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On the coordinate ring of spherical conjugacy classes

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Abstract Let G be a simple algebraic group over an algebraically closed field k of characteristic zero and $\mathcal{O}$ be a spherical conjugacy class of G. We determine the decomposition of the coordinate ring $k[\mathcal{O}]$ of $\mathcal{O}$ into simple G-modules.

1 Introduction

Let G be a simple algebraic group over an algebraically closed field k of characteristic zero, with Lie algebra $\mathfrak{g}$. In this paper we study the G-module structure of the coordinate ring $k[\mathcal{O}]$, where $\mathcal{O}$ is a spherical conjugacy class of G (we recall that a conjugacy class $\mathcal{O}$ in G is called *spherical* if a Borel subgroup of G has a dense orbit on $\mathcal{O}$: see Definition 3.1). There has been a lot of work related to this field. Initially, spherical G-orbits have been studied in the context of Lie algebras [33,35]. More recently, in [9], we put our attention to spherical conjugacy classes in G. Since k is of characteristic zero, there is no essential difference between the (spherical) nilpotent orbits in $\mathfrak{g}$ and the (spherical) unipotent conjugacy classes in G. In the group context there is the possibility to use the Bruhat decomposition, whereas the nilpotent orbits in $\mathfrak{g}$ are conical varieties and therefore $k[\mathcal{O}]$ is naturally graded. The latter also lead to the notion of height of a nilpotent orbit.

To fix the notation, G is a simple simply-connected algebraic group over k, $\mathfrak{g}$ its Lie algebra, B a Borel subgroup of G, T a maximal torus of B, B^- the Borel subgroup opposite to B, $\{\alpha_1, \ldots, \alpha_n\}$ the set of simple roots with respect to the choice of (T, B). Let W be the Weyl group of G and let us denote by s_i the reflection corresponding to the simple root α_i: $\ell(w)$ is the length of the element $w \in W$ and $\mathrm{rk}(1-w)$ is the rank of $1-w$ in the geometric representation of W.

There are two main characterizations of spherical orbits:

(1) (Panyushev [33,36]) A nilpotent orbit $\mathcal{O} \subset \mathfrak{g}$ is spherical if and only if its height is at most 3, which means 2 or 3 if $\mathcal{O}$ is not the zero orbit;

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(2) (Cantarini et al. [9]) A conjugacy class $\mathcal{O} \subset G$ is spherical if and only if $\dim \mathcal{O} = \ell(w) + rk(1-w)$, where $w = w(\mathcal{O})$ is the unique element of W such that $\mathcal{O} \cap BwB$ is dense in $\mathcal{O}$ (we observe that the classification given in [9] over the complex numbers, holds in general for k).

The latter characterization was achieved while proving the De Concini–Kac–Procesi conjecture on the quantized enveloping algebra $\mathcal{U}_\varepsilon(\mathfrak{g})$ (introduced in [14]) for simple $\mathcal{U}_\varepsilon(\mathfrak{g})$-modules over spherical conjugacy classes of G: our main tool was the representation theory of the quantized Borel subalgebra B_ε introduced in [15]. The representation theory of $\mathcal{U}_\varepsilon(\mathfrak{g})$ is related to the stratification of G given by conjugacy classes, while the representation theory of B_ε is related to the stratification $\{X_w \mid w \in W\}$ of B^-, where $X_w = B^- \cap BwB$ for every $w \in W$ (each X_w is an affine variety of dimension $n + \ell(w)$). We proved that for every spherical conjugacy class $\mathcal{O}$ in G, there exists $w \in W$ such that $\mathcal{O} \cap X_w \neq \varnothing$ and $\ell(w) + rk(1-w) = \dim \mathcal{O}$: this then allows to prove the De Concini–Kac–Procesi conjecture for simple $\mathcal{U}_\varepsilon(\mathfrak{g})$-modules over elements in $\mathcal{O}$. In fact we proved also a result in the opposite direction, giving therefore the above mentioned characterization of spherical conjugacy classes in terms of the Weyl group [9, Theorem 25]. Moreover w is always an involution (see [9, Remark 4], [10, Theorem 2.7]). From this result we conjectured that, for a spherical $\mathcal{O}$, the decomposition of the ring $k[\mathcal{O}]$ of regular functions on $\mathcal{O}$ (to which we refer as to *the coordinate ring* of $\mathcal{O}$) as a G-module should be strictly related to $w(\mathcal{O})$. This is the motivation for the present paper.

We recall that $k[\mathcal{O}]$ is multiplicity-free, so that in order to obtain the decomposition of $k[\mathcal{O}]$ into simple components one has just to determine which simple modules occur in $k[\mathcal{O}]$:

$$k[\mathcal{O}] \underset{G}{\cong} \bigoplus_{\lambda \in \lambda(\mathcal{O})} V(\lambda)$$

where for each dominant weight λ, $V(\lambda)$ is the simple G-module of highest weight λ (if $\lambda \in \lambda(\mathcal{O})$ we say that λ *occurs* in $k[\mathcal{O}]$).

The decomposition of the coordinate ring $k[X]$ for G-varieties X has been investigated by various authors. If λ is a non-zero highest weight, and $v \in V(\lambda)$ is a non-zero highest weight vector, then $k[G.v]$ is isomorphic to $\oplus_{n \geq 0} V(n\lambda)^*$ [45, Theorem 2]. In particular this determines $k[\mathcal{O}]$ for the minimal unipotent orbit of G. For a unipotent class in G (equivalently nilpotent orbit in $\mathfrak{g}$) McGovern ([30, Theorem 3.1], with $k = \mathbb{C}$, the complex numbers) describes $\mathbb{C}[\mathcal{O}]$ in terms of induced building blocks from a certain Levi subgroup of G (via sheaf cohomology on G/Q, Q a parabolic subgroup of G associated to $\mathcal{O}$): it is then possible to obtain multiplicities of simple G-modules in $\mathbb{C}[\mathcal{O}]$ as an alternating sum of certain partition functions. In the same paper the author gives a formula for $\mathbb{C}[\hat{\mathcal{O}}]$, where $\hat{\mathcal{O}}$ is the simply-connected cover of $\mathcal{O}$ [30, Theorem 4.1]. Then in [31] there are tables for the sets of simple modules in $\mathbb{C}[\hat{\mathcal{O}}]$ for spherical unipotent classes in the classical groups (and conjecturally in the exceptional groups). For type F_4 the monoid $\lambda(\mathcal{O})$ has been described in [7], over $\mathbb{C}$, for all spherical unipotent classes. For the maximal spherical unipotent class $\mathcal{O}$ in E_8, it has been shown in [2, Theorem 1.1], that every simple G-module occurs in $\mathbb{C}[\mathcal{O}]$ (so that $\mathcal{O}$ is a model orbit). In [36], Panyushev gives tables for the sets of simple modules for (spherical) nilpotent orbits of height 2 (and conjecturally for height 3). In [28] the author describes explicitly the structure of complex principal model homogeneous spaces. For semisimple spherical classes, the description of $\lambda(\mathcal{O})$ may be deduced from the tables in [26]. See also [46, Théorème 3], where symmetric varieties are considered.

The main result of this paper is the following:

Theorem *Assume $\mathcal{O}$ is a spherical conjugacy class in G, and let $w = w(\mathcal{O})$. Then a dominant weight λ occurs in $k[\mathcal{O}]$ if and only if $w(\lambda) = -\lambda$ and $\lambda(S_{\mathcal{O}}) = 1$.*

Here $S_{\mathcal{O}}$ is a certain (finite) elementary abelian 2-subgroup of T which we determine for every spherical conjugacy class, describing therefore explicitly $\lambda(\mathcal{O})$: see Tables 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25 and 26. In particular we completely solve the problem of determining the simple modules occurring in $k[\mathcal{O}]$ for unipotent classes [22, 8.13, Remark 2], and obtain the decomposition of $k[\mathcal{O}]$ for conjugacy classes of mixed elements.

Our proof is based on the deformation result obtained by Brion in [4] and independently by Vinberg in [44]. We have $k[\mathcal{O}] = k[G/H] = k[G]^H$, where H is the centralizer of an element of $\mathcal{O}$ in G. There exists a flat deformation of G/H to a quotient G/H_0, where H_0 contains the unipotent radical U^- of B^-. We determine the decomposition of $k[G/H_0]$ into simple components (i.e. we determine $\lambda(G/H_0)$), relating the group H_0 with H via the theory of elementary embeddings [5,29]. We then prove the crucial fact that $\lambda(\mathcal{O})$ is saturated [34, Sect. 1.3], so that $k[G/H] = k[G/H_0]$ as G-modules. We also determine the decomposition of the coordinate ring $k[\hat{\mathcal{O}}]$ for the simply-connected cover $\hat{\mathcal{O}}$ of $\mathcal{O}$, and of $k[\overline{\mathcal{O}}]$.

The paper is structured as follows. In Sect. 2 we introduce the notation. In Sect. 3 we recall some basic facts about spherical varieties and we prove the main theorem. In Sect. 4 we determine the group $S_{\mathcal{O}}$ for the spherical conjugacy classes in the various groups, determining therefore the monoid $\lambda(\mathcal{O})$, and also $\lambda(\hat{O})$. In Sect. 5 we consider the coordinate ring $k[\overline{\mathcal{O}}]$ of the closure of $\mathcal{O}$. It is well known that $k[\overline{\mathcal{O}}] = k[\mathcal{O}]$ if and only if $\overline{\mathcal{O}}$ is normal: we list all cases in which the spherical conjugacy class $\mathcal{O}$ has normal closure and we determine $\lambda(\overline{\mathcal{O}})$ for the classes with non-normal closure. In Sect. 6 we consider the case when G in not necessarily simply-connected.

2 Preliminaries

We denote by $\mathbb{C}$ the complex numbers, by $\mathbb{R}$ the reals, by $\mathbb{Z}$ the integers and by $\mathbb{N}$ the natural numbers.

Let $A = (a_{ij})$ be a finite indecomposable Cartan matrix of rank n. To A there is associated a root system Φ, a simple Lie algebra $\mathfrak{g}$ and a simple simply-connected algebraic group G over k. We fix a maximal torus T of G, and a Borel subgroup B containing T: B^- is the Borel subgroup opposite to B, U (respectively U^-) is the unipotent radical of B (respectively of B^-). If χ is a character of T, we still denote by χ the character of B which extends χ. We denote by $\mathfrak{h}$ the Lie algebra of T. Then Φ is the set of roots relative to T, and B determines the set of positive roots Φ^+, and the simple roots $\Delta = \{\alpha_1, \dots, \alpha_n\}$. We fix a total ordering on Φ^+ compatible with the height function. We shall use the numbering and the description of the simple roots in terms of the canonical basis $(e_1, \dots, e_k)$ of an appropriate $\mathbb{R}^k$ as in [3], Planches I–IX. For the exceptional groups, we shall write $\beta = (m_1, \dots, m_n)$ for $\beta = m_1\alpha_1 + \cdots + m_n\alpha_n$. We denote by P the weight lattice, by P^+ the monoid of dominant weights and by W the Weyl group; s_i is the simple reflection associated to α_i, $\{\omega_1, \dots, \omega_n\}$ are the fundamental weights, w_0 is the longest element of W. In the expression $\lambda = \sum_i k_i n_i \omega_i$ we always assume k_i's and n_i's in $\mathbb{N}$. The real space $E = \mathbb{R}P$ is a Euclidean space, endowed with the scalar product $(\alpha_i, \alpha_j) = d_i a_{ij}$. Here $\{d_1, \dots, d_n\}$ are relatively prime positive integers such that if D is the diagonal matrix with entries $d_1, \dots, d_n$, then DA is symmetric.

