The overlap algebra of regular opens

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Abstract

Overlap algebras are complete lattices enriched with an extra primitive relation, called "overlap". The new notion of overlap relation satisfies a set of axioms intended to capture, in a positive way, the properties which hold of two elements with non-zero infimum. For each set, its powerset is an example of overlap algebra where two subsets overlap each other when their intersection is inhabited. Moreover, atomic overlap algebras are naturally isomorphic to the powerset of the set of their atoms. Overlap algebras can be seen as particular open (or overt) locales and, from a classical point of view, they essentially coincide with complete Boolean algebras. Contrary to the latter, overlap algebras offers a negation-free framework suitable, among other things, for the development of point-free topology. A lot of topology can be done "inside" the language of overlap algebra. In particular, we prove that the collection of all regular open subsets of a topological space is an example of overlap algebra which, under natural hypotheses, is atomless. Since they are a constructive counterpart to complete Boolean algebras and, at the same time, they have a more powerful axiomatization than Heyting algebras, overlap algebras are expected to turn out useful both in constructive mathematics and for applications in computer science.

1 Introduction

The notion of overlap algebra, which has been recently introduced in [6] by the second author, is an algebraic version of the structure $(\mathcal{P}(S), \subseteq, \between)$ on the powerset of S in which also the notion of "overlap" is axiomatized: two subsets U and V of a set S overlap each other, written $U \between V$, if their intersection is inhabited. Hence, the notion of overlap is a primitive and positive counterpart to what is usually referred to as "non-empty intersection". Like the symbol $\leq$ in order theory calligraphically resembles its corresponding $\subseteq$ in set theory, we write $\gtrless$ as an algebraic version of $\between$. The new primitive $\gtrless$ increases the expressive power of the language of lattices and allows the development, for instance, of a lot of topology in fully algebraic terms and in a positive way (no negation or complement needed). All the results obtained this way are independent from foundations, in the sense that they also hold in a constructive (that is, intuitionistic and predicative) framework.

It can be proved that an atomic overlap algebra is exactly the powerset of a set. The present paper shows that the notion of overlap algebra is much more general. In fact, we prove that the collection of all regular open subsets of a topological space is an overlap algebra which, in general, is atomless.

To confirm how powerful the language of overlap algebras is, we formulate and prove the above result "inside" the language of overlap algebras, that is, in a completely algebraic way. To do this, we need to replace all topological notions involved by suitable (overlap-)algebraic versions. This suggests that one can do a lot of topology in an element-free and complement-free way. Our notion

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of overlap-topology can be seen as a suitable positive version of Tarski's closure algebra (see, for instance, [4]).

In [1] and [2] the notion of quasi o-algebra was employed in the semantics of intuitionistic first-order and tense logics.

2 Basic notions and definitions

2.1 Overlap algebras

The definitions of complete Boolean and Heyting algebras are algebraic formulations of the structure $(\mathcal{P}(S), \subseteq)$ from classical and intuitionistic points of view, respectively. In the same way, the definition of overlap algebra we are now to give is an axiomatization of the structure $(\mathcal{P}(S), \subseteq, \between)$ where, for U and V subsets, $U \between V$ is $(\exists a \in S)(a \in U \cap V)$. We have selected a set of positive properties that link $\between$ with inclusion, intersection and union with no reference to complement; this fits with our general attitude of doing Mathematics in a constructive way: predicative and intuitionistic, positive and compatible with several foundations. Note that $U \between V$ is equivalent to $U \cap V \neq \emptyset$ only if classical logic is available.

Definition 2.1 *An* overlap algebra*, or* o-algebra*, is a structure* $(\mathcal{P}, \leq, \gtrless)$ *where:* $\mathcal{P}$ *is a collection, with objects* $p, q, \ldots$*;* $(\mathcal{P}, \leq)$ *is a complete lattice;* $p \gtrless q$ *is a binary relation on* $\mathcal{P}$*, which satisfies:*

- *symmetry:* $$p \gtrless q \implies q \gtrless p$$
- *preservation of infimum:* $$p \gtrless q \implies p \gtrless (p \wedge q)$$
- *splitting of supremum:* $$p \gtrless \bigvee_{i \in \mathrm{I}} q_i \iff (\exists i \in \mathrm{I})(p \gtrless q_i)$$
- *density:* $$\forall r(p \gtrless r \Longrightarrow q \gtrless r) \Longrightarrow p \leq q$$
- *properness:* $$1 \gtrless 1$$

for any set I *and any* p*,* q*,* q_i *in* $\mathcal{P}$*.*

We call quasi o-algebra *a structure satisfying all the above axioms but density.*

Clearly, for any set X, the structure $(\mathcal{P}(X), \subseteq, \between)$ on the powerset of X is an example of o-algebra and, actually, the motivating one. A natural question is whether there are examples not isomorphic to these. In [6] it is shown that atomic overlap algebras coincide with powersets up to isomorphism in any natural sense. The intuition of an atom as a minimal non-zero element can be expressed within the language of o-algebras in a very elegant way, namely: an element m of an o-algebra $\mathcal{P}$ is an atom provided that:

$$m \gtrless p \iff m \leq p$$

for every $p : \mathcal{P}$. An o-algebra $\mathcal{P}$ is atomic if each element p is the supremum of the family of all atoms m such that $m \leq p$. In this case, $\mathcal{P}$ is isomorphic to $(\mathcal{P}(S), \subseteq, \between)$, S being the set of atoms.

One of the aims of this paper is to show that there are natural examples of non atomic overlap algebras; in fact, the regular open subsets of any topological space form an overlap algebra which, in general, is non-atomic, even atomless. In a sense, this could look trivial since it is well know that regular open subsets form a complete Boolean algebra and, as it is shown by proposition 5.1, the latter is classically the same as an o-algebra. However, all that holds only within a classical framework, while our results remain true also with respect to intuitionistic and predicative foundations (see section 5 for further details).

Here we list some of the basic properties of an overlap algebra.

Proposition 2.2 *In every quasi o-algebra all the following hold:*

1. $(p \gtrless r)\ \ \&\ \ (p \leq q)\ \Longrightarrow\ (q \gtrless r)$

2. $p \gtrless r \wedge q\ \Longleftrightarrow\ p \wedge r \gtrless q$

3. $p \gtrless p\ \Longleftrightarrow\ \exists q(p \gtrless q)$

4. $p \gtrless q\ \Longleftrightarrow\ p \wedge q \gtrless p \wedge q\ \Longleftrightarrow\ (\exists r \leq p \wedge q)\ r \gtrless r$

5. $(p \wedge q = 0) \Longrightarrow \neg(p \gtrless q)$

for any p, q and r in $\mathcal{P}$. In an o-algebra also the converse to item 5 holds.

Proof (1) From $p \leq q$ one has $p \vee q = q$; on the other hand, $p \gtrless r$ yields $p \vee q \gtrless r$ ($\gtrless$ splits suprema); thus $q \gtrless r$. (2) Suppose $p \wedge r \gtrless q$; then $p \wedge r \gtrless (p \wedge r) \wedge q$ ($\gtrless$ respects infima); this yields $p \gtrless r \wedge q$ by the previous item since $p \wedge r \leq p$ and $(p \wedge r) \wedge q \leq r \wedge q$. (3) If $p \gtrless q$ for some q, then $p \gtrless p \wedge q$ ($\gtrless$ respects infima) and hence $p \gtrless p$, thanks to the first item. (4) Easy. (5) First note that $\neg(0 \gtrless 0)$ since $\gtrless$ splits suprema and 0 is the supremum of the empty family; thus $p \gtrless q$ and $p \wedge q = 0$ together easily lead to a contradiction (use the previous items).

