation groups.

5 ($m = 1$ or $m = 3$)**.** As in this case the order of ω is at most 2 (Proposition 1), every representation of ω has the shape

$$\begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}$$

with 2-dimensional matrices M_{ij} such that

$$\begin{aligned} &a)\ M_{11}^2 + M_{12} M_{21} = I, \qquad &&b)\ M_{11} M_{12} = M_{12} M_{22}, \\ &c)\ M_{22}^2 + M_{21} M_{12} = I, \qquad &&d)\ M_{21} M_{11} = M_{22} M_{21}. \end{aligned} \tag{13}$$

Let $i = 1$. Then $(\omega\varphi_1)^3 = 1$ and $\varphi_1 = 1$ force $\omega = 1$.

Let $i = 2$. Using k-coordinates we have that φ_2 fixes each point of the k-line $L = \{(X, O) : X \in \mathsf{k}\}$. So from $\omega\varphi_2^2\omega = \varphi_2\omega\varphi_2$ we infer that $\omega\varphi_2$ leaves the subspace $\varphi_2\omega(L) = \{(M_{11}X, EM_{21}X) : X \in \mathsf{k}\}$ point-wise fixed and this leads to

$$a)\ M_{11}^2 + M_{12}E^2M_{21} = M_{11}; \qquad b)\ M_{21}M_{11} + M_{22}E^2M_{21} = EM_{21}. \tag{14}$$

Thus (13.a) and (14.a) give

$$M_{11} = I + M_{12}EM_{21}, \tag{15}$$

whereas (13.d) and (14.b) yield

$$(M_{22} + I)EM_{21} = O. \tag{16}$$

Now we distinguish three cases according to $\mathsf{rank}M_{21}$.
Let $M_{21} = O$. Then $M_{11} = I$ by (15) and $M_{22}^2 = (M_{22}E)^3 = I$ force $M_{22} = I$. Thus, in view of Remark 1 and Remark 2, we may take $M_{12} = O$, $I + J$, or I according as whether $\mathsf{rank}M_{12} = 0, 1,$ or 2.
Let $\mathsf{rank}M_{21} = 1$, then we may assume $M_{21} = I + J$. Conditions (15) and (13.a) imply $(M_{12}EM_{21})^2 = M_{12}M_{21}$ and we find

$$M_{12} = \begin{pmatrix} a & ab \\ b & ab \end{pmatrix}.$$

On the other hand (13.d) gives

$$M_{11} = \begin{pmatrix} x & y \\ x+1 & y+1 \end{pmatrix} \quad \text{and} \quad M_{22} = \begin{pmatrix} z & z+1 \\ t & t+1 \end{pmatrix}$$

with $x \neq y$ and $z \neq t$, ω being invertible. So $M_{22} = I$ by (16) and $M_{12}(I + J) = O$ by (13.c), which means $a(b + 1) = b(a + 1) = 0$, i.e. $M_{12} = O$, or $M_{12} = I + J$. Using (15) we see that the first case leads to $M_{11} = I$, the latter to $M_{11} = J$.
Let $\mathsf{rank}M_{21} = 2$, then we may take $M_{21} = I$ and consequently $M_{22} = M_{11} = I$ thanks to (16) and (14.b), and $M_{12} = O$ by (15).

Let $i = 3$. We have

Lemma. $\omega\varphi_3 + \varphi_3\omega = \omega + 1$.

Proof. The given conditions on φ_3 and ω force $\rho := \left(\varphi_3\omega\varphi_3^2 + \varphi_3^2\omega\varphi_3\right)^2$ to be the zero map. On the other hand, using $\varphi_3^2 = \varphi_3 + 1$, we find

$$\rho = (\varphi_3\omega + \omega\varphi_3)^2 = \omega\varphi_3^2 + \varphi_3^2 + \varphi_3\omega\varphi_3 + \varphi_3^2\omega = \omega\varphi_3 + \varphi_3^2 + \varphi_3\omega\varphi_3 + \varphi_3\omega.$$

Therefore $\omega\varphi_3 + \varphi_3\omega = \varphi_3^2 + \varphi_3\omega\varphi_3$, which in turn gives the claimed identity, provided we multiply both sides by φ_3^2. $\square$

As $\omega^2 = 1$ there is a subspace L of U such that ω induces the identity both on L and on U/L. If $\omega \neq 1$ such a subspace is unique and the above lemma says that L must be a k-line. So, up to an element in $\mathbf{GL}_2(\mathsf{k})$, we may assume that $L = \{(X,0) : X \in \mathsf{k}\}$ which means $M_{11} = M_{22} = I$ and $M_{21} = O$. Then $M_{12} \in \{J, JE, JE^2\}$ and we may take $M_{12} = J$. Summing up we may represent ω by one of the following

$$\begin{cases} R^{11} = \begin{pmatrix} I & O \\ O & I \end{pmatrix}, & \text{if } i = 1; \\ R^{21} = \begin{pmatrix} I & O \\ O & I \end{pmatrix},\; R^{22} = \begin{pmatrix} I & I+J \\ O & I \end{pmatrix},\quad R^{23} = \begin{pmatrix} I & O \\ I+J & I \end{pmatrix}, & \\ R^{24} = \begin{pmatrix} I & I \\ O & I \end{pmatrix},\; R^{25} = \begin{pmatrix} J & I+J \\ I+J & I \end{pmatrix},\; R^{26} = \begin{pmatrix} I & O \\ I & I \end{pmatrix}, & \text{if } i = 2; \\ R^{31} = \begin{pmatrix} I & O \\ O & I \end{pmatrix},\quad R^{32} = \begin{pmatrix} I & J \\ O & I \end{pmatrix}, & \text{if } i = 3. \end{cases} \tag{17}$$