If V is a G-module, $v \in V$, $f \in V^*$, then the matrix coefficient $c_{f,v} : G \to k$ is defined by $c_{f,v}(g) = f(g.v)$ for $g \in G$. We consider the action of $G \times G$ on $k[G]$

$$((g, g_1).f)(c) = f(g^{-1}cg_1)$$

for $c, g, g_1 \in G$, $f \in k[G]$. The algebraic version of the Peter-Weyl theorem gives the decomposition

$$k[G] = \bigoplus_{\lambda \in P^+} V(-w_0\lambda)^* \otimes V(-w_0\lambda) \tag{2.1}$$

We put $\Pi = \{1, \ldots, n\}$ and we fix a Chevalley basis $\{h_i, i \in \Pi; e_\alpha, \alpha \in \Phi\}$ of $\mathfrak{g}$. We shall denote by $\check{\omega}_i$, for $i = 1, \ldots, n$, the elements in $\mathfrak{h}$ defined by $\alpha_j(\check{\omega}_i) = \delta_{ij}$ (recall that $\omega_j(h_i) = \delta_{ij}$) for $j = 1, \ldots, n$. As usual we put $\langle x, y \rangle = \frac{2(x,y)}{(y,y)}$.

We use the notation $x_\alpha(\xi)$, $h_\alpha(z)$, for $\alpha \in \Phi$, $\xi \in k$, $z \in k^*$ as in [11,43]. For $\alpha \in \Phi$ we put $X_\alpha = \{x_\alpha(\xi) \mid \xi \in k\}$, the root-subgroup corresponding to α, and $H_\alpha = \{h_\alpha(z) \mid z \in k^*\}$. We identify W with N/T, where N is the normalizer of T: given an element $w \in W$ we shall denote a representative of w in N by $\dot{w}$. We choose the x_α's so that, for all $\alpha \in \Phi$, $n_\alpha = x_\alpha(1)x_{-\alpha}(-1)x_\alpha(1)$ lies in N and has image the reflection s_α in W. Then

$$x_\alpha(\xi)x_{-\alpha}(-\xi^{-1})x_\alpha(\xi) = h_\alpha(\xi)n_\alpha, \quad n_\alpha^2 = h_\alpha(-1) \tag{2.2}$$

for every $\xi \in k^*$, $\alpha \in \Phi$ [41, Proposition 11.2.1].

We put $T^w = \{t \in T \mid wtw^{-1} = t\}$, $T_2 = \{t \in T \mid t^2 = 1\}$. In particular $T^w = T_2$ if $w = w_0 = -1$.

For algebraic groups we use the notation in [12,19]. In particular, for $J \subseteq \Pi$, $\Delta_J = \{\alpha_j \mid j \in J\}$, Φ_J is the corresponding root system, W_J the Weyl group, P_J the standard parabolic subgroup of G, $L_J = T\langle X_\alpha \mid \alpha \in \Phi_J\rangle$ the standard Levi subgroup of P_J. For $z \in W$ we put $U_z = U \cap z^{-1}U^-z$. Then the unipotent radical R_uP_J of P_J is $U_{w_0w_J}$, where w_J is the longest element of W_J. Moreover $U \cap L_J = U_{w_J}$ is a maximal unipotent subgroup of L_J.

If Ψ is a subsystem of type X_r of Φ and H is the subgroup generated by X_α, $\alpha \in \Psi$, we say that H is a X_r-subgroup of G.

If X is an algebraic variety, we denote by $k[X]$ the ring of regular functions on X. If X is a multiplicity-free G-variety, then we denote by $\lambda(X)$ the set of dominant weights occurring in $k[X]$, i.e. $\lambda \in P^+$ such that $k[X]$ contains (a copy of) $V(\lambda)$. If $x \in X$ we denote by $G.x$ the G-orbit of x and by G_x the isotropy subgroup of x in G. If the homogeneous space G/H is spherical, we say that H is a spherical subgroup of G.

If x is an element of a group K and $H \leq K$, we shall also denote by $C(x)$ the centralizer of x in K, and by $C_H(x)$ the centralizer of x in H. If $x, y \in K$, then $x \sim y$ means that x, y are conjugate in K. For unipotent classes in exceptional groups we use the notation in [12]. We use the description of centralizers of involutions as in [21].

3 The main theorem

In this section we prove the main result which states that the monoid of weights of a spherical conjugacy class is saturated. We recall

Definition 3.1 Let X be a G-variety. Then X is called *spherical* if X is normal and B has a dense (open) orbit on X.

Let $\mathcal{O}$ be a spherical conjugacy class. Our aim is to determine $\lambda(\mathcal{O})$. For this purpose if H is the centralizer of an element in $\mathcal{O}$, we have $k[\mathcal{O}] = k[G/H] = k[G]^H$ and, from (2.1),

$$k[G]^H = \bigoplus_{\lambda\in\lambda(\mathcal{O})} V(-w_0\lambda)^* \otimes u_\lambda$$

where $0 \neq u_\lambda \in V(-w_0\lambda)^H$ [37, Theorem 3.12]. We start by considering in general a spherical homogeneous space G/H. Without loss of generality we may assume BH dense in G. By [4, Theorem 1], there exists a (flat) deformation of G/H to a homogeneous (spherical) space G/H_0, where H_0 contains a maximal unipotent subgroup of G: such an homogeneous space is called *horospherical*, and H_0 a horospherical contraction of H (see also [44]). An *elementary embedding* of G/H is a pair (X, x) where X is a normal algebraic G-variety, $x \in X$ is such that $G.x$ is dense in X, $G_x = H$ and $X\setminus G.x$ is a G-orbit of codimension 1 [6, 2.2]. In [4] Brion constructs a $G \times k^*$-variety and a flat $G \times k^*$-morphism $p : Z \to k$ (where G acts trivially on k and k^* acts via homotheties) such that $p^{-1}(k^*) \cong G/H \times k^*$ and $p^{-1}(0) \cong G/H_0$ [4, Theoreme 1], [6, Sect. 3.11]. One may consider Z as an elementary embedding (Z, z) of $(G \times k^*)/(H \times 1)$, with closed orbit $(G \times k^*)/(H_0 \times k^*)$; $H \times 1$ is the isotropy subgroup of z, $H_0 \times k^*$ is the isotropy subgroup of an element in the closed orbit [6, Proof of Corollaire 3.7]. Let $P = P_J$ be the parabolic subgroup *associated* to H, $P = \{g \in G \mid gBH = BH\}$, and let L be a Levi subgroup (which we may assume equal to L_J, by taking an appropriate conjugate of H instead of H) of P *adapted* to H [6, 2.9]: in particular

$$P \cap H = L \cap H, \quad L' \leq H \tag{3.3}$$

Then $P \times k^*$ is the parabolic subgroup of $G \times k^*$ associated to $H \times 1$ and $L \times k^*$ is a Levi subgroup adapted to $H \times 1$ ([6], Corollaire 3.7 and its proof).

By [6, Proposition 3.10, i)], we have $H_0 \times k^* = (R_u Q \times 1)(L \times k^* \cap H_0 \times k^*)$ where Q is the opposite parabolic subgroup of P with respect to L, so that

$$H_0 = (R_u Q)(L \cap H_0) \tag{3.4}$$

We show that $L \cap H = L \cap H_0$. Let $L = CL'$, where C is the connected component of the centre of L. Then L' is contained also in H_0, by [6, Théorème 3.6].

By [6, Proposition 3.4], Z contains an open $P \times k^*$-stable subset isomorphic to $R_u P \times W$ where W is $L\times k^*$-stable and meets the closed orbit, and (W, z) is an elementary embedding of the torus $(C\times k^*)/(C\cap H\times 1)$ [5, Proof of Lemme 4.2]. Then $f = p_{|W} : W \to k$ is a $(C\times k^*)$-equivariant flat morphism such that $f^{-1}(k^*) \cong C/C\cap H \times k^*$ and $f^{-1}(0) \cong C/H_0\cap C$. So the coordinate rings of these orbits are isomorphic C-modules and it follows that the isotropy groups of all points of W are the same. In particular

$$C \cap H = C \cap H_0 \tag{3.5}$$

With the above notation we prove

Theorem 3.2 *Let H be a spherical subgroup of G such that BH is dense in G and $L = L_J$ is a Levi subgroup adapted to H. Then $H_0 = R_u Q\,(L \cap H) = \langle U^-, U_{w_J}, C \cap H\rangle$.*

Proof. By (3.5) we have

$$L \cap H_0 = L'C \cap H_0 = L'(C \cap H_0) = L'(C \cap H) = L'C \cap H = L \cap H$$

so that by (3.4) we conclude. □

Definition 3.3 We put $\tilde{\lambda}(G/H) = \lambda(G/H_0)$.

Note that $\lambda(G/H) \le \tilde{\lambda}(G/H)$ since BH is dense in G, and more generally $\mathbb{Z}\lambda(G/H) \cap P^+ \le \tilde{\lambda}(G/H)$ ([34], part 2 of the proof of Proposition 1.5). Moreover

$$\lambda(G/H_0) = \{\lambda \in P^+ \mid \lambda(T \cap H) = 1\} \tag{3.6}$$

since $\prod_{j\in J} H_{\alpha_j} \le H$ and $X_{\alpha_j}.v_{-\lambda} = v_{-\lambda}$ if $(\lambda, \alpha_j) = 0$ (here $v_{-\lambda}$ is a lowest weight vector of weight $-\lambda$ in $V(-w_0\lambda)$). Also $B \cap H \le P \cap H = L \cap H$, so that $B \cap H = U_{w_J}(T \cap H)$. If $\lambda \in \tilde{\lambda}(G/H)$, then $F_\lambda : BH/H \to k$, $b^{-1}H \mapsto \lambda(b)$ is a regular function on BH/H, and therefore a B-eigenvector of weight λ in $k(G/H)$. In case G/H is quasi affine (as for conjugacy classes), then $\mathbb{Z}\lambda(G/H) \cap P^+ = \tilde{\lambda}(G/H)$ since $k(G/H) = \operatorname{Frac} k[G/H]$, as in [34], Proposition 1.5. I do not know if $\mathbb{Z}\lambda(G/H) \cap P^+ = \tilde{\lambda}(G/H)$ holds in general.

Lemma 3.4 *Suppose F in Frac $k[G/H]$ is a B-eigenvector of weight λ and $m\lambda$ lies in $\lambda(G/H)$ for a positive integer m. Then F lies in $k[G/H]$.*

Proof. There exists a B-eigenvector $F_1 \in k[G/H]$ of weight $m\lambda$. Then F^m/F_1 is invariant under B (as its weight is 0). So F^m/F_1 is constant, as G/H is spherical. In other words, F^m is regular on G/H. We conclude that F is in $k[G/H]$, since $k[G/H]$ is integrally closed [16, Lemma 1.8]. □

Let $\mathcal{O}$ be a spherical conjugacy class of G. We recall that $w = w(\mathcal{O})$ is the unique element (an involution) of W such that $BwB \cap \mathcal{O}$ is (open) dense in $\mathcal{O}$. Let v be the dense B-orbit in $\mathcal{O}$. Then BG_y is dense in G for any $y \in$ v. The parabolic subgroup $P = P_J$ associated to G_y coincides with $\{g \in G \mid g.\mathrm{v} = \mathrm{v}\}$. Moreover v $= \mathcal{O} \cap BwB$ [9, Corollary 26], and it is affine, as an orbit of a soluble algebraic group.

We have $w = w_0 w_J$, the subset J is invariant under ϑ, where ϑ is the symmetry of Π induced by $-w_0$, and w_0 and w_J act in the same way on Φ_J (see [10] the discussion at the end of Sect. 3, Corollary 4.2, Remark 4.3 and Proposition 4.15).

Since all Levi subgroups of P are conjugate under R_uP, we may choose $y \in$ v such that the standard Levi subgroup L_J is adapted to G_y. For the rest of this section we fix such a y, and we put $H = G_y$, $P = P_J$, $L = L_J$. By Theorem 3.2, we have

$$H_0 = \langle U^-, U_{w_J}, C_y \rangle = \langle U^-, U_{w_J}, T_y \rangle \tag{3.7}$$

and $\tilde{\lambda}(\mathcal{O}) = \lambda(G/H_0)$.