Finally, suppose $r \gtrless p \wedge q$ for an arbitrary r; then, in particular, $p \gtrless q$; but $\neg(p \gtrless q)$ by hypothesis: a contradiction; hence (ex falso quodlibet) $r \gtrless 0$; so (by density) $p \wedge q \leq 0$. q.e.d.

We say that an element p in $\mathcal{P}$ is *inhabited* if $p \gtrless p$. It follows from the above discussion that in any o-algebra 0 is the unique non-inhabited element (so, classically, inhabited elements are precisely those different from 0). Moreover, item 4 says that $p \gtrless q$ holds if and only if $p \wedge q$ is inhabited; hence, from a classical point of view, $p \gtrless q$ becomes definable by $p \wedge q \neq 0$ (item 5 and its converse). Item 1, together with density, implies that $p \leq q$ can be thought as a defined notion since it is tantamount to $\forall r(p \gtrless r \Longrightarrow q \gtrless r)$.

Proposition 2.3 *In every o-algebra the lattice $(\mathcal{P}, \leq)$ is an open (or overt) locale, that is, the following infinite distributive law holds:*

$$p \wedge \big(\bigvee_{i \in \mathrm{I}} q_i\big) \;=\; \bigvee_{i \in \mathrm{I}} (p \wedge q_i) \tag{1}$$

and there exists a unary predicate Pos *(the* positivity predicate*), defined as* $\mathrm{Pos}(p) = p \gtrless p$*, such that:*

$$\mathrm{Pos}(1) \quad , \quad \mathrm{Pos}(p)\ \&\ \big(p \leq \bigvee_{i \in I} q_i\big) \Longrightarrow (\exists i \in I)\mathrm{Pos}(q_i) \quad \textit{and} \quad \big(\mathrm{Pos}(p) \Longrightarrow (p \leq q)\big) \Longrightarrow (p \leq q)$$

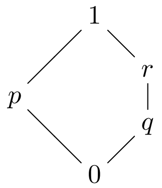
*(*I *set and p, q, q_i in $\mathcal{P}$).*

Proof The inequality $\bigvee_{i \in \mathrm{I}} (p \wedge q_i) \leq p \wedge (\bigvee_{i \in \mathrm{I}} q_i)$ holds in an arbitrary complete lattice. We prove its reverse by density. If $r \gtrless p \wedge (\bigvee_{i \in \mathrm{I}} q_i)$, then $r \wedge p \gtrless \bigvee_{i \in \mathrm{I}} q_i$; so $r \wedge p \gtrless q_i$ for some i, that is, $r \gtrless p \wedge q_i$ for some i; hence $r \gtrless \bigvee_{i \in \mathrm{I}} (p \wedge q_i)$.

By unfolding definitions, the first two conditions on Pos are easy. The other one is proved via density: if $p \gtrless r$, then $p \gtrless p$, that is, $\mathrm{Pos}(p)$; so $p \leq q$, hence $q \gtrless r$. q.e.d.

Given an open locale $(\mathcal{P}, \leq, \mathrm{Pos})$, the structure $(\mathcal{P}, \leq, \gtrless)$, where $p \gtrless q = \mathrm{Pos}(p \wedge q)$, is not an o-algebra, in general. In fact, o-algebras can be characterized as those open locales whose positivity predicate Pos satisfies $\forall r\big(\mathrm{Pos}(p \wedge r) \Longrightarrow \mathrm{Pos}(q \wedge r)\big) \Longrightarrow (p \leq q)$ (density).

Contrary to the case of o-algebras, a quasi o-algebra needs not to be distributive. As an example, let us consider the non-distributive lattice N_5:



with respect to the following overlap relation: $x \gtrless y$ if $\{x, y\} \subseteq \{p, 1\}$ (in other words, the inhabited elements are p and 1). It is easy, though boring, to verify that $\gtrless$ satisfies all the axioms of an overlap relation except for density.

We say that a quasi o-algebra $(\mathcal{P}, \leq, \gtrless)$ is *distributive* if $(\mathcal{P}, \leq)$ is a locale, that is, it satisfies the infinite distributive law (1). Thus a *distributive quasi o-algebra* can be described as a locale endowed with a predicate Pos which satisfies Pos(1) and Pos(p) & $\big(p \leq \bigvee_{i\in I} q_i\big) \Longrightarrow (\exists i \in I)\mathrm{Pos}(q_i)$. In other words, a distributive quasi o-algebra is an open locale except for not necessarily satisfying the so-called positivity axiom: $\big(\mathrm{Pos}(p) \Longrightarrow (p \leq q)\big) \Longrightarrow (p \leq q)$.

2.2 Reduction and saturation operators

Let $\mathcal{P}$ be a complete lattice and F an operator on $\mathcal{P}$. We say that F is

- *monotonic* (or *monotonically increasing*) if $\mathrm{F}\,p \leq \mathrm{F}\,q$ whenever $p \leq q$;
- *idempotent* if $\mathrm{F}\,\mathrm{F}\,p = \mathrm{F}\,p$, for any p.

The collection $Fix(\mathrm{F}) = \{\mathrm{F}\,p : p \text{ in } \mathcal{P}\}$ of all fixed points of a monotonic and idempotent operator is a complete lattice with respect to the following operations:

$$\bigvee_{i\in \mathrm{I}}^{\mathrm{F}} \mathrm{F}\,p_i \;=\; \mathrm{F}\,(\bigvee_{i\in \mathrm{I}} \mathrm{F}\,p_i) \qquad \text{and} \qquad \bigwedge_{i\in \mathrm{I}}^{\mathrm{F}} \mathrm{F}\,p_i \;=\; \mathrm{F}\,(\bigwedge_{i\in \mathrm{I}} \mathrm{F}\,p_i)\;. \tag{2}$$

In particular, the order of $Fix(\mathrm{F})$, which is defined as usual by $\mathrm{F}\,p \leq^{\mathrm{F}} \mathrm{F}\,q$ iff $\mathrm{F}\,p \wedge^{\mathrm{F}} \mathrm{F}\,q = \mathrm{F}\,p$, is that inherited from $\mathcal{P}$, that is:

$$\mathrm{F}\,p \leq^{\mathrm{F}} \mathrm{F}\,q \qquad \Longleftrightarrow \qquad \mathrm{F}\,p \leq \mathrm{F}\,q \tag{3}$$

(which we shall refer to as $\leq = \leq^{\mathrm{F}}$). To prove this, firstly note that $\mathrm{F}\,(\mathrm{F}\,p \wedge \mathrm{F}\,q) \leq \mathrm{F}\,p \wedge \mathrm{F}\,q$; for $\mathrm{F}\,p \wedge \mathrm{F}\,q \leq \mathrm{F}\,p$ yields $\mathrm{F}\,(\mathrm{F}\,p \wedge \mathrm{F}\,q) \leq \mathrm{F}\,p$ by monotonicity and idempotence (and similarly for q). So $\mathrm{F}\,p \leq^{\mathrm{F}} \mathrm{F}\,q$ iff (by definition of $\leq^{\mathrm{F}}$) $\mathrm{F}\,p = \mathrm{F}\,p \wedge^{\mathrm{F}} \mathrm{F}\,q$ iff (by definition of $\wedge^{\mathrm{F}}$) $\mathrm{F}\,p = \mathrm{F}\,(\mathrm{F}\,p \wedge \mathrm{F}\,q)$ iff (by the previous discussion) $\mathrm{F}\,p \leq \mathrm{F}\,(\mathrm{F}\,p \wedge \mathrm{F}\,q)$ iff (by the previous discussion, monotonicity and idempotence) $\mathrm{F}\,p \leq \mathrm{F}\,p \wedge \mathrm{F}\,q$ iff $\mathrm{F}\,p \leq \mathrm{F}\,q$.