Thus Remarks 1 and 2 guarantee that there are at least nine non-isomorphic subgroups G_1^{ij} of $\mathcal{G}_1$ and nine non-isomorphic subgroups G_3^{ij} of $\mathcal{G}_3$ fulfilling the conditions required in §1. Using analogous notation such as in the previous paragraph, by Proposition 3 we have

$$G_m^{ij} = \left\langle g_{\gamma_i, 0_V}, g_{\delta, \mathbf{u}}, g_{1,\mathbf{w}} : \mathbf{w} \in \mathbf{W}_2^{ij} \right\rangle,$$

with $\mathbf{u} = (\mathsf{u}_1, \mathsf{u}_2, \mathsf{u}_3, 0_U)$ and

$$\begin{array}{ll} \mathsf{u}_1 + \mathsf{u}_2 = \varphi_i \omega \varphi_i^2(\mathsf{u}_3) \in \ker(\omega + \mathrm{id}_U),\ (\varphi_i^2 + \mathrm{id}_U)(\mathsf{u}_1) = (\varphi_i + \omega)(\mathsf{u}_2), & \text{if } m = 1; \\ \mathsf{u}_1 \in \ker(\varphi_i^2 \omega + \omega \varphi_i^2),\ \psi_i(\mathsf{u}_1) = \mathsf{u}_2 = \mathsf{u}_3 = 0_U, & \text{if } m = 3. \end{array}$$

Moreover G_m^{ij} splits over $\mathbf{W}_m^{ij}$ if there exist $\mathsf{x} \in U$ such that

$$\begin{array}{ll} \mathsf{u}_1 + \mathsf{u}_2 = (\omega + \mathrm{id}_U)(\mathsf{x}),\ \psi_i(\mathsf{u}_1) = \psi_i(\mathsf{x}), & \text{if } m = 1; \\ \mathsf{u}_1 = \left(\omega + \varphi_i + \varphi_i^2 + \omega\varphi_i\omega + \varphi_i^2\omega + \varphi_i\omega\varphi_i^2\right)(\mathsf{x}) & \text{if } m = 3. \end{array}$$

This confines non-splitting extensions to the following cases, where we put $P = I + J$ and, using k-coordinates, $\mathsf{u}_h = (U_h', U_h'')$, $h = 1, 2$,

$$m = 1 : \begin{cases} (i,j) = (1,1), & \text{if } \mathsf{u}_1 \neq \mathsf{u}_2; \\ (i,j) = (2,1) \text{ with } U_2'' = E^2 U_1'', & \text{if } \mathsf{u}_1 \neq \mathsf{u}_2; \\ (i,j) = (2,2) \text{ with } U_1'' = U_2'' = O, & \text{if } PU_1' \neq PU_2'; \\ (i,j) = (2,3) \text{ with } PU_1' = PU_2' = EU_1'' + E^2 U_2'', & \text{if } U_1' \neq U_2', \text{ or } U_2'' \neq EU_1''; \\ (i,j) = (2,6) \text{ with } U_1' = U_2' = EU_1'' + E^2 U_2'', & \text{if } U_2'' \neq EU_1''; \\ (i,j) = (3,1) \text{ with } \mathsf{u}_2 = E^2 \mathsf{u}_1, & \text{if } \mathsf{u}_2 \neq 0_U; \end{cases}$$

$$m = 3 : \begin{cases} (i,j) = (2,1) \text{ with } \mathsf{u}_1 = (O,I), (O,E), \text{ or } (O,E^2); \\ (i,j) = (2,2) \text{ with } \mathsf{u}_1 = (O,E); \\ (i,j) = (2,3) \text{ with } \mathsf{u}_1 = (O,I), \text{ or } (O,E^2); \\ (i,j) = (3,1) \text{ with } \mathsf{u}_1 \neq 0_U; \\ (i,j) = (3,2) \text{ with } \mathsf{u}_1 = (I,O), (E,O), \text{ or } (E^2,O). \end{cases}$$