We shall now relate H with centralizers of elements in v $\cap\, wB$. By the Bruhat decomposition, y is of the form $y = u\dot{w}b$, where $u \in R_uP$ and $b \in B$. We put $x_1 = u^{-1}yu = \dot{w}bu$. By [10, Corollary 4.13], $U_{w_J}(T^w)^\circ \le C(x_1)$. Moreover, since $L' \le C(y)$, by [10, Lemma 3.4], and commutation of y with $X_{\pm\alpha_i}$ for $i \in J$, we get $L' \le C(x_1)$ (see also the proof of [10], Proposition 4.15).

Proposition 3.5 *Let x be in $\mathcal{O} \cap wB$. Then $T_x = T_y$ and $T \cap H^\circ = T \cap C(x)^\circ$.*

Proof. We observe that $C_{TU_w}(x) \le T$ by the Bruhat decomposition and $C_{TU_w}(y) \le T$, since L is adapted to $C(y)$. Now $x_1 = u^{-1}yu = y^u$ implies

$$T_{x_1} = C_T(x_1) = C_{TU_w}(x_1) \le T \cap T^u = C_T(u)$$
$$T_y = C_T(y) = C_{TU_w}(y) \le T \cap T^{u^{-1}} = C_T(u^{-1}) = C_T(u)$$

therefore if $t \in T_y$, then $t = t^u \in T_{x_1}$ and similarly if $t \in T_{x_1}$, then $t = t^{u^{-1}} \in T_y$. Hence $T_y = T_{x_1}$, and $T \cap C(y)^\circ = T \cap C(x_1)^\circ$. To conclude note that $\mathcal{O} \cap wB$ is the T-orbit of x_1. □

Remark 3.6 In fact $C_L(x) = C_L(y)$ for every $x \in \mathcal{O} \cap wB$, since $L' \leq C(x)$.

Remark 3.7 In general it is not true that L_J is adapted to $C(x)$ for $x \in \mathcal{O} \cap wB$. For example if $\mathcal{O}$ is the minimal unipotent class, and u is a non-identity element in $X_{-\beta}$, where β is the highest root, then $C(u) \geq U^-$, so that there is a unique Levi subgroup of P adapted to $C(u)$ [6, Proposition 3.9], and this is L_J. Since $u \notin wB$, there is no element $x \in wB$ such that L_J is adapted to $C(x)$.

From Theorem 3.2 we get

Corollary 3.8 *Let $\mathcal{O}$ be a spherical conjugacy class, $w = w(\mathcal{O})$ and x any element in $\mathcal{O} \cap wB$. Then $H_0 = \langle U^-, U_{w_J}, T_x \rangle$, $w = w_0 w_J$.* □

By Proposition 3.5, we may put $T_{\mathcal{O}} = T_x$, for $x \in \mathcal{O} \cap wB$. Then $T_{\mathcal{O}} = T_y$ and $(T^w)^\circ \leq T_{\mathcal{O}} \leq T^w$ by [9], step 2 in the proof of Theorem 5.

We shall need the description of the monoid of weights λ such that $w(\lambda) = -\lambda$. In the next lemma we consider more generally w of the form $w = w_0 w_J$, where J is ϑ-invariant.

Lemma 3.9 *Let $J \subseteq \Pi$ be ϑ-invariant and $w = w_0 w_J$. The dominant weight λ satisfies $w(\lambda) = -\lambda$ if and only if $\lambda = \sum_{i \in \Pi \setminus J} n_i \omega_i$ with $n_{\vartheta(i)} = n_i$ for all $i \in \Pi \setminus J$. Moreover, $w(\lambda) = -\lambda$ implies $w_0(\lambda) = -\lambda$.*

Proof. Let $\lambda \in P^+$, $\lambda = \sum n_i \omega_i$, $n_i \in \mathbb{N}$. For $i \in \Pi \setminus J$ we have $w_J(\omega_i) = \omega_i$, so that $w(\omega_i) = -\omega_{\vartheta(i)}$.

It is clear that if $\lambda = \sum_{i \in \Pi \setminus J} n_i \omega_i$ with $n_i = n_{\vartheta(i)}$ for every $i \in \Pi \setminus J$, then $(w+1)(\lambda) = 0$. On the other hand, assume $w(\lambda) = -\lambda$. Then $w_J(\lambda) = -w_0\lambda$ and, by [20, Theorem 1.12 (a)], we get $-w_0\lambda = \lambda$ and $(\lambda, \alpha_j) = 0$ for every $j \in J$. Hence $n_j = 0$ for every $j \in J$. Moreover, from $\lambda = \sum_{i \in \Pi \setminus J} n_i \omega_i$ and $-w_0\lambda = \lambda$ it follows $n_{\vartheta(i)} = n_i$ for all $i \in \Pi \setminus J$. □

Remark 3.10 If S is a ϑ-orbit in $\Pi \setminus J$, and we put $\omega_S = \sum_{i \in S} \omega_i$ then we have seen that $\{\omega_S \mid S \in (\Pi \setminus J)/\vartheta\}$ is a basis of the monoid $\{\lambda \in P^+ \mid w(\lambda) = -\lambda\}$, where $(\Pi \setminus J)/\vartheta$ is the set of ϑ-orbits in $\Pi \setminus J$. If we also assume that w acts trivially on Φ_J (as in the case of $w = w(\mathcal{O})$), then $\{\omega_S \mid S \in (\Pi \setminus J)/\vartheta\}$ is a basis of $\ker(w+1)$ in E, and so a basis of the free abelian group $\{\lambda \in P \mid w(\lambda) = -\lambda\}$.

We describe $\tilde{\lambda}(\mathcal{O})$. For this purpose we denote by $S_{\mathcal{O}}$ any supplement of $(T^w)^\circ$ in $T_{\mathcal{O}}$ (i.e. $S_{\mathcal{O}}(T^w)^\circ = T_{\mathcal{O}}$). We also put $P^+_w = \{\lambda \in P^+ \mid w(\lambda) = -\lambda\}$. By Lemma 3.9 each element of P^+_w satisfies $-w_0\lambda = \lambda$, so that in particular any subset X of P^+_w is *symmetric*, i.e. $-w_0(X) = X$ [32, 4.2], [10, Theorem 4.17].

Theorem 3.11 *Let $\mathcal{O}$ be a spherical conjugacy class, $w = w(\mathcal{O})$ and let $S_{\mathcal{O}}$ be any supplement of $(T^w)^\circ$ in $T_{\mathcal{O}}$. Then*

$$\tilde{\lambda}(\mathcal{O}) = \{\lambda \in P^+_w \mid \lambda(S_{\mathcal{O}}) = 1\}$$

Proof. By (3.6), $\tilde{\lambda}(\mathcal{O}) = \{\lambda \in P^+ \mid \lambda(T_{\mathcal{O}}) = 1\}$. Since $(T^w)^\circ \leq T_{\mathcal{O}}$, a necessary condition for $\lambda \in P^+$ to be in $\tilde{\lambda}(\mathcal{O})$ is that $\lambda(t\,t^w) = 1$ for every $t \in T$, as $(T^w)^\circ = \{t\,t^w \mid t \in T\}$. This condition is equivalent to $(w+1)\lambda = 0$, so that $\tilde{\lambda}(\mathcal{O}) \leq P^+_w$. Let $\lambda \in P^+_w$: then $\lambda \in \tilde{\lambda}(\mathcal{O}) \Longleftrightarrow \lambda(S_{\mathcal{O}}) = 1$. □

We shall prove the crucial fact that $\tilde{\lambda}(\mathcal{O}) = \lambda(\mathcal{O})$, so that the monoid $\lambda(\mathcal{O})$ is *saturated* (that is $\mathbb{Z}\lambda(\mathcal{O}) \cap P^+ = \lambda(\mathcal{O})$, [34, Definition 1.3]. In the following, x is a fixed element in $\mathcal{O} \cap wB$ and $\dot{w}$ a representative of w in N such that $x = \dot{w}u$, $u \in U$. If $u = \prod_{\alpha \in \Phi^+} x_\alpha(k_\alpha)$, and $i \in \Pi$, we say that α_i *occurs* in x if $k_{\alpha_i} \neq 0$. This is independent of the chosen total ordering on Φ^+.

For the closure $\overline{\mathcal{O}}$ of $\mathcal{O}$ in G, the monoid $\lambda(\overline{\mathcal{O}})$ of dominant weights occurring in $k[\overline{\mathcal{O}}]$ is a submonoid of $\lambda(\mathcal{O})$. We start with

Proposition 3.12 *Let $\lambda \in P^+$. Then $(1-w)\lambda$ lies in $\lambda(\overline{\mathcal{O}})$.*

Proof. Let $f \in V(\lambda)^*_{-w\lambda}$, $v \in V(\lambda)_\lambda$ with $f(\dot{w}.v) = 1$. Then $c_{f,v}(t^{-1}gt) = c_{t.f,t.v}(g) = ((1-w)\lambda)(t)c_{f,v}(g)$ for every $t \in T$, $g \in G$. For every $z, z_1 \in U$ we have

$$c_{f,v}(z_1xz) = f(z_1\dot{w}\,uz.v) = f(z_1\dot{w}\,.v) = f(\dot{w}\,.v) = 1$$

since $z_1\dot{w}\,.v = \dot{w}\,.v + v_1$, where v_1 is a sum of weight vectors of weights strictly greater than $w\lambda$. Therefore for every $t \in T$, $z \in U$ we have

$$c_{f,v}(t^{-1}z^{-1}xzt) = ((1-w)\lambda)(t) \tag{3.8}$$

Since $B.x$ is dense in $\overline{\mathcal{O}}$, by (3.8) the restriction of $c_{f,v}$ to $\overline{\mathcal{O}}$ is a (non-zero) B-eigenvector of weight $(1-w)\lambda$ in $k[\overline{\mathcal{O}}]$. Hence $(1-w)\lambda \in \lambda(\overline{\mathcal{O}})$. □

Corollary 3.13 *Let $\lambda \in P^+_w$. Then 2λ lies in $\lambda(\overline{\mathcal{O}})$.* □

Corollary 3.14 *Let $\lambda \in P^+$. Then $(1-w)\lambda \in \lambda(\mathcal{O})$. If moreover $\lambda \in P^+_w$, then 2λ lies in $\lambda(\mathcal{O})$.*

Proof. This follows from the fact that $\lambda(\overline{\mathcal{O}}) \leq \lambda(\mathcal{O})$. □

We have shown that

$$2P^+_w \leq (1-w)P^+ \leq \lambda(\overline{\mathcal{O}}) \leq \lambda(\mathcal{O}) \leq \tilde{\lambda}(\mathcal{O}) \leq P^+_w \tag{3.9}$$

We can prove that $\lambda(\mathcal{O})$ is saturated.

Theorem 3.15 *Let $\mathcal{O}$ be a spherical conjugacy class. Then $\lambda(\mathcal{O})$ is saturated.*

Proof. Let $\lambda \in \tilde{\lambda}(\mathcal{O})$. We put $F(b^{-1}xb) = \lambda(b)$ for $b \in B$. We observed that F is well-defined since $C_B(x) = T_xU_{w_J}$ and gives rise to a B-eigenvector of weight λ in $k(\mathcal{O})$. Since $\mathcal{O}$ is quasi affine, we conclude that λ lies in $\lambda(\mathcal{O})$ by Theorem 3.11, Corollary 3.14 and Lemma 3.4. □

Theorem 3.15 in particular proves Conjecture 5.12 (and 5.10 and 5.11) in [36].

To deal with $\lambda(\overline{\mathcal{O}})$, in Sect. 5 we shall make use of

Proposition 3.16 *Let $\lambda \in P^+$, $i \in \Pi\backslash J$ be such that α_i occurs in x and $(\lambda, \alpha_i) \neq 0$. Then $(1-w)\lambda - \alpha_i \in \lambda(\overline{\mathcal{O}})$.*

Proof. Since $\langle\lambda, \alpha_i\rangle \neq 0$, $\lambda - \alpha_i$ is a weight of $V(\lambda)$. We construct two matrix coefficients. We fix a non-zero $v \in V(\lambda)_{\lambda-\alpha_i}$. By [43, Lemma 72], there exists a (unique) $v_\lambda \in V(\lambda)_\lambda$ such that $x_{\alpha_i}(k).v = v + kv_\lambda$ for every $k \in k$. Then we choose $f \in V(\lambda)^*_{-w\lambda}$ such that $f(\dot{w}.v_\lambda) = 1$.