Definition 2.4 *A monotonic and idempotent operator* F *is a*

- saturation[1] *if* $p \leq \mathrm{F}\,p$, *for any* p *(*F *is* expansive*);*
- reduction *if* $\mathrm{F}\,p \leq p$, *for any* p *(*F *is* reductive*).*

It is not hard to show that an operator C is a saturation if and only if it satisfies the identity $\mathrm{C}\,p \leq \mathrm{C}\,q \Longleftrightarrow p \leq \mathrm{C}\,q$; similarly, an operator I is a reduction if and only if $\mathrm{I}\,p \leq \mathrm{I}\,q \Longleftrightarrow \mathrm{I}\,p \leq q$. Note that $\mathrm{C}\,1 = 1$ and $\mathrm{I}\,0 = 0$ but $\mathrm{C}\,0 \neq 0$ and $\mathrm{I}\,1 \neq 1$, in general. Provided that C is a saturation and I a reduction (as always in this paper), equations (2) simplify to:

$$\bigvee_{i\in \mathrm{I}}^{\mathrm{C}} \mathrm{C}\,p_i \;=\; \mathrm{C}\,(\bigvee p_i) \qquad \text{and} \qquad \bigwedge_{i\in \mathrm{I}}^{\mathrm{C}} \mathrm{C}\,p_i \;=\; \bigwedge_{i\in \mathrm{I}} \mathrm{C}\,p_i$$

$$\bigvee_{i\in \mathrm{I}}^{\mathrm{I}} \mathrm{I}\,p_i \;=\; \bigvee \mathrm{I}\,p_i \qquad \text{and} \qquad \bigwedge_{i\in \mathrm{I}}^{\mathrm{I}} \mathrm{I}\,p_i \;=\; \mathrm{I}\,(\bigwedge_{i\in \mathrm{I}} p_i)$$

respectively.

Definition 2.5 *Let* $\mathcal{P}$ *be a complete lattice and let* C *and* I *be a saturation and a reduction, respectively, on* $\mathcal{P}$. *We say that* p *in* $\mathcal{P}$ *is* regular *if* $p = \mathrm{I}\,\mathrm{C}\,p$. *We write* $Reg(\mathcal{P})$ *for the collection of all regular elements of* $\mathcal{P}$.

[1] A *closure* operator in the usual sense (see [4]) is a saturation which, in addition, preserves finite joins.

A regular subset D of a topological space X is one which satisfies: $D = \mathsf{int}\,\mathsf{cl}\,D$ (in particular, D is open), int and cl being the interior and closure operator of X, respectively. This shows that our definition of *regular* is nothing else than a natural generalization of the standard notion to the case of arbitrary closure and reduction operators. From now on, we set

$$\mathrm{R} \quad = \quad \mathrm{I}\,\mathrm{C}\,\mathrm{I} \tag{4}$$

which is clearly a monotonic operator.

Lemma 2.6 *For any two operators* C, I *of saturation and reduction, respectively, on the same complete lattice* $\mathcal{P}$, *the following hold:*

1. $\mathrm{I} \leq \mathrm{I}\,\mathrm{C}\,\mathrm{I}$ *and* $\mathrm{C}\,\mathrm{I}\,\mathrm{C} \leq \mathrm{C}$ *;*
2. $\mathrm{I}\,\mathrm{C}\,\mathrm{I}\,\mathrm{C} = \mathrm{I}\,\mathrm{C}$ *and* $\mathrm{C}\,\mathrm{I}\,\mathrm{C}\,\mathrm{I} = \mathrm{C}\,\mathrm{I}$ *;*
3. $\mathrm{I}\,\mathrm{C}$ *is a saturation on* $Fix(\mathrm{I})$ *and* $\mathrm{C}\,\mathrm{I}$ *is a reduction on* $Fix(\mathrm{C})$ *.*

Proof (1) From $(\mathrm{I}\,\mathrm{p}) \leq \mathrm{C}\,(\mathrm{I}\,\mathrm{p})$ (C expansive) one gets $\mathrm{I}\,p \leq \mathrm{I}\,(\mathrm{C}\,\mathrm{I}\,p)$ (I is a reduction). The second part is proved dually. (2) $(\mathrm{I})\mathrm{C} \leq (\mathrm{I}\,\mathrm{C}\,\mathrm{I})\mathrm{C}$ by item 1, first part; $\mathrm{I}\,(\mathrm{C}\,\mathrm{I}\,\mathrm{C}) \leq \mathrm{I}\,(\mathrm{C})$ by item 1, second part (I is monotonic). Dually for the second part. (3) $\mathrm{I}\,\mathrm{C}$ is monotonic because composition of monotonic operators; it is idempotent by item 2, first part; finally it is expansive on $Fix(\mathrm{I})$ by item 1, first part. The second part is dual. q.e.d.

In particular, R is also idempotent: $\mathrm{R}\,\mathrm{R} = \mathrm{I}\mathrm{C}\mathrm{I}\mathrm{I}\mathrm{C}\mathrm{I} = \mathrm{I}\mathrm{C}\mathrm{I}\mathrm{C}\mathrm{I} = \mathrm{I}\mathrm{C}\mathrm{I} = \mathrm{R}$. So we can consider the collection $Fix(\mathrm{R})$ of all its fixed points.

Proposition 2.7 *Let* C *and* I *be a saturation and a reduction, respectively, on a complete lattice* $\mathcal{P}$*; then all the following hold:*

1. *Reg*$(\mathcal{P}) = Fix(\mathrm{I}\,\mathrm{C})$ *is a sub-collection of* $Fix(\mathrm{I})$ *(every regular element is open);*
2. p *is regular if and only if* $p = \mathrm{I}\,\mathrm{C}\,q$ *for some* q*;*
3. p *is regular if and only if* $p = \mathrm{R}\,p$ *(that is, Reg*$(\mathcal{P}) = Fix(\mathrm{R})$*);*
4. p *is regular if and only if* $p = \mathrm{R}\,q$ *for some* q.

Proof (1) Let $p = \mathrm{I}\,\mathrm{C}\,p$; then $\mathrm{I}\,p = \mathrm{I}\,\mathrm{I}\,\mathrm{C}\,p = \mathrm{I}\,\mathrm{C}\,p$ (because I is idempotent) $= p$; that is, p is I-fixed. (2) If p is regular, then it is enough to take $p = q$; vice versa, if $p = \mathrm{I}\,\mathrm{C}\,q$, then $\mathrm{I}\,\mathrm{C}\,p = \mathrm{I}\,\mathrm{C}\,\mathrm{I}\,\mathrm{C}\,q = \mathrm{I}\,\mathrm{C}\,q$ (because $\mathrm{I}\,\mathrm{C}$ is idempotent) $= p$. (3) If $p = \mathrm{I}\,\mathrm{C}\,p$, then $\mathrm{R}\,p = \mathrm{I}\,\mathrm{C}\,\mathrm{I}\,\mathrm{I}\,\mathrm{C}\,p = \mathrm{I}\,\mathrm{C}\,\mathrm{I}\,\mathrm{C}\,p = \mathrm{I}\,\mathrm{C}\,p = p$. Vice versa, if $p = \mathrm{I}\,\mathrm{C}\,(\mathrm{I}\,p)$, then p is regular by item 2. (4) If p is regular, then $p = \mathrm{I}\,\mathrm{C}\,p = \mathrm{I}\,\mathrm{C}\,\mathrm{I}\,\mathrm{C}\,p = \mathrm{R}\,(\mathrm{C}\,p)$; vice versa, if $p = \mathrm{R}\,q$, then $p = \mathrm{I}\,\mathrm{C}\,(\mathrm{I}\,q)$ and p is regular by item 2. q.e.d.