Notice that it is always $\mathbf{u} = (\mathsf{u}_1, \mathsf{u}_2, \mathsf{u}_1 + \mathsf{u}_2, 0_U)$ in case $m = 1$. Finally, up to transforming by an automorphism of U centralizing both φ_i and ω, matters can be reduced to the cases

$$m=1:\begin{cases}
\mathsf{u}_1=(O,E), & \mathsf{u}_2=(O,I), \quad \text{with } (i,j)=(2,1),(2,3),(2,6),(3,1);\\
\mathsf{u}_1=(O,O), & \mathsf{u}_2=(I,O), \quad \text{with } (i,j)=(1,1),(2,2);\\
\mathsf{u}_1=(O,O), & \mathsf{u}_2=(E^2,O), \quad \text{with } (i,j)=(2,1),(2,3);\\
\mathsf{u}_1=(O,I), & \mathsf{u}_2=(E^2,E^2), \quad \text{with } (i,j)=(2,1),(2,3);\\
\mathsf{u}_1=(I,O), & \mathsf{u}_2=(O,O), \quad \text{with } (i,j)=(1,1),(2,2);\\
\mathsf{u}_1=(I,O), & \mathsf{u}_2=(E,O), \quad \text{with } (i,j)=(1,1),(2,1);\\
\mathsf{u}_1=(I,O), & \mathsf{u}_2=(E^2,O), \quad \text{with } (i,j)=(2,2),(3,1);\\
\mathsf{u}_1=(E,E), & \mathsf{u}_2=(I,I), \quad \text{with } (i,j)=(2,1),(3,1);\\
\mathsf{u}_1=(E^2,O), & \mathsf{u}_2=(O,O), \quad \text{with } (i,j)=(2,1),(2,3);\\
\mathsf{u}_1=(E^2,E^2), & \mathsf{u}_2=(O,E), \quad \text{with } (i,j)=(2,1),(2,3);\\
\mathsf{u}_1=(E^2,E^2), & \mathsf{u}_2=(E^2,E), \quad \text{with } (i,j)=(2,1),(2,3);\\
\mathsf{u}_1=(O,I), & \mathsf{u}_2=(O,E^2), \quad \text{with } (i,j)=(2,3);\\
\mathsf{u}_1=(O,E), & \mathsf{u}_2=(E^2,I), \quad \text{with } (i,j)=(2,3);\\
\mathsf{u}_1=(O,E^2), & \mathsf{u}_2=(O,E), \quad \text{with } (i,j)=(2,3);\\
\mathsf{u}_1=(O,E^2), & \mathsf{u}_2=(E^2,E), \quad \text{with } (i,j)=(2,3);\\
\mathsf{u}_1=(I,O), & \mathsf{u}_2=(I,I), \quad \text{with } (i,j)=(2,3);\\
\mathsf{u}_1=(I,O), & \mathsf{u}_2=(I,E), \quad \text{with } (i,j)=(2,6);\\
\mathsf{u}_1=(E,O), & \mathsf{u}_2=(I,I), \quad \text{with } (i,j)=(2,3);\\
\mathsf{u}_1=(E^2,O), & \mathsf{u}_2=(I,O), \quad \text{with } (i,j)=(2,2);\\
\mathsf{u}_1=(E^2,E), & \mathsf{u}_2=(O,I), \quad \text{with } (i,j)=(2,3);\\
\mathsf{u}_1=(E^2,I), & \mathsf{u}_2=(O,E^2), \quad \text{with } (i,j)=(2,3);\\
\mathsf{u}_1=(E,E), & \mathsf{u}_2=(I,O), \quad \text{with } (i,j)=(2,3);\\
\mathsf{u}_1=(E,I), & \mathsf{u}_2=(E,O), \quad \text{with } (i,j)=(2,6);\\
\mathsf{u}_1=(E,E), & \mathsf{u}_2=(E,O), \quad \text{with } (i,j)=(2,3);\\
\mathsf{u}_1=(I,I), & \mathsf{u}_2=(I,I), \quad \text{with } (i,j)=(2,6);\\
\mathsf{u}_1=(I,I), & \mathsf{u}_2=(E,E), \quad \text{with } (i,j)=(2,3);\\
\mathsf{u}_1=(I,E^2), & \mathsf{u}_2=(I,E^2), \quad \text{with } (i,j)=(2,3);\\
\mathsf{u}_1=(I,E^2), & \mathsf{u}_2=(E,E^2), \quad \text{with } (i,j)=(2,3);\\
\mathsf{u}_1=(E^2,I), & \mathsf{u}_2=(E^2,E^2), \quad \text{with } (i,j)=(2,3);\\
\mathsf{u}_1=(E^2,E), & \mathsf{u}_2=(E^2,I), \quad \text{with } (i,j)=(2,3).
\end{cases} \tag{18}$$

$$m=3:\begin{cases}
\mathsf{u}_1=(I,O), & \text{with } (i,j)=(3,1),(3,2);\\
\mathsf{u}_1=(O,I), & \text{with } (i,j)=(2,1),(2,3);\\
\mathsf{u}_1=(O,E), & \text{with } (i,j)=(2,2);\\
\mathsf{u}_1=(O,E^2), & \text{with } (i,j)=(2,3);
\end{cases}$$

and it is a straightforward calculation to verify that none of the above extensions splits over the inertia group. On the other hand Proposition 2 guaranties that actually no two of them are equivalent as permutation groups.

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