Since α_i occurs in $x = \dot{w}\,u$, we have $u = x_{\alpha_i}(r)u'$, with $r \in k^*$, $u' \in \prod_{\beta\in\Phi^+\setminus\{\alpha_i\}} X_\beta$. Let $y, y_1 \in U$, and let $y = x_{\alpha_i}(k)y'$, $y' \in \prod_{\beta\in\Phi^+\setminus\{\alpha_i\}} X_\beta$, then

$$y_1^{-1}xy.v = y_1^{-1}\dot{w}.v + (k+r)y_1^{-1}\dot{w}.v_\lambda$$

The vector $\dot{w}.v$ has weight $w(\lambda - \alpha_i)$, so that $y_1^{-1}\dot{w}.v$ is a sum of weight vectors of weight $w(\lambda-\alpha_i) + \beta$, where β is a sum of simple roots with non-negative coefficients. Assume

$w\lambda = w(\lambda - \alpha_i) + \beta$ for a certain β. Then $w(\alpha_i) = \beta$ would be positive, a contradiction since $i \in \Pi\setminus J$. Hence $f(y_1^{-1}\dot{w}.v) = 0$. Similarly, $y_1^{-1}\dot{w}.v_\lambda = \dot{w}.v_\lambda + v'$, where v' is a sum of weight vectors of weights greater than $w\lambda$, hence $f(y_1^{-1}\dot{w}.v_\lambda) = f(\dot{w}.v_\lambda) = 1$, so that $c_{f,v}(y_1^{-1}xy) = k + r$.

The second matrix coefficient is defined dually. We fix a non-zero $f_1 \in V(-w_0\lambda)^*_{\lambda-\alpha_i}$. There exists a (unique) $f_\lambda \in V(-w_0\lambda)^*_\lambda$ such that $x_{\alpha_i}(k).f_1 = f_1 + kf_\lambda$ for every $k \in k$. Then we choose $v_1 \in V(-w_0\lambda)_{-w\lambda}$ such that $f_\lambda(\dot{w}.v_1) = 1$. Let $z, z_1 \in U$, $z_1 = x_{\alpha_i}(k_1)z'$, $z' \in \prod_{\beta\in\Phi^+\setminus\{\alpha_i\}} X_\beta$, then proceeding as before, we get $c_{f_1,v_1}(z_1^{-1}xz) = k_1$.

For $t \in T$, $z \in U$ we obtain

$$(c_{f,v} - c_{f_1,v_1})(t^{-1}z^{-1}xzt) = r\,((1-w)\lambda - \alpha_i)(t) \tag{3.10}$$

Since $B.x$ is dense in $\overline{\mathcal{O}}$, by (3.10) the restriction of $c_{f,v} - c_{f_1,v_1}$ to $\overline{\mathcal{O}}$ is a (non-zero) B-eigenvector of weight $(1-w)\lambda - \alpha_i$ in $k[\overline{\mathcal{O}}]$. Hence $(1-w)\lambda - \alpha_i \in \lambda(\overline{\mathcal{O}})$. □

Corollary 3.17 *Let $i \in \Pi\setminus J$ be such that α_i occurs in x. Then $\omega_i + \omega_{\vartheta(i)} - \alpha_i$ lies in $\lambda(\overline{\mathcal{O}})$.*

Proof. This follows from Proposition 3.16 by taking $\lambda = \omega_i$. □

We can deal with other homogeneous spaces related to $\mathcal{O}$. The simply-connected cover (or the universal covering, as in [22, p. 107]) $\hat{\mathcal{O}}$ of $\mathcal{O}$ can be identified with G/H°, since G is simply-connected.

Corollary 3.18 *Let $\mathcal{O}$ be a spherical conjugacy class, and let S be a supplement of $(T^w)^\circ$ in $T \cap C(x)^\circ$. Then $\lambda(\hat{\mathcal{O}}) = \{\lambda \in P^+_w \mid \lambda(S) = 1\}$ is saturated.*

Proof. By [16, Corollary 2.2], $\hat{\mathcal{O}}$ is quasi affine and, by [6, Propositions 5.1, 5.2], L is adapted to H°, so that $\tilde{\lambda}(\hat{\mathcal{O}}) = \tilde{\lambda}(G/H^\circ) = \{\lambda \in P^+_w \mid \lambda(S) = 1\}$, since $(T^w)^\circ \le T \cap H^\circ$. Let $\lambda \in \tilde{\lambda}(\hat{\mathcal{O}})$; then $F_\lambda : BH^\circ/H^\circ \to k$, $b^{-1}H^\circ \mapsto \lambda(b)$ is a regular function on BH°/H°, and therefore a B-eigenvector of weight λ in $k(G/H^\circ)$. By Corollary 3.14, $2\lambda \in \lambda(G/H) \le \lambda(G/H^\circ)$, and we conclude by Lemma 3.4 and Proposition 3.5. □

Corollary 3.19 *Let K be a closed subgroup of G with $H^\circ \le K \le N(H^\circ)$. Then $\lambda(G/K) = \tilde{\lambda}(G/K)$ (and $\lambda(G/K)$ is saturated).*

Proof. Since L is adapted to H, we get $N(H) = N(H^\circ) = H(C \cap N(H))$ by [6, Corollaire 5.2], P is the parabolic subgroup corresponding to $N(H)$ and L is adapted to $N(H)$ (by the proof of [6], Proposition 5.2 a). Clearly the same holds for K, since $BH = BK$.

By Corollary 3.18, $\lambda \in \lambda(G/H^\circ) \Leftrightarrow \lambda(T \cap H^\circ) = 1$. We prove that $\lambda \in \lambda(G/K) \Leftrightarrow \lambda(T \cap K) = 1$. In one direction $\lambda \in \lambda(G/K) \Rightarrow \lambda(T \cap K) = 1$, since $\lambda(G/K) \le \tilde{\lambda}(G/K)$. So assume $\lambda(T \cap K) = 1$. Then $\lambda(T \cap H^\circ) = 1$, so that $\lambda \in \lambda(G/H^\circ)$, and in particular $w_0\lambda = -\lambda$. Let v be a non-zero vector in $V(\lambda)^{H^0}$, and let $v = v_{-\lambda} + v'$, with $v_{-\lambda} \in V(\lambda)_{-\lambda}$, $v' \in \sum_{\mu > -\lambda} V(\lambda)_\mu$: then $v_{-\lambda} \ne 0$, since BH° is dense in G.

Since $V(\lambda)^{H^0}$ is 1-dimensional, there is a character γ of K, trivial on H°, such that $k.v = \gamma(k)v$ for $k \in K$. Since $K = H^\circ(T \cap K)$, v is K-invariant if and only if $\gamma(T \cap K) = 1$. But $v_{-\lambda} \ne 0$ implies $\gamma(k) = -\lambda(k)$ for every $k \in T \cap K$ so that v is K-invariant if and only if $\lambda(T \cap K) = 1$, and we are done. □

Remark 3.20 In general K is not quasi affine: for instance the centralizer H of $x_{-\beta}(1)$, β the highest root, contains U^-, and $T \le N(H)$. Then $N(H)$ is epimorphic, i.e. the minimal quasi affine subgroup of G containing $N(H)$ is G [16, p. 19, ex. 2]. To our knowledge, it was known that $\lambda(G/K)$ is saturated for symmetric varieties G/K, due to the work of Vust [46].

Proposition 3.21 *We have*

$$H/H^\circ \cong T_y/T\cap H^\circ = T_x/T\cap C(x)^\circ$$

Proof. We have $H = H^\circ(H\cap T) = H^\circ T_y$. Hence we get an epimorphism $\pi : T_y \to H/H^\circ$, inducing an isomorphism $\overline{\pi} : T_y/T\cap H^\circ \to H/H^\circ$, and we conclude by Proposition 3.5. $\square$

Corollary 3.22 *If T^w is connected, then H is connected.*

Proof. This follows from $(T^w)^\circ \le T\cap C(x)^\circ \le T_x \le T^w = (T^w)^\circ$ and Proposition 3.21. $\square$

Due to the fact that T is 2-divisible, we have the decomposition $T = (T^w)^\circ(S^w)^\circ$ where $S^w = \{t \in T \mid t^w = t^{-1}\}$. Let $t \in T^w$, $t = sz$, with $s \in (T^w)^\circ$, $z \in (S^w)^\circ$. Then $z = ts^{-1} \in T^w\cap(S^w)^\circ \le T^w\cap S^w \le T_2$, the elementary abelian 2-subgroup of T of rank n. We note that $(T^w)^\circ \cap (S^w)^\circ$ is finite, even though in general not trivial. Therefore $z \in T_2$, and $T^w \le (T^w)^\circ T_2$. In particular we have

$$T^w = (T^w)^\circ(T^w\cap(S^w)^\circ) = (T^w)^\circ(T^w\cap T_2)$$

and

$$T_x = (T^w)^\circ(C(x)\cap(S^w)^\circ) = (T^w)^\circ(C(x)\cap T_2)$$

Moreover every subgroup M of T_2 is a complemented group (i.e. for every subgroup X of M there exists a subgroup Y such that $XY = M$ and $X\cap Y = 1$), hence we may find a subgroup R of T_2 such that $T^w = (T^w)^\circ \times R$. Then $T_x = (T^w)^\circ \times (R\cap C(x))$ and $T\cap C(x)^\circ = (T^w)^\circ \times (R\cap C(x)^\circ)$. We put $S_{\mathcal{O}} = R\cap C(x)$, $S_{\hat{\mathcal{O}}} = R\cap C(x)^\circ$. We have therefore proved

Theorem 3.23 *Let $\mathcal{O}$ be a spherical conjugacy class, $w = w(\mathcal{O})$. Then*

$$\lambda(\mathcal{O}) = \{\lambda \in P_w^+ \mid \lambda(S_{\mathcal{O}}) = 1\}, \quad \lambda(\hat{\mathcal{O}}) = \{\lambda \in P_w^+ \mid \lambda(S_{\hat{\mathcal{O}}}) = 1\}$$

$\square$

From Proposition 3.21 it follows that H always splits over H°: if Y is a complement of $R\cap C(x)^\circ$ in $R\cap C(x)$, then Y is a complement of H° in H.

4 Description of $\boldsymbol{\lambda(\mathcal{O})}$ and $\boldsymbol{\lambda(\hat{\mathcal{O}})}$

In this section we explicitly determine the monoids $\lambda(\mathcal{O})$ and $\lambda(\hat{\mathcal{O}})$ for every spherical conjugacy class $\mathcal{O}$. The monoids $\lambda(\mathcal{O})$ have been described for all nilpotent orbits of height 2 in [36]. Moreover, the table in [36] describes the saturation of $\lambda(\mathcal{O})$ for nilpotent orbits of height 3: since, by Theorem 3.15, $\lambda(\mathcal{O})$ is saturated, the description of $\lambda(\mathcal{O})$ is known for all unipotent spherical conjugacy classes. The strategy in [36] follows Hesselink's approach [17], which exploits a resolution of singularities $G\times^Q V \to \overline{\mathcal{O}}$, where Q is a certain parabolic subgroup and V a reducible Q-module. Here we describe $\lambda(\mathcal{O})$ and $\lambda(\hat{\mathcal{O}})$ by means of Theorem 3.23. Unfortunately, we found some mistakes in the tables in [31].

From our discussion it is clear that to determine $\lambda(\mathcal{O})$ the most favourable case is when T^w is connected, so that $T_x = T^w = (T^w)^\circ$. In this case then $\lambda(\mathcal{O}) = \lambda(\hat{\mathcal{O}}) = P_w^+ = \{\sum_{i\in\Pi\setminus J} n_i\omega_i \mid n_{\vartheta(i)} = n_i\}$. We note that of course we have $Z(G) \le T_x$, so that it is also straightforward to determine $\lambda(\mathcal{O})$ even when $T^w = (T^w)^\circ Z(G)$, so that $T_x = T^w$.