The identity $\mathrm{R}\,p = \mathrm{I}\,\mathrm{C}\,(\mathrm{I}\,p)$ makes it evident that the regular elements are exactly the $\mathrm{I}\,\mathrm{C}$-fixed elements over $Fix(\mathrm{I})$. That $\mathrm{I}\,\mathrm{C}$ is a saturation over the I-fixed elements (Lemma 2.6) means that:

$$\mathrm{R}\,p \leq \mathrm{R}\,q \quad \Longleftrightarrow \quad \mathrm{I}\,p \leq \mathrm{R}\,q \quad (\Longleftrightarrow \quad \mathrm{I}\,p \leq \mathrm{C}\,\mathrm{I}\,q \quad \Longleftrightarrow \quad \mathrm{C}\,\mathrm{I}\,p \leq \mathrm{C}\,\mathrm{I}\,q) \tag{5}$$

(the last two equivalences following from the properties of I and C, respectively); thus $\mathrm{R}\,p$ *is the least regular element greater than* $\mathrm{I}\,p$.

Besides those mentioned above, R satisfies also some simple but useful derived properties, such as: $\mathrm{R} = \mathrm{R}\,\mathrm{I} = \mathrm{I}\,\mathrm{R} = \mathrm{I}\,\mathrm{C}\,\mathrm{R} = \mathrm{R}\,\mathrm{C}\,\mathrm{I}$, $\mathrm{R}\,\mathrm{C} = \mathrm{I}\,\mathrm{C}$ and $\mathrm{C}\,\mathrm{R} = \mathrm{C}\,\mathrm{I}$.

2.3 Overlap topologies

The following definition introduces the notion of overlap topology; this can be seen as a positive analogue of a closure algebra (complete Boolean algebra endowed with a closure operator). We assume both a saturation and a reduction as primitive notions. Actually, as it is shown by proposition 5.2, the saturation is definable from the reduction, although in an impredicative way. However, the converse (that the reduction is definable from the saturation) does not hold since complementation is not available.

Definition 2.8 *An* overlap topology *(or,* o-topology*) is a triple* $(\mathcal{P}, \mathrm{C}, \mathrm{I})$*, where* $\mathcal{P}$ *is a distributive quasi o-algebra,* C *is a saturation on* $\mathcal{P}$ *and* I *is a reduction on* $\mathcal{P}$ *which satisfy:*

- *compatibility:* $\qquad \mathrm{I}\,p \between \mathrm{C}\,q \implies \mathrm{I}\,p \between q$
- $\wedge = \wedge^{\mathrm{I}}$*:* $\qquad \mathrm{I}\,p \wedge \mathrm{I}\,q \;\leq\; \mathrm{I}\,(p \wedge q)$
- $\mathrm{C} - \mathrm{I}$ *density:* $\qquad \forall r(p \between \mathrm{I}\,r \implies q \between \mathrm{I}\,r) \implies p \leq \mathrm{C}\,q$
- *properness:* $\qquad \mathrm{I}\,1 \between \mathrm{I}\,1$

for any p *and* q *in* $\mathcal{P}$*.*

Note that we do not require $\mathcal{P}$ to satisfy density, since none of the results presented below needs it. This definition generalizes the notion of topological space; indeed, if X is a topological space, and int and cl are the standard interior and closure operators on $\mathcal{P}(X)$, then $(\mathcal{P}(X), \mathsf{cl}, \mathsf{int})$ is an o-topology. To see why compatibility holds, let x be a point in $\mathsf{int}\,D \cap \mathsf{cl}\,E$ $(D, E \subseteq X)$; thus $\mathsf{int}\,D$ is an open neighbourhood of x, hence $\mathsf{int}\,D \between E$ since $x \in \mathsf{cl}\,E$. Similarly for $\mathrm{C} - \mathrm{I}$ density: assume $F \between \mathsf{int}\,D \Longrightarrow E \between \mathsf{int}\,D$ for any D; take $x \in F$ and let $\mathsf{int}\,D$ be an open neighbourhood of x; so $F \between \mathsf{int}\,D$, hence $E \between \mathsf{int}\,D$; thus $x \in \mathsf{cl}\,E$. Note that the usual condition $\mathsf{cl} = -\,\mathsf{int}\,-$ is not assumed here. Actually, in a classical foundation, compatibility and $\mathrm{C} - \mathrm{I}$ density together become equivalent to $\mathrm{C} = -\mathrm{I}-$ (see proposition 5.3). Of course, hypothesis $\wedge = \wedge^{\mathrm{I}}$ holds because the intersection of two open subsets is open. Finally, properness is a positive way to express that $\mathrm{I}\,1 \neq 0$ $(\mathsf{int}\,X \neq \emptyset)$.

In any o-topology, $\mathrm{I}\,p \wedge \mathrm{I}\,q$ is in fact equal to $\mathrm{I}\,(p \wedge q)$ because $\mathrm{I}\,(p \wedge q) \leq \mathrm{I}\,p \wedge \mathrm{I}\,q$ follows from the monotonicity of I. The converse of "compatibility" also holds (because $q \leq \mathrm{C}\,q$). Finally, the assertion $p \leq \mathrm{C}\,q$ is in fact equivalent to $\forall r(p \between \mathrm{I}\,r \Longrightarrow q \between \mathrm{I}\,r)$: if $p \leq \mathrm{C}\,q$ and $p \between \mathrm{I}\,r$, then $\mathrm{C}\,q \between \mathrm{I}\,r$ and $q \between \mathrm{I}\,r$ by compatibility.

It is possible to prove the algebraic version of the well known fact that the open subsets of a topological space form a locale (or complete Heyting algebra). As $Fix(\mathrm{I})$ is a complete lattice, we only need to show that $Fix(\mathrm{I})$ satisfies the infinite distributive law (1): $\mathrm{I}\,p \wedge^{\mathrm{I}} (\bigvee^{\mathrm{I}} \mathrm{I}\,q_i) =$ (unfolding definitions and using $\wedge = \wedge^{\mathrm{I}}$) $\mathrm{I}\,p \wedge (\bigvee \mathrm{I}\,q_i) =$ (by distributivity of $\mathcal{P}$) $\bigvee(\mathrm{I}\,p \wedge \mathrm{I}\,q_i) = \bigvee(\mathrm{I}\,p \wedge^{\mathrm{I}} \mathrm{I}\,q_i)$ $= \bigvee^{\mathrm{I}} (\mathrm{I}\,p \wedge^{\mathrm{I}} \mathrm{I}\,q_i)$.

3 The overlap algebra of regular opens

We are now going to prove our main theorem which states that the regular elements of an o-topology form an overlap algebra. In view of proposition 5.1, this can be seen as a positive, constructive version of the classical fact (particularly important for constructing Boolean-valued models) that the regular open subsets of a topological space form a complete Boolean algebra.

Theorem 3.1 *Let* $(\mathcal{P}, \mathrm{C}, \mathrm{I})$ *be an o-topology over* $(\mathcal{P}, \leq, \between)$*. Then* $(Reg(\mathcal{P}), \leq, \between)$ *is an o-algebra.*

The proof is divided into two parts: the first one about the order and the second one about the overlap relation.