In general it is quite cumbersome to determine T_x. Our strategy will be to determine T^w as $T^w = (T^w)^\circ \times R$, and then determine $R \cap C(x)$. To deal with unipotent classes, we shall usually start from the maximal one, (corresponding to w_0), and then deal with the remaining classes by an inductive procedure. In some cases we shall use an explicit form of an element x (in $\mathcal{O} \cap wB$), while in some other cases we shall determine $T \cap C(x)$ by analysing the form of eventual involutions in $T_x \backslash Z(G)(T^w)^\circ$. Note that when T^w is connected (or $T^w = (T^w)^\circ Z(G)$), it is not necessary to have an explicit description of $x \in \mathcal{O} \cap wB$ (however in certain cases it will be necessary to have such a description in Sect. 6).

We use the fact that if $G_1 \subset G_2$ are reductive algebraic groups and u is a unipotent element in G_1 such that the conjugacy class of u in G_2 is spherical, then the conjugacy class of u in G_1 is spherical [33, Corollary 2.3, Theorem 3.1].

The character group $X(T^w)$ is isomorphic to $P/(1-w)P$, since $P = X(T)$. Therefore T^w is connected if and only if $P/(1-w)P$ is torsion free. We are reduced to calculate elementary divisors of the endomorphism $1-w$ of P. We shall use the following results.

Lemma 4.1 *Assume the positive roots $\beta_i, \dots, \beta_\ell$ are long and pairwise orthogonal. Then, for $\xi_1, \dots, \xi_\ell \in k^*$ and $g = x_{\beta_1}(-\xi_1^{-1}) \cdots x_{\beta_\ell}(-\xi_\ell^{-1})$ we have*

$$gx_{-\beta_1}(\xi_1) \cdots x_{-\beta_\ell}(\xi_\ell)g^{-1} = n_{\beta_1} \cdots n_{\beta_\ell} h x_{\beta_1}(2\xi_1^{-1}) \cdots x_{\beta_\ell}(2\xi_\ell^{-1})$$

for a certain $h \in T$.

Proof. By (2.2) we have $x_\alpha(-\xi^{-1})x_{-\alpha}(\xi)x_\alpha(\xi^{-1}) = n_\alpha h_\alpha(-\xi)x_\alpha(2\xi^{-1})$. Hence we get the result with $h = h_{\beta_1}(-\xi_1) \cdots h_{\beta_\ell}(-\xi_\ell)$. □

Proposition 4.2 *Let $\alpha \in \Phi$. Then T^{s_α} is connected except in the following cases*:

(i) *G is of type A_1*;
(ii) *G is of type C_n and α is long*;
(iii) *G is of type B_2 and α is long.*

In these cases we have $T^{s_\alpha} = (T^{s_\alpha})^\circ \times Z(G)$.

Proof. It is enough to determine in which cases the non-zero elementary divisor of $1 - s_i$ is not 1. Since $(1-s_i)\omega_j = \delta_{ij}\alpha_i$ and $\alpha_i = \sum_k a_{ik}\omega_k$, this happens only for G of type A_1 and $i = 1$, C_n and $i = n$, or B_2 and $i = 1$ [18, p. 59]. In these cases the non-zero elementary divisor is 2, and $T^{s_{\alpha_i}} = (T^{s_{\alpha_i}})^\circ \times Z(G)$. □

Lemma 4.3 *Let M be a connected algebraic group, S a torus of M, g a semisimple element in $C_M(S)$. Then $\langle S, g\rangle$ is contained in a torus of M.*

Proof. See [18, Corollary 22.3 B]. □

Lemma 4.4 *Assume K is a connected spherical subgroup of G with no non-trivial characters. Then the monoid $\lambda(G/K)$ is free.*

Proof. We recall that we are assuming G simply-connected, so that by [16], Theorem 20.2, ${}^U k[G/K]$ is a polynomial algebra. But ${}^U k[G/K]$ is the monoid algebra of $\lambda(G/K)$ and the monoid algebra is factorial if and only if $\lambda(G/K)$ is free (see the proof of [32], Proposition 2). □

Lemma 4.5 *Let V be a G-module, $g \in G$, such that the image Q of the endomorphism $p(g)$ of V is 1 dimensional for a certain polynomial p. Assume $M \leq C(g)$ has no non-trivial characters. Then M acts trivially on Q.*

Proof. This is clear. □

Let $S = \{i, \vartheta(i)\}$ be a ϑ-orbit in $\Pi \setminus J$ consisting of 2 elements. We put $H_S = \{h_{\alpha_i}(z)h_{\alpha_{\vartheta(i)}}(z^{-1}) \mid z \in k^*\}$. Let $\mathcal{S}_1$ be the set of ϑ-orbits in $\Pi \setminus J$ consisting of 2 elements. Then, by Remark 3.10, $\Delta_J \cup \{\alpha_i - \alpha_{\vartheta(i)}\}_{\mathcal{S}_1}$ is a basis of $\ker(1 - w)$ and

$$(T^w)^\circ = \prod_{j \in J} H_{\alpha_j} \times \prod_{S \in \mathcal{S}_1} H_S \tag{4.11}$$

We put $\Psi_J = \{\beta \in \Phi \mid w(\beta) = -\beta\}$. Then Ψ_J is a root system in $\mathrm{Im}(1 - w)$ [40, Proposition 2], and $w_{|\mathrm{Im}(1-w)}$ is -1. If $K = C((T^w)^\circ)'$, then K is semisimple with root system Ψ_J and maximal torus $T(K) := T \cap K = (S^w)^\circ$.

If g is in $Z(G)$, then $\mathcal{O}_g = \{g\}$, $w = 1$ and $k[\mathcal{O}_g] = k$. For each spherical non-central conjugacy class $\mathcal{O}$ we give the corresponding J and w as a product of commuting reflections (writing $\mathcal{O} \longleftrightarrow J \longleftrightarrow w$) using the tables in [9]. We give tables with corresponding $\lambda(\mathcal{O})$ and $\lambda(\hat{\mathcal{O}})$ (for semisimple classes we also give the type of the centralizer of elements in $\mathcal{O}$). In the cases when $\lambda(\hat{\mathcal{O}}) = \lambda(\mathcal{O})$, we leave a blank entry. For length reasons we shall give proofs only for some classes. Just for notational convenience, we work over the complex numbers, but the tables hold over any algebraically closed field k of characteristic zero. In [9] for the classical groups we gave representative of semisimple conjugacy classes in $SL(n)$, $Sp(n)$ and $SO(n)$. Here we shall give an expression in terms of exp.

We recall that a quasi affine homogeneous space G/H is called a *model homogeneous space* if $k[G/H]$ contains every simple G-module with multiplicity one. Moreover the model homogeneous space G/H is called *principal* if $\dim G/N(H)$ is maximal (see [28], Definition 3.1).

4.1 Type A_n, $n \geq 1$

Let $m = \left[\frac{n+1}{2}\right]$, $\beta_i = e_i - e_{n+2-i}$, for $i = 1, \ldots, m$. For $\ell = 1, \ldots, m - 1$ we put $J_\ell = \{\ell + 1, \ldots, n - \ell\}$, $J_m = \varnothing$.

4.1.1 Unipotent classes in A_n

If we denote by X_i the unipotent class $(2^i, 1^{n+1-2i})$, then

$$X_\ell \longleftrightarrow J_\ell \longleftrightarrow s_{\beta_1} \cdots s_{\beta_\ell}$$

for $\ell = 1, \ldots, m$ (here $w_0 = s_{\beta_1} \cdots s_{\beta_m}$).

In this case T^w is almost always connected. There is only one case when it is not connected, namely when n is odd, $n + 1 = 2m$, and $w = w_0$. However in this case we have $T^{w_0} = (T^{w_0})^\circ Z(G) = (T^{w_0})^\circ \times \langle h_{\alpha_m}(-1) \rangle$.

We get

Table 1 $\lambda(\mathcal{O})$, $\lambda(\hat{\mathcal{O}})$ for unipotent classes in A_n

$\mathcal{O}$	$\lambda(\mathcal{O})$	$\lambda(\hat{\mathcal{O}})$
$(2^\ell, 1^{n+1-2\ell})$ $\ell = 1, \ldots, m-1$	$\sum_{k=1}^{\ell} n_k(\omega_k + \omega_{n-k+1})$	
$(2^m, 1)$ $n = 2m$	$\sum_{k=1}^{m} n_k(\omega_k + \omega_{n-k+1})$	
(2^m) $n+1 = 2m$	$\sum_{k=1}^{m-1} n_k(\omega_k + \omega_{n-k+1}) + 2n_m\omega_m$	$\sum_{k=1}^{m-1} n_k(\omega_k + \omega_{n-k+1}) + n_m\omega_m$

In particular $\hat{X}_1$ is a model homogeneous space for $SL(2)$, and in fact the principal one, by [28, 3.3 (1)].

4.1.2 *Semisimple classes in A_n*

Following the notation in [9], Tables 1 and 5 we get

$$T_1 A_{\ell-1} A_{n-\ell} \longleftrightarrow J_\ell \longleftrightarrow s_{\beta_1} \cdots s_{\beta_\ell}$$

for $\ell = 1, \ldots, m$.

We get

Table 2 $\lambda(\mathcal{O})$ for semisimple classes in A_n

$\mathcal{O}$	H	$\lambda(\mathcal{O})$
$\exp(\zeta\check{\omega}_\ell)$ $\zeta \in \mathbb{C}\backslash 2\pi i\mathbb{Z}$ $\ell = 1, \ldots, m-1$	$T_1 A_{\ell-1} A_{n-\ell}$	$\sum_{k=1}^{\ell} n_k(\omega_k + \omega_{n-k+1})$
$\exp(\zeta\check{\omega}_m)$ $\zeta \in \mathbb{C}\backslash 2\pi i\mathbb{Z}$ $n = 2m$	$T_1 A_{m-1} A_m$	$\sum_{k=1}^{m} n_k(\omega_k + \omega_{n-k+1})$
$\exp(\zeta\check{\omega}_m)$ $\zeta \in \mathbb{C}\backslash 2\pi i\mathbb{Z}$ $n+1 = 2m$	$T_1 A_{m-1} A_{m-1}$	$\sum_{k=1}^{m-1} n_k(\omega_k + \omega_{n-k+1}) + 2n_m\omega_m$

4.2 Type C_n, $n \geq 2$

We have $\omega_\ell = e_1 + \cdots + e_\ell$ for $\ell = 1, \ldots, n$ and $Z(G) = \langle z \rangle$, where $z = \prod_{i=1}^{[\frac{n+1}{2}]} h_{\alpha_{2i-1}}(-1)$.

4.2.1 *Unipotent classes in C_n*

For $i = 1, \ldots, n$ we denote by X_i the unipotent class $(2^i, 1^{2n-2i})$ and we put $\beta_i = 2e_i$, $J_i = \{i+1, \ldots, n\}$ ($J_n = \varnothing$). Then

$$X_\ell \longleftrightarrow J_\ell \longleftrightarrow s_{\beta_1} \cdots s_{\beta_\ell}$$

for $\ell = 1, \ldots, n$ (here $w_0 = s_{\beta_1} \cdots s_{\beta_n}$).