The complete lattice of regular opens. As R is monotonic and idempotent, the collection $Reg(\mathcal{P})$ is a complete lattice with respect to the operations described in equation (2). In the case of R, those conditions become:

$$\bigvee_{i\in\mathrm{I}}^{\mathrm{R}} \mathrm{R}\,p_i \;=\; \mathrm{I\,C}\bigvee_{i\in\mathrm{I}} \mathrm{I}\,p_i \qquad\text{and}\qquad \bigwedge_{i\in\mathrm{I}}^{\mathrm{R}} \mathrm{R}\,p_i \;=\; \mathrm{I}\bigwedge_{i\in\mathrm{I}} \mathrm{C\,I}\,p_i\,. \tag{6}$$

Indeed, $\bigvee^{\mathrm{R}} \mathrm{R}\,p_i = \mathrm{I\,C}\,(\mathrm{I}\bigvee \mathrm{I\,C\,I}\,p_i) = \mathrm{I\,C}\bigvee^{\mathrm{I}} \mathrm{I\,C}\,(\mathrm{I}\,p_i) = \mathrm{I\,C}\bigvee^{\mathrm{I}} \mathrm{I}\,p_i$ (because $\mathrm{I\,C}$ is a saturation on $Fix(\mathrm{I})$) $= \mathrm{I\,C}\bigvee \mathrm{I}\,p_i$ (because I is a reduction). Similarly, $\bigwedge^{\mathrm{R}} \mathrm{R}\,p_i = \mathrm{I\,C}\,(\mathrm{I}\bigwedge \mathrm{I\,C\,I}\,p_i) = \mathrm{I\,C}\,(\bigwedge^{\mathrm{I}} \mathrm{I\,C}\,(\mathrm{I}\,p_i)) = \bigwedge^{\mathrm{I}} \mathrm{I}\,(\mathrm{C\,I}\,p_i) = \mathrm{I}\bigwedge \mathrm{C\,I}\,p_i$.

Note that the operations induced by the operator R are exactly the operations induced by $\mathrm{I\,C}$ as a saturation on $Fix(\mathrm{I})$. Also note that

$$\mathrm{R}\,p \wedge^{\mathrm{R}} \mathrm{R}\,q \;=\; \mathrm{R}\,p \wedge \mathrm{R}\,q \tag{7}$$

because $\wedge = \wedge^{\mathrm{I}}$. Finally note that $0^{\mathrm{R}} = \mathrm{R}\,0 = \mathrm{I\,C\,I}\,0 = \mathrm{I\,C}\,0 \neq 0$, in general. However $\mathrm{R}\,0 = 0$ in the case that $\mathcal{P}$ is an o-algebra: if $\mathrm{I\,C}\,0 \between \mathrm{I\,C}\,0$, then $\mathrm{I\,C}\,0 \between \mathrm{C}\,0$ (because $\mathrm{I\,C}\,0 \leq \mathrm{C}\,0$) and $\mathrm{I\,C}\,0 \between 0$ (by compatibility), which is impossible; thus $\neg(\mathrm{I\,C}\,0 \between \mathrm{I\,C}\,0)$, hence $\mathrm{I\,C}\,0 = \mathrm{I\,C}\,0 \wedge \mathrm{I\,C}\,0 = 0$ (item 5 of proposition 2.2 in the case of o-algebras).

The overlap relation. We show now that $Reg(\mathcal{P})$ is an o-algebra with respect to the same relation $\between$ of $\mathcal{P}$. Firstly note that

$$\mathrm{R}\,p \between \mathrm{R}\,q \qquad\Longleftrightarrow\qquad \mathrm{R}\,p \between \mathrm{I}\,q \qquad\Longleftrightarrow\qquad \mathrm{I}\,p \between \mathrm{I}\,q \tag{8}$$

for any p and q in $\mathcal{P}$. For $\mathrm{I\,C\,I}\,p \between \mathrm{I\,C\,I}\,q$ iff (compatibility) $\mathrm{I\,C\,I}\,p \between \mathrm{C\,I\,C\,I}\,q$ iff ($\mathrm{C\,I}$ is idempotent) $\mathrm{I\,C\,I}\,p \between \mathrm{C\,I}\,q$ iff (compatibility) $\mathrm{I\,C\,I}\,p \between \mathrm{I}\,q$ iff (symmetry of $\between$) $\mathrm{I}\,q \between \mathrm{I\,C\,I}\,p$ iff (analogously to the first part of this proof) $\mathrm{I}\,q \between \mathrm{I}\,p$.

$\between$ preserves $\wedge^{\mathrm{R}}$: $\mathrm{R}\,p \between \mathrm{R}\,q \implies \mathrm{R}\,p \between \mathrm{R}\,p \wedge^{\mathrm{R}} \mathrm{R}\,q$

Since $\between$ preserves $\wedge$ in $\mathcal{P}$, we know that $\mathrm{R}\,p \between \mathrm{R}\,q \Longrightarrow \mathrm{R}\,p \between \mathrm{R}\,p \wedge \mathrm{R}\,q$ and we can use equation (7).

$\between$ splits $\bigvee^{\mathrm{R}}$: $\mathrm{R}\,p \between \bigvee_{i\in\mathrm{I}}{}^{\mathrm{R}} \mathrm{R}\,q_i \iff (\exists i \in \mathrm{I})(\mathrm{R}\,p \between \mathrm{R}\,q_i)$

By display (6), the left-hand side is $\mathrm{R}\,p \between \mathrm{I\,C}\,(\bigvee_{i\in\mathrm{I}} \mathrm{I}\,q_i)$; hence $\mathrm{R}\,p \between \mathrm{C}\,(\bigvee_{i\in\mathrm{I}} \mathrm{I}\,q_i)$. By compatibility, we have $\mathrm{R}\,p \between \bigvee_{i\in\mathrm{I}} \mathrm{I}\,q_i$ and thus $\mathrm{R}\,p \between \mathrm{I}\,q_i$, for some i (because $\between$ splits $\bigvee$ in $\mathcal{P}$). This is the right-hand side, thanks to display (8). The other direction is by monotonicity of $\between$, since $\mathrm{R}\,q_i \leq \bigvee_{i\in\mathrm{I}}^{\mathrm{R}} \mathrm{R}\,q_i$ and $\leq = \leq^{\mathrm{F}}$.

Density: $\forall r(\mathrm{R}\,p \between \mathrm{R}\,r \implies \mathrm{R}\,q \between \mathrm{R}\,r) \implies \mathrm{R}\,p \leq \mathrm{R}\,q$

This is an immediate corollary of the $\mathrm{C}-\mathrm{I}$ density: thanks to equations (5) and (8), density reduces to $\forall r(\mathrm{I}\,p \between \mathrm{I}\,r \Longrightarrow \mathrm{I}\,q \between \mathrm{I}\,r) \Longrightarrow \mathrm{I}\,p \leq \mathrm{C\,I}\,q$ which is an instance of $\mathrm{C}-\mathrm{I}$ density applied to $\mathrm{I}\,p$ and $\mathrm{I}\,q$.

Properness: $1^{\mathrm{R}} \between 1^{\mathrm{R}}$

The top element of $Reg(\mathcal{P})$ is $\mathrm{R}\,1$ and we have $\mathrm{R}\,1 \between \mathrm{R}\,1$ iff $\mathrm{I}\,1 \between \mathrm{I}\,1$ (properness of the o-topology).

3.1 About atoms

We have already said that an element m of an o-algebra $\mathcal{P}$ is an *atom* if $m \gtrless p \Longleftrightarrow m \leq p$, for any p in P. This fits well with the usual intuition of an atom as a minimal non-zero element. So $m = \mathrm{R}\, m$ is an atom of the o-algebra $Reg(\mathrm{R})$ if and only if

$$m \gtrless \mathrm{I}\, p \qquad \Longleftrightarrow \qquad m \leq \mathrm{C}\,\mathrm{I}\, p$$

by equations (8) and (5). We want to show that $Reg(\mathcal{P})$ is atomless, under the following assumption:

$$\mathrm{I}\, p \gtrless \mathrm{I}\, p \implies \exists q(\mathrm{I}\, q \gtrless \mathrm{I}\, q \;\&\; \mathrm{I}\, q < \mathrm{I}\, p) \tag{9}$$

(every inhabited open set has a proper inhabited open subset) which, obviously, is satisfied by a large class of topological spaces. Indeed, suppose m is an atom of $Reg(\mathcal{P})$; thus, in particular, m is open ($m = \mathrm{I}\, m$). Consequently, it is $m \leq \mathrm{C}\,\mathrm{I}\, m$; so we have $m \gtrless \mathrm{I}\, m$ and also $\mathrm{I}\, m \gtrless \mathrm{I}\, m$. Now we can apply condition (9) and get: $\exists q(\mathrm{I}\, q \gtrless \mathrm{I}\, q \;\&\; \mathrm{I}\, q < \mathrm{I}\, m)$. In particular $m = \mathrm{I}\, m \gtrless \mathrm{I}\, q$ (by monotonicity of $\gtrless$), hence $m \leq \mathrm{I}\, q$ by the definition of atom. Clearly, this contradicts $\mathrm{I}\, q < m = \mathrm{I}\, m$.