Lemma 4.6 *Let $w = s_{\beta_1} \cdots s_{\beta_\ell}$ for $\ell = 1, \ldots, n$. Then*

$$T^w = (T^w)^\circ \times R, \quad R = \langle h_{\alpha_1}(-1)\rangle \times \cdots \times \langle h_{\alpha_\ell}(-1)\rangle$$

Proof. For $\ell = 1, \ldots, n$ we have $(1-w)P = \mathbb{Z}\langle 2\omega_1, \ldots, 2\omega_\ell\rangle$. □

Proposition 4.7 *For $\ell = 1, \ldots, n$ we have*

$$\lambda(X_\ell) = \{2n_1\omega_1 + \cdots + 2n_\ell\omega_\ell \mid n_k \in \mathbb{N}\}$$

Proof. In [9] we exhibit the element $x_{-\beta_1}(1) \cdots x_{-\beta_\ell}(1) \in \mathcal{O} \cap BwB \cap B^-$. By Lemma 4.1, we can choose

$$x = n_{\beta_1} \cdots n_{\beta_\ell} h\, x_{\beta_1}(2) \cdots x_{\beta_\ell}(2) \in \mathcal{O} \cap wB$$

for a certain $h \in T$. Let now $t \in R$. Then $t \in C(x) \Leftrightarrow \beta_i(t) = 1$ for $i = 1, \ldots, \ell$. But $\mathbb{Z}\langle \beta_1, \ldots, \beta_\ell\rangle = \mathbb{Z}\langle 2\omega_1, \ldots, 2\omega_\ell\rangle$, so that $R \leq T_x$, and $T_x = T^w$. □

Proposition 4.8 *For $\ell = 1, \ldots, n$ we have*

$$\lambda(\hat{X}_\ell) = \{2n_1\omega_1 + \cdots + 2n_{\ell-1}\omega_{\ell-1} + n_\ell\omega_\ell \mid n_k \in \mathbb{N}\}$$

Proof. We have $R \cap C(x)^\circ = \langle h_{\alpha_1}(-1), \ldots, h_{\alpha_{\ell-1}}(-1)\rangle$. In fact, for $i = 1 \ldots, \ell - 1$

$$e_{\alpha_i} - e_{-\alpha_i} \in C_{\mathfrak{g}}(\langle x_{\beta_1}(\xi) \cdots x_{\beta_\ell}(\xi)\rangle)$$

for every $\xi \in \mathbb{C}$, so that $h_{\alpha_i}(-1) = \exp(\pi(e_{\alpha_i} - e_{-\alpha_i})) \in C(x)^\circ$. On the other hand the reductive part of $C(x)$ is of type $Sp(2n-2\ell) \times O(\ell)$, so that $C(x)/C(x)^\circ$ has order 2, and we are done. □

Hence

Table 3 $\lambda(\mathcal{O})$, $\lambda(\hat{\mathcal{O}})$ for unipotent classes in C_n

$\mathcal{O}$	$\lambda(\mathcal{O})$	$\lambda(\hat{\mathcal{O}})$
$(2^\ell, 1^{2n-2\ell})$ $\ell = 1, \ldots, n$	$\sum_{i=1}^{\ell} 2n_i\omega_i$	$\sum_{i=1}^{\ell-1} 2n_i\omega_i + n_\ell\omega_\ell$

4.2.2 Semisimple classes in C_n

Let $p = [\frac{n}{2}]$. We put $\gamma_\ell = e_{2\ell-1} + e_{2\ell}$, $K_\ell = \{1, 3, \ldots, 2\ell-1, 2\ell+1, 2\ell+2, \ldots, n\}$ for $\ell = 1, \ldots, p$. Then, following the notation in [9], Tables 1 and 5 we have

$$\begin{array}{llll} C_\ell C_{n-\ell}, \quad \ell = 1, \ldots, p & \longleftrightarrow K_\ell & \longleftrightarrow & s_{\gamma_1} \cdots s_{\gamma_\ell} \\ T_1 C_{n-1} & \longleftrightarrow J_2 & \longleftrightarrow & s_{\beta_1} s_{\beta_2} \\ T_1 \tilde{A}_{n-1} & \longleftrightarrow \varnothing & \longleftrightarrow & w_0 \end{array}$$

We get

Table 4 $\lambda(\mathcal{O})$ for semisimple classes in C_n

$\mathcal{O}$	H	$\lambda(\mathcal{O})$
$\exp(\zeta\check{\omega}_n)$ $\zeta\in\mathbb{C}\backslash 2\pi i\mathbb{Z}$	$T_1\tilde{A}_{n-1}$	$\sum_{k=1}^{n} 2n_k\omega_k$
$\exp(\zeta\check{\omega}_1)$ $\zeta\in\mathbb{C}\backslash \pi i\mathbb{Z}$	T_1C_{n-1}	$2n_1\omega_1+n_2\omega_2$
$\exp(\pi i\check{\omega}_\ell)$ $\ell=1,\dots,[\frac{n}{2}]$	$C_\ell C_{n-\ell}$	$\sum_{i=1}^{\ell} n_{2i}\,\omega_{2i}$

4.2.3 Mixed classes in C_n

We put $p=[\frac{n}{2}]$. From [9], Table 4, we get

$$\begin{array}{llll} \sigma_p x_{\alpha_n}(1) & \longleftrightarrow \varnothing & \longleftrightarrow w_0 \\ \sigma_k x_{\alpha_n}(1),\ k=1,\dots,p-1 & \longleftrightarrow J_{2k+1} & \longleftrightarrow s_{\beta_1}\cdots s_{\beta_{2k+1}} \\ \sigma_k x_{\beta_1}(1),\ k=1,\dots,p & \longleftrightarrow J_{2k} & \longleftrightarrow s_{\beta_1}\cdots s_{\beta_{2k}} \end{array}$$

Note that when n is even, then $\sigma_p x_{\beta_1}(1)\sim z\sigma_p x_{\alpha_n}(1)$. We obtain

Table 5 $\lambda(\mathcal{O})$, $\lambda(\hat{\mathcal{O}})$ for mixed classes in C_n

$\mathcal{O}$	$\lambda(\mathcal{O})$	$\lambda(\hat{\mathcal{O}})$
$\sigma_p x_{\alpha_n}(1)$	$\sum_{i=1}^{n} n_i\omega_i,\ \sum_{i=1}^{[\frac{n+1}{2}]} n_{2i-1}$ even	$\sum_{i=1}^{n} n_i\omega_i$
$\sigma_\ell x_{\alpha_n}(1)$ $\ell=1,\dots,[\frac{n}{2}]-1$	$\sum_{i=1}^{2\ell+1} n_i\omega_i,\ \sum_{i=1}^{\ell+1} n_{2i-1}$ even	$\sum_{i=1}^{2\ell+1} n_i\omega_i$
$\sigma_\ell x_{\beta_1}(1)$ $\ell=1,\dots,[\frac{n}{2}]$	$\sum_{i=1}^{2\ell} n_i\omega_i,\ \sum_{i=1}^{\ell} n_{2i-1}$ even	$\sum_{i=1}^{2\ell} n_i\omega_i$

In particular $\hat{\mathcal{O}}_{\sigma_p x_{\alpha_n}(1)}$ is a model homogeneous space, and in fact the principal one, by [28, 3.3 (3)].

4.3 Type D_n, $n\geq 4$

To deal with types D_n and B_n, we denote by X_i the unipotent class which in $SO(s)$ has canonical form $(2^{2i},1^{s-4i})$, $i=1,\dots,\left[\frac{s}{4}\right]$ (for $s=4m$, $i=m$ there are 2 classes of this form: X_m and X'_m, the *very even* classes) and by Z_i the unipotent class $(3,2^{2(i-1)},1^{s-4i+1})$, $i=1,\dots,1+\left[\frac{s-3}{4}\right]$.

Assume G is of type D_n, $n\geq 4$. Let $m=\left[\frac{n}{2}\right]$. We have $\omega_i=e_1+\cdots+e_i$ for $i=1,\dots,n-2$, $\omega_{n-1}=\frac{1}{2}(e_1+\cdots+e_{n-1})-\frac{1}{2}e_n$, $\omega_n=\frac{1}{2}(e_1+\cdots+e_n)$. We put $\beta_i=e_{2i-1}+e_{2i}$, $\delta_i=e_{2i-1}-e_{2i}$ for $i=1,\dots,m$. For $\ell=1,\dots,m-1$ we put $J_\ell=\{2\ell+1,\dots,n\}$, $J_m=\varnothing$, $K_\ell=J_\ell\cup\{1,3,\dots,2\ell-1\}$ for $\ell=1,\dots,m$.

4.3.1 Unipotent classes in D_n, n even, $n = 2m$

The centre of G is $\langle \prod_{i=1}^{m} h_{\alpha_{2i-1}}(-1), h_{\alpha_{n-1}}(-1)h_{\alpha_n}(-1)\rangle$. From [9] we get

$$\begin{array}{lll} Z_\ell,\ \ell=1,\dots,m \longleftrightarrow J_\ell & \longleftrightarrow s_{\beta_1}s_{\delta_1}\cdots s_{\beta_\ell}s_{\delta_\ell} \\ X_\ell,\ \ell=1,\dots,m \longleftrightarrow K_\ell & \longleftrightarrow s_{\beta_1}\cdots s_{\beta_\ell} \\ X'_m \qquad\qquad \longleftrightarrow \{1,3,\dots,n-3,n\} & \longleftrightarrow s_{\beta_1}\cdots s_{\beta_{m-1}}s_{\alpha_{n-1}} \end{array}$$

We just point out that

$$T_x = \begin{cases} Z(G) & \text{for}\quad x\in Z_m\cap wB \\ (T^w)^\circ \times \langle \prod_{i=1}^{\ell} h_{\alpha_{2i-1}}(-1)\rangle & \text{for}\quad x\in Z_\ell\cap wB,\ \ell=1,\dots m-1 \end{cases}$$

We get

Table 6 $\lambda(\mathcal{O})$, $\lambda(\hat{\mathcal{O}})$ for unipotent classes in D_n, $n = 2m$

$\mathcal{O}$	$\lambda(\mathcal{O})$	$\lambda(\hat{\mathcal{O}})$
$(2^{2\ell}, 1^{2n-4\ell})$ $\ell = 1,\dots,m-1$	$\sum_{i=1}^{\ell} n_{2i}\,\omega_{2i}$	
(2^{2m})	$\sum_{i=1}^{m-1} n_{2i}\,\omega_{2i} + 2n_n\omega_n$	$\sum_{i=1}^{m-1} n_{2i}\,\omega_{2i} + n_n\omega_n$
$(2^{2m})'$	$\sum_{i=1}^{m-1} n_{2i}\,\omega_{2i} + 2n_{n-1}\omega_{n-1}$	$\sum_{i=1}^{m-1} n_{2i}\,\omega_{2i} + n_{n-1}\omega_{n-1}$
$(3, 2^{2(\ell-1)}, 1^{2n-4\ell+1})$ $\ell = 1,\dots,m-1$	$\sum_{i=1}^{2\ell} n_i\omega_i$, $\sum_{i=1}^{\ell} n_{2i-1}$ even	$\sum_{i=1}^{2\ell} n_i\omega_i$
$(3, 2^{2(m-1)}, 1)$	$\sum_{i=1}^{n} n_i\omega_i$, $\sum_{i=1}^{m} n_{2i-1}$ even, $n_{n-1}+n_n$ even	$\sum_{i=1}^{n} n_i\omega_i$

In particular $\hat{Z}_m$ is a model homogeneous space, and in fact the principal one, by [28, 3.3 (4)].

4.3.2 Semisimple classes in D_n, n even $n = 2m$

Following the notation in [9], Tables 1 and 5 we have

$$\begin{array}{lll} D_\ell D_{n-\ell} & \longleftrightarrow J_\ell,\quad \ell=1,\dots,m & \longleftrightarrow s_{\beta_1}s_{\delta_1}\cdots s_{\beta_\ell}s_{\delta_\ell} \\ T_1A_{n-1} & \longleftrightarrow K_m, & \longleftrightarrow s_{\beta_1}\cdots s_{\beta_m} \\ (T_1A_{n-1})' & \longleftrightarrow \{1,3,\dots,n-3,n\} & \longleftrightarrow s_{\beta_1}\cdots s_{\beta_{m-1}}s_{\alpha_{n-1}} \end{array}$$

There are two families of classes of semisimple elements with centralizer of type T_1A_{n-1}: to distinguish them we wrote T_1A_{n-1} and $(T_1A_{n-1})'$. We get

Table 7 $\lambda(\mathcal{O})$ for semisimple classes in D_n, $n = 2m$

$\mathcal{O}$	H	$\lambda(\mathcal{O})$
$\exp(\zeta\check{\omega}_1)$ $\zeta \in \mathbb{C}\backslash 2\pi i\mathbb{Z}$	$T_1 D_{n-1}$	$2n_1\omega_1 + n_2\omega_2$
$\exp(\pi i\check{\omega}_\ell)$ $\ell = 2, \ldots, m-1$	$D_\ell D_{n-\ell}$	$\sum_{i=1}^{2\ell-1} 2n_i\omega_i + n_{2\ell}\omega_{2\ell}$
$\exp(\pi i\check{\omega}_m)$	$D_m D_m$	$\sum_{i=1}^{n} 2n_i\omega_i$
$\exp(\zeta\check{\omega}_n)$ $\zeta \in \mathbb{C}\backslash 2\pi i\mathbb{Z}$	$T_1 A_{n-1}$	$\sum_{i=1}^{m-1} n_{2i}\,\omega_{2i} + 2n_n\omega_n$
$\exp(\zeta\check{\omega}_{n-1})$ $\zeta \in \mathbb{C}\backslash 2\pi i\mathbb{Z}$	$(T_1 A_{n-1})'$	$\sum_{i=1}^{m-1} n_{2i}\,\omega_{2i} + 2n_{n-1}\omega_{n-1}$

4.3.3 Unipotent classes in D_n, n odd, $n = 2m+1$

The centre of G is $\langle(\prod_{j=1}^{m} h_{\alpha_{2j-1}}(-1))h_{\alpha_{n-1}}(i)h_{\alpha_n}(-i)\rangle$. From [9] we have

$$\begin{array}{l} Z_\ell,\ \ell = 1, \ldots, m \longleftrightarrow J_\ell \longleftrightarrow s_{\beta_1}s_{\delta_1}\cdots s_{\beta_\ell}s_{\delta_\ell} \\ X_\ell,\ \ell = 1, \ldots, m \longleftrightarrow K_\ell \longleftrightarrow s_{\beta_1}\cdots s_{\beta_\ell} \end{array}$$

We get

Table 8 $\lambda(\mathcal{O})$, $\lambda(\hat{\mathcal{O}})$ for unipotent classes in D_n, $n = 2m+1$

$\mathcal{O}$	$\lambda(\mathcal{O})$	$\lambda(\hat{\mathcal{O}})$
$(2^{2\ell}, 1^{2n-4\ell})$ $\ell = 1, \ldots, m-1$	$\sum_{i=1}^{\ell} n_{2i}\,\omega_{2i}$	
$(2^{2m}, 1^2)$	$\sum_{i=1}^{m-1} n_{2i}\,\omega_{2i} + n_{n-1}(\omega_{n-1} + \omega_n)$	
$(3, 2^{2(\ell-1)}, 1^{2n-4\ell+1})$ $\ell = 1, \ldots, m-1$	$\sum_{i=1}^{2\ell} n_i\omega_i$, $\sum_{i=1}^{\ell} n_{2i-1}$ even	$\sum_{i=1}^{2\ell} n_i\omega_i$
$(3, 2^{2(m-1)}, 1^3)$	$\sum_{i=1}^{n-2} n_i\omega_i + n_{n-1}(\omega_{n-1} + \omega_n)$, $\sum_{i=1}^{m} n_{2i-1}$ even	$\sum_{i=1}^{n-2} n_i\omega_i + n_{n-1}(\omega_{n-1} + \omega_n)$

4.3.4 Semisimple classes in D_n, n odd, $n = 2m+1$

$$\begin{array}{lll} D_\ell D_{n-\ell}, & \ell = 1, \ldots, m \longleftrightarrow J_\ell & \longleftrightarrow s_{\beta_1}s_{\delta_1}\cdots s_{\beta_\ell}s_{\delta_\ell} \\ T_1 A_{n-1} & \qquad\qquad\ \ \longleftrightarrow K_m & \longleftrightarrow s_{\beta_1}\cdots s_{\beta_m} \end{array}$$

We obtain

Table 9 $\lambda(\mathcal{O})$ for semisimple classes in D_n, $n = 2m+1$

$\mathcal{O}$	H	$\lambda(\mathcal{O})$
$\exp(\zeta\check{\omega}_1)$ $\zeta \in \mathbb{C}\setminus 2\pi i\mathbb{Z}$	$T_1 D_{n-1}$	$2n_1\omega_1 + n_2\omega_2$
$\exp(\pi i\check{\omega}_\ell)$ $\ell = 2, \ldots, m-1$	$D_\ell D_{n-\ell}$	$\sum_{i=1}^{2\ell-1} 2n_i\omega_i + n_{2\ell}\omega_{2\ell}$
$\exp(\pi i\check{\omega}_m)$	$D_m D_{m+1}$	$\sum_{i=1}^{n-2} 2n_i\omega_i + n_{n-1}(\omega_{n-1}+\omega_n)$
$\exp(\zeta\check{\omega}_n)$ $\zeta \in \mathbb{C}\setminus 2\pi i\mathbb{Z}$	$T_1 A_{n-1}$	$\sum_{i=1}^{m-1} n_{2i}\,\omega_{2i} + n_{n-1}(\omega_{n-1}+\omega_n)$

4.4 Type B_n, $n \geq 2$

We put $m = [\frac{n}{2}]$. The centre of G is $\langle h_{\alpha_n}(-1)\rangle$. We have $\omega_i = e_1 + \cdots + e_i$ for $i = 1, \ldots, n-1$, $\omega_n = \frac{1}{2}(e_1 + \cdots + e_n)$. We put $\gamma_\ell = e_\ell$, $M_\ell = \{\ell+1, \ldots, n\}$ for $\ell = 1, \ldots, n$ and $\beta_i = e_{2i-1} + e_{2i}$, $\delta_i = e_{2i-1} - e_{2i}$, $J_\ell = \{2\ell+1, \ldots, n\}$, $K_\ell = J_\ell \cup \{1, 3, \ldots, 2\ell-1\}$ for $i = 1, \ldots, m$.

4.4.1 Unipotent classes in B_n, n even, $n = 2m$

$$\begin{array}{l} Z_\ell,\ \ell = 1, \ldots, m \longleftrightarrow J_\ell \longleftrightarrow s_{\beta_1}s_{\delta_1}\cdots s_{\beta_\ell}s_{\delta_\ell} \\ X_\ell,\ \ell = 1, \ldots, m \longleftrightarrow K_\ell \longleftrightarrow s_{\beta_1}\cdots s_{\beta_\ell} \end{array}$$

We obtain

Table 10 $\lambda(\mathcal{O})$, $\lambda(\hat{\mathcal{O}})$ for unipotent classes in B_n, $n = 2m$

$\mathcal{O}$	$\lambda(\mathcal{O})$	$\lambda(\hat{\mathcal{O}})$
$(2^{2\ell}, 1^{2n+1-4\ell})$ $\ell = 1, \ldots, m-1$	$\sum_{i=1}^{\ell} n_{2i}\,\omega_{2i}$	
$(2^{2m}, 1)$	$\sum_{i=1}^{m-1} n_{2i}\,\omega_{2i} + 2n_n\omega_n$	$\sum_{i=1}^{m} n_{2i}\,\omega_{2i}$
$(3, 2^{2(\ell-1)}, 1^{2n+2-4\ell})$ $\ell = 1, \ldots, m-1$	$\sum_{i=1}^{2\ell} n_i\omega_i$, $\sum_{i=1}^{\ell} n_{2i-1}$ even	$\sum_{i=1}^{2\ell} n_i\omega_i$
$(3, 2^{2(m-1)}, 1^2)$	$\sum_{i=1}^{n} n_i\omega_i$, $\sum_{i=1}^{m} n_{2i-1}$ even, n_n even	$\sum_{i=1}^{n} n_i\omega_i$, n_n even

4.4.2 Semisimple classes in B_n, n even $n = 2m$

Following the notation in [9], Tables 1 and 5 we have

$$\begin{array}{lll} D_\ell B_{n-\ell},\ \ell=1,\ldots,m & \longleftrightarrow J_\ell & \longleftrightarrow s_{\beta_1}s_{\delta_1}\cdots s_{\beta_\ell}s_{\delta_\ell} \\ D_\ell B_{n-\ell},\ \ell=m+1,\ldots,n & \longleftrightarrow M_{2(n-\ell)+1} & \longleftrightarrow s_{\gamma_1}s_{\gamma_2}\cdots s_{\gamma_{2(n-\ell)+1}} \\ T_1A_{n-1} & \longleftrightarrow \varnothing & \longleftrightarrow w_0 \end{array}$$

We obtain

Table 11 $\lambda(\mathcal{O})$ for semisimple classes in B_n, $n=2m$

$\mathcal{O}$	H	$\lambda(\mathcal{O})$
$\exp(\zeta\check{\omega}_1)$ $\zeta\in\mathbb{C}\setminus 2\pi i\mathbb{Z}$, $m\geq 2$	T_1B_{n-1}	$2n_1\omega_1+n_2\omega_2$
$\exp(\zeta\check{\omega}_1)$ $\zeta\in\mathbb{C}\setminus 2\pi i\mathbb{Z}$, $m=1$	T_1B_1	$2n_1\omega_1+2n_2\omega_2$
$\exp(\pi i\check{\omega}_\ell)$ $\ell=2,\ldots,m-1$	$D_\ell B_{n-\ell}$	$\sum_{i=1}^{2\ell-1}2n_i\omega_i+n_{2\ell}\omega_{2\ell}$
$\exp(\pi i\check{\omega}_m)$	D_mB_m	$\sum_{i=1}^{n}2n_i\omega_i$
$\exp(\pi i\check{\omega}_\ell)$ $\ell=m+1,\ldots,n$	$D_\ell B_{n-\ell}$	$\sum_{i=1}^{2(n-\ell)}2n_i\omega_i+n_{2(n-\ell)+1}\omega_{2(n-\ell)+1}$
$\exp(\zeta\check{\omega}_n)$ $\zeta\in\mathbb{C}\setminus\pi i\mathbb{Z}$	T_1A_{n-1}	$\sum_{i=1}^{n}n_i\omega_i$, n_n even

4.4.3 *Mixed classes in B_n, n even, $n=2m$*

From [9], Table 4, we get

$$\begin{array}{lll} \sigma_n x_{\beta_1}(1)\cdots x_{\beta_m}(1) & \longleftrightarrow \varnothing & \longleftrightarrow w_0 \\ \sigma_n x_{\beta_1}(1)\cdots x_{\beta_\ell}(1),\ \ell=1,\ldots,m-1 & \longleftrightarrow M_{2\ell+1} & \longleftrightarrow s_{\gamma_1}\cdots s_{\gamma_{2\ell+1}} \end{array}$$

We obtain

Table 12 $\lambda(\mathcal{O})$, $\lambda(\hat{\mathcal{O}})$ for mixed classes in B_n, $n=2m$

$\mathcal{O}$	$\lambda(\mathcal{O})$	$\lambda(\hat{\mathcal{O}})$
$\sigma_n x_{\beta_1}(1)\cdots x_{\beta_\ell}(1)$ $\ell=1,\cdots,m-1$	$\sum_{i=1}^{2\ell+1}n_i\omega_i$	
$\sigma_n x_{\beta_1}(1)\cdots x_{\beta_m}(1)$	$\sum_{i=1}^{n}n_i\omega_i$, n_n even	$\sum_{i=1}^{n}n_i\omega_i$

In particular $\hat{\mathcal{O}}_{\sigma_n x_{\beta_1}(1)\cdots x_{\beta_m}(1)}$ is a model homogeneous space, and in fact the principal one, by [28, 3.3 (2)].