3.2 Regular elements as a sublocale of the opens

According to [3], a sublocale is the collection of fixed points of a *nucleus* on the given ambient locale. A nucleus is nothing else than a saturation operator which distributes over binary meets ($Fix(\mathrm{I})$ is not a sublocale of $\mathcal{P}$). Theorem (3.1) yields that $(Reg(\mathcal{P}), \leq)$ is a locale. This section is devoted to show that, in fact, it is a sublocale of $Fix(\mathrm{I})$.

Lemma 3.2 *Let $(\mathcal{P}, \mathrm{C}, \mathrm{I})$ be an o-topology (even without $\mathrm{C} - \mathrm{I}$ density). Then:*

$$(\mathrm{I}\, r \gtrless \mathrm{I}\, r) \;\;\&\;\; \mathrm{I}\, r \leq (\mathrm{C}\,\mathrm{I}\, p \wedge \mathrm{C}\,\mathrm{I}\, q) \implies \mathrm{I}\, r \gtrless (\mathrm{I}\, p \wedge \mathrm{I}\, q)$$

for any p, q and r in $\mathcal{P}$.

Proof From the second premise one gets $\mathrm{I}\, r \leq \mathrm{C}\,\mathrm{I}\, p$, which, together with the first premise, gives $\mathrm{I}\, r \gtrless \mathrm{C}\,\mathrm{I}\, p$, by monotonicity of $\gtrless$. By compatibility, also $\mathrm{I}\, r \gtrless \mathrm{I}\, p$ holds; hence $\mathrm{I}\, r \gtrless \mathrm{I}\, r \wedge \mathrm{I}\, p$, because $\gtrless$ preserves $\wedge$.

The second premise yields also $\mathrm{I}\, r \leq \mathrm{C}\,\mathrm{I}\, q$; combining this with $\mathrm{I}\, r \gtrless \mathrm{I}\, r \wedge \mathrm{I}\, p$, one gets $\mathrm{I}\, r \wedge \mathrm{I}\, p \gtrless \mathrm{C}\,\mathrm{I}\, q$ (use symmetry and monotonicity of $\gtrless$). Now, thanks to $\wedge = \wedge^{\mathrm{I}}$ and compatibility, one gets $\mathrm{I}\, r \wedge \mathrm{I}\, p \gtrless \mathrm{I}\, q$, hence the conclusion. q.e.d.

Proposition 3.3 *If $(\mathcal{P}, \mathrm{C}, \mathrm{I})$ is an o-topology, then $Reg(\mathcal{P})$ is a sublocale of $Fix(\mathrm{I})$.*

Proof We know (proposition 2.7 and equations (5) and (6)) that $Reg(\mathcal{P})$ can be seen as the collection of $\mathrm{I}\,\mathrm{C}$-fixed elements of $Fix(\mathrm{I})$. Thus, asserting that $Reg(\mathcal{P})$ is a sublocale of $Fix(\mathrm{I})$ is tantamount to claim that $\mathrm{I}\,\mathrm{C}$ is a nucleus on the latter. By Lemma 2.6, $\mathrm{I}\,\mathrm{C}$ is a saturation on $Fix(\mathrm{I})$; hence, in order to prove our claim, we have only to check that $\mathrm{I}\,\mathrm{C}\,(\mathrm{I}\, p \wedge^{\mathrm{I}} \mathrm{I}\, q) = \mathrm{I}\,\mathrm{C}\,\mathrm{I}\, p \wedge^{\mathrm{I}} \mathrm{I}\,\mathrm{C}\,\mathrm{I}\, q$ for any p and q in $\mathcal{P}$. Thanks to the assumption $\wedge = \wedge^{\mathrm{I}}$, the latter can be rewritten as

$$\mathrm{R}\,(p \wedge q) \;=\; \mathrm{R}\, p \wedge \mathrm{R}\, q \tag{10}$$

which is what we are now going to prove. In fact, we shall prove that $\mathrm{R}\, p \wedge \mathrm{R}\, q \leq \mathrm{R}\,(p \wedge q)$ (the other direction is true by monotonicity of R). By definition, our claim is $\mathrm{I}\,\mathrm{C}\,\mathrm{I}\, p \wedge \mathrm{I}\,\mathrm{C}\,\mathrm{I}\, q \leq \mathrm{I}\,\mathrm{C}\,\mathrm{I}\,(p \wedge q)$, which is equivalent to $\mathrm{I}\,(\mathrm{C}\,\mathrm{I}\, p \wedge \mathrm{C}\,\mathrm{I}\, q) \leq \mathrm{I}\,\mathrm{C}\,\mathrm{I}\,(p \wedge q)$ (because $\wedge = \wedge^{\mathrm{I}}$) and then to $\mathrm{I}\,(\mathrm{C}\,\mathrm{I}\, p \wedge \mathrm{C}\,\mathrm{I}\, q) \leq \mathrm{C}\,\mathrm{I}\,(p \wedge q)$ (because I is a reduction). In view of $\mathrm{C} - \mathrm{I}$ density, we shall check that: $\mathrm{I}\,(\mathrm{C}\,\mathrm{I}\, p \wedge \mathrm{C}\,\mathrm{I}\, q) \gtrless \mathrm{I}\, r \Longrightarrow \mathrm{I}\,(p \wedge q) \gtrless \mathrm{I}\, r$, for an arbitrary r in $\mathcal{P}$.

Thus suppose $\mathrm{I}\,(\mathrm{C}\,\mathrm{I}\, p \wedge \mathrm{C}\,\mathrm{I}\, q) \gtrless \mathrm{I}\, r$; hence $\mathrm{I}\,(r \wedge \mathrm{C}\,\mathrm{I}\, p \wedge \mathrm{C}\,\mathrm{I}\, q) \gtrless \mathrm{I}\,(r \wedge \mathrm{C}\,\mathrm{I}\, p \wedge \mathrm{C}\,\mathrm{I}\, q)$ (because $\gtrless$ preserves $\wedge$ and $\wedge = \wedge^{\mathrm{I}}$) and $\mathrm{I}\,(r \wedge \mathrm{C}\,\mathrm{I}\, p \wedge \mathrm{C}\,\mathrm{I}\, q) \leq (\mathrm{C}\,\mathrm{I}\, p \wedge \mathrm{C}\,\mathrm{I}\, q)$ (because I is a reduction). By Lemma 3.2, $\mathrm{I}\,(r \wedge \mathrm{C}\,\mathrm{I}\, p \wedge \mathrm{C}\,\mathrm{I}\, q) \gtrless \mathrm{I}\, p \wedge \mathrm{I}\, q$ and then $\mathrm{I}\, r \gtrless \mathrm{I}\, p \wedge \mathrm{I}\, q$ (because $\mathrm{I}\,(r \wedge \mathrm{C}\,\mathrm{I}\, p \wedge \mathrm{C}\,\mathrm{I}\, q) \leq \mathrm{I}\, r$). q.e.d.

4 Some overlap-topology: regular spaces

This section represents an example of the expressive power of the overlap relation. What we are going to show is how to translate the definition of regular space in the language of o-topologies.

One of the most common definition of regular space is: *a topological space X is* regular *if for any open subset D and any point $x \in D$, there exists an open subset E such that $x \epsilon E$ and* $\mathsf{cl}\, E \subseteq D$. Note that a space is regular if and only if *for any subset F and any open D, if $F \between D$, then there exists an open E such that $F \between E$ and* $\mathsf{cl}\, E \subseteq D$ (if X is regular and $x \in F \cap D$, then, by regularity, there exists an open subset E such that $x \in E$ (hence $x \in F \cap E$) and $\mathsf{cl}\, E \subseteq D$; vice versa, take $F = \{x\}$).

The latter characterization, as it does not mention points, can literally translated into the language of o-topologies.

Definition 4.1 *An o-topology* $(\mathcal{P}, \mathrm{C}, \mathrm{I})$ *is* regular *if:*

$$r \gtrless \mathrm{I}\, p \implies \exists q (r \gtrless \mathrm{I}\, q \ \& \ \mathrm{C}\,\mathrm{I}\, q \leq \mathrm{I}\, p)$$

for any p and r.

Another way to get this definition is to start from the following equivalent characterization of regularity: *a topological space is regular if and only if every open subset D is the union of all those open subsets E whose closure* $\mathsf{cl}\, E$ *is contained in D, that is $D = \bigcup\{E \ : \ \mathsf{cl}\, E \subseteq D\}$*. In the language of o-topologies, we have:

$$\mathrm{I}\, p = \bigvee \{\mathrm{I}\, q \ : \ \mathrm{C}\,\mathrm{I}\, q \leq \mathrm{I}\, p\}$$

for any p. The fact that the latter and definition 4.1 are in fact equivalent can be proved within the language of o-topologies: if $r \gtrless \mathrm{I}\, p$ and $\mathrm{I}\, p = \bigvee\{\mathrm{I}\, q \ : \ \mathrm{C}\,\mathrm{I}\, q \leq \mathrm{I}\, p\}$, then there exists q such that $r \gtrless \mathrm{I}\, q$ and $\mathrm{C}\,\mathrm{I}\, q \leq \mathrm{I}\, p$; vice versa, for any r, if $r \gtrless \mathrm{I}\, p$, then there exists q such that $r \gtrless \mathrm{I}\, q$ and $\mathrm{C}\,\mathrm{I}\, q \leq \mathrm{I}\, p$ by regularity, so $r \gtrless \bigvee\{\mathrm{I}\, q \ : \ \mathrm{C}\,\mathrm{I}\, q \leq \mathrm{I}\, p\}$; by density, $\mathrm{I}\, p \leq \bigvee\{\mathrm{I}\, q \ : \ \mathrm{C}\,\mathrm{I}\, q \leq \mathrm{I}\, p\}$, hence $\mathrm{I}\, p = \bigvee\{\mathrm{I}\, q \ : \ \mathrm{C}\,\mathrm{I}\, q \leq \mathrm{I}\, p\}$.

5 Some remarks about foundations

We intentionally wrote this paper without speaking about foundations. At the same time, it was our desire to make all definitions and results meaningful whatever the foundational point of view of the reader was. In particular, we had in mind three kind of frameworks: Zermelo-Fraenkel set theory with Choice, Martin-Löf Type Theory and Topos Theory. Since the meaning of even a single mathematical term depends on foundations, the only way to fulfill our task was to keep a minimalist attitude and to avoid both the Axiom of Choice and the Powerset Axiom and the Principle of Excluded Middle.

Here we want to discuss what the notions of o-algebra and o-topology actually look like from some different foundations.

First of all, it is worthwhile to analyze the notion of complete lattice. A quick look trough the paper shows that by the adjective "complete" we have understood the existence of the least upper bound for any *set-indexed* family. Now the notion of a set-indexed family is susceptible to various interpretations: if the powerset axiom is available every family is set-indexed, provided that the carrier of the o-algebra is a set; this is not true in a predicative (no powerset axiom) approach, of course. Moreover, within a predicative approach, it can happen (in fact, it is most often the case) that the carrier of an o-algebra is not a set. In this case, the universal quantification in the "density" axiom is troublesome. As a consequence, a predicativist should require the lattice to be set-based: a complete lattice $\mathcal{P}$ is set-based on a set S if there exists a set-indexed family $\{g(a) : a \in S\}$ such that, for any p in $\mathcal{P}$, $p = \bigvee\{g(a) : a \in S \ \& \ g(a) \leq p\}$. In this case, "density" becomes equivalent to $(\forall a \in S)\big(p \gtrless g(a) \implies q \gtrless g(a)\big) \implies p \leq q$.

A similar discussion can be done with respect to "$\mathrm{C} - \mathrm{I}$ density". This condition becomes predicatively meaningful if $Fix(\mathrm{I})$ is set-based. Of course, this is the case if I is the interior operator of a topological space which admits a predicatively acceptable set as a base. Note that if $Fix(\mathrm{I})$ is set-based, then also $Reg(\mathcal{P})$ is set-based: let $\{\mathrm{I}\,g(a) : a \in S\}$ be a base for $Fix(\mathrm{I})$; then, for any p in $\mathcal{P}$, we have $\mathrm{R}\,p = \mathrm{I}\,\mathrm{C}\,\mathrm{I}\,p = \mathrm{I}\,\mathrm{C} \bigvee^{\mathrm{I}} \{\mathrm{I}\,g(a) : \mathrm{I}\,g(a) \leq \mathrm{I}\,p\} = \mathrm{I}\,\mathrm{C} \bigvee \{\mathrm{I}\,g(a) : \mathrm{I}\,g(a) \leq \mathrm{I}\,p\}$ $= \bigvee^{\mathrm{R}} \{\mathrm{R}\,g(a) : \mathrm{I}\,g(a) \leq \mathrm{I}\,p\}$. This shows that $\mathrm{R}\,p \leq \bigvee^{\mathrm{R}} \{\mathrm{R}\,g(a) : \mathrm{R}\,g(a) \leq \mathrm{R}\,p\}$ because $\mathrm{I}\,g(a) \leq \mathrm{I}\,p$ yields $\mathrm{R}\,g(a) \leq \mathrm{R}\,p$; besides, $\bigvee^{\mathrm{R}} \{\mathrm{R}\,g(a) : \mathrm{R}\,g(a) \leq \mathrm{R}\,p\} \leq \mathrm{R}\,p$ always holds. Summing up, $\mathrm{R}\,p = \bigvee^{\mathrm{R}} \{\mathrm{R}\,g(a) : \mathrm{R}\,g(a) \leq \mathrm{R}\,p\}$ and $\{\mathrm{R}\,g(a) : a \in S\}$ generates $Reg(\mathcal{P})$.

We now want to show that o-algebras and complete Boolean algebras are essentially the same notion provided that a classical metalanguage is adopted.

Proposition 5.1 *Assuming the Principle of Excluded Middle, if $(P, \leq)$ is a complete Boolean algebra, then $(P, \leq, \gtrless)$ is an o-algebra, where $p \gtrless q$ is $p \wedge q \neq 0$.*

Assuming the Principle of Excluded Middle and the Powerset Axiom, if $(P, \leq, \gtrless)$ is an o-algebra, then $(P, \leq)$ is a complete Boolean algebra.

Proof The binary relation $p \wedge q \neq 0$ is symmetric and preserves infima (in the sense of definition 2.1); also, $1 \neq 0$ (properness). Moreover, $p \wedge \bigvee_{i \in I} q_i \neq 0$ iff $\bigvee_{i \in I}(p \wedge q_i) \neq 0$ iff (by classical logic) $(\exists i \in I)$ $(p \wedge q_i \neq 0)$ (splitting of suprema). Finally, in order to check the validity of the density axiom, suppose $r \wedge p \neq 0 \Longrightarrow r \wedge q \neq 0$, for any r; in particular, $-q \wedge p \neq 0 \Longrightarrow -q \wedge q \neq 0$ then, equivalently, $-q \wedge q = 0 \Longrightarrow -q \wedge p = 0$; thus $-q \wedge p = 0$, hence $p \leq q$.

Since $(P, \leq)$ is a locale, one can define an implication in the usual impredicative way: $p \to q = \bigvee \{r : r \wedge p \leq q\}$ and, accordingly, a pseudo-complement $-p = p \to 0$. Our claim is that $- - p \leq p$ holds for any p. By density, this is equivalent to prove that $(r \gtrless - - p) \Longrightarrow (r \gtrless p)$ (for any r); this is tantamount to say that $\neg(r \gtrless p) \Longrightarrow \neg(r \gtrless - - p)$. This can be read as: $r \wedge p = 0$ $\Longrightarrow r \wedge - - p = 0$; in other words, our claim is: $r \leq -p \Longrightarrow r \leq - - -p$, which is obvious since $- - -p = -p$. q.e.d.

Summing up, an impredicative definition is needed to turn an o-algebra into a Heyting o-algebra (that is, an o-algebra enriched with an operation of implication which is right adjoint to infimum), then classical reasoning makes the notions of Heyting o-algebra and complete Boolean algebra coincide.

We now want to prove some facts about o-topologies in classical or impredicative foundations. We have already observed that $\mathrm{C} - \mathrm{I}$ density expresses equivalence between $p \leq \mathrm{C}\,q$ and $\forall r(p \gtrless \mathrm{I}\,r \Longrightarrow q \gtrless \mathrm{I}\,r)$. This yields that $\mathrm{C}\,q = \bigvee \{p : \forall r(p \gtrless \mathrm{I}\,r \Longrightarrow q \gtrless \mathrm{I}\,r)\}$ which can be read as an impredicative definition of C.

Proposition 5.2 *Assuming the Powerset Axiom, o-topologies can be characterized as those structures $(\mathcal{P}, \mathrm{I})$, where $\mathcal{P}$ is a distributive quasi o-algebra and I is a reduction on $\mathcal{P}$ satisfying $\wedge = \wedge^{I}$ and $\mathrm{I}\,1 \gtrless \mathrm{I}\,1$. Here C is introduced via the above impredicative definition.*

Proof We must show that C is a saturation operator and that it is compatible with I ($\mathrm{C} - \mathrm{I}$ density being trivial). Let W_q be the family $\{p : \forall r(p \gtrless \mathrm{I}\,r \Longrightarrow q \gtrless \mathrm{I}\,r)\}$; so $\mathrm{C}\,q = \bigvee W_q$.

Compatibility: if $\mathrm{I}\,r \gtrless \mathrm{C}\,q = \bigvee W_q$, then there exists p in W_q such that $\mathrm{I}\,r \gtrless p$; so $\mathrm{I}\,r \gtrless q$ (because p belongs to W_q).

C is expansive: $q \leq \mathrm{C}\,q$ because q belongs to W_q (trivially).

C is idempotent: $\mathrm{C}\,\mathrm{C}\,q$ belongs to W_q (by compatibility applied twice); hence $\mathrm{C}\,\mathrm{C}\,q \leq \mathrm{C}\,q$ and $\mathrm{C}\,\mathrm{C}\,q = \mathrm{C}\,q$ because C is expansive.

C is monotonic: if $q_1 \leq q_2$, then W_{q_1} is a sub-family of W_{q_2} ($p \gtrless \mathrm{I}\,r \Longrightarrow q_1 \gtrless \mathrm{I}\,r$ and $q_1 \leq q_2$ imply $p \gtrless \mathrm{I}\,r \Longrightarrow q_2 \gtrless \mathrm{I}\,r$). q.e.d.

Proposition 5.3 *Assume the Powerset Axiom and the Principle of Excluded Middle. Let $(\mathcal{P}, \leq, \gtrless)$ be a distributive quasi o-algebra and C and I be a saturation and a reduction on $\mathcal{P}$. Then compatibility between C and I is equivalent to $\mathrm{C} \leq -\mathrm{I}-$, while $\mathrm{C} - \mathrm{I}$ density is equivalent to $-\mathrm{I}- \leq \mathrm{C}$.*

PROOF Thanks to the Powerset Axiom, $(\mathcal{P}, \leq)$ is a complete Heyting algebra; for p in $\mathcal{P}$, let $-p$ be the pseudo-complement of p. Also, the Principle of Excluded Middle makes $p \between q$ equivalent to $p \wedge q \neq 0$ (see proposition 2.2).

As $\mathrm{I} - q \leq -q$, we have $\mathrm{I} - q \wedge q = 0$ and $\neg(\mathrm{I} - q \between q)$; hence (by compatibility) $\neg(\mathrm{I} - q \between \mathrm{C}\, q)$, that is, $\mathrm{C}\, q \wedge \mathrm{I} - q = 0$; this implies $\mathrm{C}\, q \leq -\mathrm{I} - q$. Vice versa, if $\mathrm{I}\, p \between \mathrm{C}\, q$, then $\mathrm{I}\, p \between -\mathrm{I} - q$, that is, $\mathrm{I}\, p \wedge -\mathrm{I} - q \neq 0$; now, assuming $\neg(\mathrm{I}\, p \between q)$ leads to a contradiction because $\mathrm{I}\, p \wedge q = 0$ iff $\mathrm{I}\, p \leq -q$ iff $\mathrm{I}\, p \leq \mathrm{I} - q$ and the latter yields $\mathrm{I}\, p \wedge -\mathrm{I} - q = 0$.

For any p, $-\mathrm{I} - q \between \mathrm{I}\, p$ yields $q \between \mathrm{I}\, p$ (by the argument above); hence $-\mathrm{I} - q \leq \mathrm{C}\, q$ by $\mathrm{C} - \mathrm{I}$ density. Vice versa, suppose $\forall r(p \between \mathrm{I}\, r \implies q \between \mathrm{I}\, r)$; in particular, as $q \between \mathrm{I} - q$ does not hold ($q \wedge \mathrm{I} - q = 0$), we have $\neg(p \between \mathrm{I} - q)$, that is, $p \wedge \mathrm{I} - q = 0$, hence $p \leq -\mathrm{I} - q$; thus $p \leq \mathrm{C}\, q$. q.e.d.

Hence, the axioms of an o-topology are a positive (that is, with no reference to negation) way to express $\mathrm{C} = -\mathrm{I}-$ ($\mathsf{cl} = -\,\mathsf{int}\,-$).

In view of these results the novelty of the notion of overlap algebra is seen better from the point of view of non-classical foundations. In particular, the above negation-free treatment of topology should be of a certain interest for intuitionistic mathematicians and for computer scientists which, we think, could appreciate the algebraic flavor of the matter.

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References

[1] F. Ciraulo, *A constructive investigation of satisfiability*, submitted.

[2] F. Ciraulo and G. Sambin, *Tense logic within a constructive metatheory*, Preprint n. 341, Dipartimento di Matematica ed Applicazioni, Università degli Studi di Palermo.

[3] P. T. Johnstone, *Stone spaces*, Cambridge University Press, 1982.

[4] J. C. C. McKinsey and A. Tarski, *The Algebra of Topology*, The Annals of Mathematics, 2nd Ser., Vol. 45, No. 1 (Jan., 1944), pp. 141-191.

[5] S. Sadocco, *Algebrization of the power set. An axiomatic treatment of the existential quantifier*, PhD thesis, Università di Siena, 2002.

[6] G. Sambin, *The Basic Picture. Structures for Constructive Topology (including two papers with P. Martin-Löf and with V. Capretta).* Oxford University Press, to appear in 2010.