4.4.4 Unipotent classes in B_n, n odd, $n = 2m+1$

$$\begin{array}{llll} Z_\ell & \longleftrightarrow J_\ell, & \ell = 1,\dots,m & \longleftrightarrow s_{\beta_1}s_{\delta_1}\cdots s_{\beta_\ell}s_{\delta_\ell} \\ Z_{m+1} & \longleftrightarrow \varnothing & & \longleftrightarrow w_0 = s_{\beta_1}s_{\delta_1}\cdots s_{\beta_m}s_{\delta_m}s_{\alpha_n} \\ X_\ell & \longleftrightarrow K_\ell, & \ell = 1,\dots,m & \longleftrightarrow s_{\beta_1}\cdots s_{\beta_\ell} \end{array}$$

Lemma 4.9 *Let $w = s_{\beta_1}\cdots s_{\beta_\ell}$ for $\ell = 1,\dots,m$. Then T^w is connected.*

Proof. For $\ell = 1,\dots,m$ we have $(1-w)P = \mathbb{Z}\langle\beta_1,\dots,\beta_\ell\rangle = \mathbb{Z}\langle\omega_{2i} \mid i = 1,\dots,\ell\rangle$. □

Lemma 4.10 *Let $w = s_{\beta_1}\cdots s_{\beta_\ell}s_{\delta_1}\cdots s_{\delta_\ell}$ for $\ell = 1,\dots,m$. Then*

$$T^w = (T^w)^\circ \times \langle h_{\alpha_1}(-1)\rangle \times \cdots \times \langle h_{\alpha_{2\ell-1}}(-1)\rangle$$

Proof. For $\ell = 1,\dots,m$ we have $(1-w)P = \mathbb{Z}\langle 2\omega_1,\dots,2\omega_{2\ell-1},\omega_{2\ell}\rangle$. □

For $\ell = 1$ we get $T^w = \langle h_{\alpha_1}(-1)\rangle \times (T^w)^\circ$. In [9] we exhibit the element $x_{-\beta_1}(1)x_{-\delta_1}(1) \in \mathcal{O}\cap BwB\cap B^-$. We may therefore choose $x = n_{\beta_1}n_{\delta_1}h\,x_{\beta_1}(2)x_{\delta_1}(2)$ for a certain $h \in T$. Then $h_{\alpha_1}(-1) \in C(x)$, so that $T_x = T^w$.

Next we consider Z_{m+1}. We claim that $T_x = Z(G)$. Suppose for a contradiction that there is an involution $\sigma \in T_x \setminus Z(G)$. Then $x \in K = C(\sigma)$, and K is the almost direct product K_1K_2, of type D_kB_{n-k}, for some $k = 1,\dots,n$. We get an orthogonal decomposition $E = E_1 \oplus E_2$ and a decomposition $x = x_1x_2 \in K_1K_2$. Then $-1 = w_0 = (w_1, w_2)$, where w_i is the element of the Weyl group of K_i corresponding to x_i (the class of x_i in K_i is spherical). It follows that each $w_i = -1$, and k is even. Then x_1 is in the class $Z_{k/2}$ of K_1 and x_2 in the class $Z_{m+1-k/2}$ of K_2. However, the product x_1x_2 is not in the class Z_{m+1} of G (since in x_1x_2 there are two rows with 3 boxes), a contradiction. Hence $T_x = Z(G)$.

We now deal with Z_ℓ, $\ell = 2,\dots,m$. Here Ψ_J has basis $\{\alpha_1,\dots,\alpha_{2\ell-1},\gamma_{2\ell}\}$, and $C((T^w)^\circ)'$ is of type $B_{2\ell}$. From the construction in [9], proof of Theorem 2.11, we can find x in the $D_{2\ell}$-subgroup K of $C((T^w)^\circ)'$ generated by the long roots, that is the $D_{2\ell}$-subgroup with basis $\{\alpha_1,\dots,\alpha_{2\ell-1},\beta_\ell\}$. We have

$$Z(K) = Z(G)\times\langle\sigma\rangle, \quad \sigma = \prod_{i=1}^{\ell} h_{\alpha_{2i-1}}(-1)$$

By Lemma 4.10, $T_x = (T^w)^\circ \times (T_x \cap R)$, where $R = \langle h_{\alpha_1}(-1)\rangle \times \cdots \times \langle h_{\alpha_{2\ell-1}}(-1)\rangle \le K$. Since x lies in the maximal spherical unipotent class of $D_{2\ell}$, from the result obtained for this class, we have $T_x \cap R = R \cap Z(K) = \langle\sigma\rangle$, hence $T_x = (T^w)^\circ \times \langle\sigma\rangle$. We have proved

Proposition 4.11 *For $\ell = 1,\dots,m$ we have*

$$\lambda(Z_\ell) = \left\{\sum_{i=1}^{2\ell} n_i\omega_i \mid n_k \in \mathbb{N},\ \sum_{i=1}^{\ell} n_{2i-1} \text{ even}\right\}$$

Moreover

$$\lambda(Z_{m+1}) = \left\{\sum_{i=1}^{n} n_i\omega_i \mid n_k \in \mathbb{N},\ \ n_n \text{ even}\right\}$$

For the simply-connected cover we obtain

Proposition 4.12 *For $\ell = 1, \dots, m$ we have*

$$\lambda(\hat{Z}_\ell) = \left\{ \sum_{i=1}^{2\ell} n_i \omega_i \mid n_k \in \mathbb{N} \right\}$$

Moreover

$$\lambda(\hat{Z}_{m+1}) = \left\{ \sum_{i=1}^{n} n_i \omega_i \mid n_k \in \mathbb{N} \right\}$$

Proof. Let $u \in Z_\ell$, with $\ell = 1, \dots, m+1$. If $C(u)^\circ = RC$ with $R = R_u(C(u))$, C connected reductive, then C is of type $C_{\ell-1} D_{n-2\ell+1}$ [12, Sect. 13.1]. In particular C is semisimple since $n - 2\ell + 1$ is even. Hence $\lambda(\hat{Z}_\ell)$ is free by Lemma 4.4.

For $\ell = m+1$, we have $Z(G) \not\leq C(x)^\circ$. In fact, we can take $u = x_{\alpha_1}(1) x_{\alpha_3}(1) \cdots x_{\alpha_n}(1)$ in Z_{m+1}. Then $S = H_{\check{\omega}_2} H_{\check{\omega}_4} \cdots H_{\check{\omega}_{n-1}}$ is a maximal torus of $C(u)^\circ$, where for $h \in \mathfrak{h}$ we put $H_h = \exp \mathbb{C}h$. Since $Z(G) \cap S = \{1\}$, we get $C(u) = C(u)^\circ \times Z(G)$ by Lemma 4.3. We are left to deal with $\ell = 1$. However for each ℓ, the image Q of $(u-1)^2$ in $V(\omega_1)$ (which is the natural module for B_n) has dimension 1, so $C(u)^\circ$ acts trivially on Q by Lemma 4.5, and $\omega_1 \in \lambda(\hat{Z}_\ell)$. □

We summarize the results obtained in

Table 13 $\lambda(\mathcal{O})$, $\lambda(\hat{\mathcal{O}})$ for unipotent classes in B_n, $n = 2m+1$

$\mathcal{O}$	$\lambda(\mathcal{O})$	$\lambda(\hat{\mathcal{O}})$
$(2^{2\ell}, 1^{2n+1-4\ell})$ $\ell = 1, \dots, m$	$\sum_{i=1}^{\ell} n_{2i}\, \omega_{2i}$	
$(3, 2^{2(\ell-1)}, 1^{2n+2-4\ell})$ $\ell = 1, \dots, m$	$\sum_{i=1}^{2\ell} n_i \omega_i,\ \sum_{i=1}^{\ell} n_{2i-1}$ even	$\sum_{i=1}^{2\ell} n_i \omega_i$
$(3, 2^{2m})$	$\sum_{i=1}^{n} n_i \omega_i,\ n_n$ even	$\sum_{i=1}^{n} n_i \omega_i$

In particular $\hat{Z}_{m+1}$ is a model homogeneous space, and in fact the principal one, by [28, 3.3 (2)].

In Sect. 5, we shall determine the decomposition of the coordinate ring of the closure of Z_{m+1}. For this purpose we shall use the fact that if $x \in Z_{m+1} \cap w_0 B$, then α_{n-1} occurs in x (see the discussion before Proposition 3.12). This can be checked by using the representative of Z_{m+1} in $SO(2n+1)$ given in [9, Proof of Theorem 12].

4.4.5 Semisimple classes in B_n, n odd $n = 2m+1$

Following the notation in [9], Tables 1 and 5 we get

$$\begin{array}{llll}
D_\ell B_{n-\ell},\ \ell = 1, \dots, m & \longleftrightarrow J_\ell & \longleftrightarrow s_{\beta_1} s_{\delta_1} \cdots s_{\beta_\ell} s_{\delta_\ell} \\
D_\ell B_{n-\ell},\ \ell = m+1, \dots, n & \longleftrightarrow M_{2(n-\ell)+1} & \longleftrightarrow s_{\gamma_1} s_{\gamma_2} \cdots s_{\gamma_{2(n-\ell)+1}} \\
T_1 A_{n-1} & \longleftrightarrow \varnothing & \longleftrightarrow w_0
\end{array}$$

and we obtain

Table 14 $\lambda(\mathcal{O})$ for semisimple classes in B_n, $n = 2m+1$

$\mathcal{O}$	H	$\lambda(\mathcal{O})$
$\exp(\zeta\check{\omega}_1)$ $\zeta \in \mathbb{C}\backslash 2\pi i\mathbb{Z}$	$T_1 B_{n-1}$	$2n_1\omega_1 + n_2\omega_2$
$\exp(\pi i\check{\omega}_\ell)$ $\ell = 2, \dots, m$	$D_\ell B_{n-\ell}$	$\sum_{i=1}^{2\ell-1} 2n_i\omega_i + n_{2\ell}\omega_{2\ell}$
$\exp(\pi i\check{\omega}_\ell)$ $\ell = m+2, \dots, n$	$D_\ell B_{n-\ell}$	$\sum_{i=1}^{2(n-\ell)} 2n_i\omega_i + n_{2(n-\ell)+1}\omega_{2(n-\ell)+1}$
$\exp(\pi i\check{\omega}_{m+1})$	$D_{m+1} B_m$	$\sum_{i=1}^{n} 2n_i\omega_i$
$\exp(\zeta\check{\omega}_n)$ $\zeta \in \mathbb{C}\backslash \pi i\mathbb{Z}$	$T_1 A_{n-1}$	$\sum_{i=1}^{n} n_i\omega_i,\ n_n$ even

4.4.6 Mixed classes in B_n, n odd, $n = 2m+1$

From [9], Table 4, we get

$$\sigma_n x_{\beta_1}(1)\cdots x_{\beta_\ell}(1),\ \ \ell = 1, \dots, m \longleftrightarrow M_{2\ell+1} \longleftrightarrow s_{\gamma_1}\cdots s_{\gamma_{2\ell+1}}$$

and we obtain

Table 15 $\lambda(\mathcal{O})$, $\lambda(\hat{\mathcal{O}})$ for mixed classes in B_n, $n = 2m+1$

$\mathcal{O}$	$\lambda(\mathcal{O}) = \lambda(\hat{\mathcal{O}})$
$\sigma_n x_{\beta_1}(1)\cdots x_{\beta_\ell}(1)$ $\ell = 1, \cdots, m-1$	$\sum_{i=1}^{2\ell+1} n_i\omega_i$
$\sigma_n x_{\beta_1}(1)\cdots x_{\beta_m}(1)$	$\sum_{i=1}^{n} n_i\omega_i,\ n_n$ even

4.5 Type E_6

We put

$$\beta_1 = (1, 2, 2, 3, 2, 1),\ \beta_2 = (1, 0, 1, 1, 1, 1)$$
$$\beta_3 = (0, 0, 1, 1, 1, 0),\ \beta_4 = (0, 0, 0, 1, 0, 0)$$

4.5.1 Unipotent classes in E_6

$$\begin{array}{lllll} A_1 & \longleftrightarrow & \{1, 3, 4, 5, 6\} & \longleftrightarrow & s_{\beta_1} \\ 2A_1 & \longleftrightarrow & \{3, 4, 5\} & \longleftrightarrow & s_{\beta_1}s_{\beta_2} \\ 3A_1 & \longleftrightarrow & \varnothing & \longleftrightarrow & w_0 = s_{\beta_1}\cdots s_{\beta_4} \end{array}$$

We obtain

Table 16 $\lambda(\mathcal{O})$, $\lambda(\hat{\mathcal{O}})$ for unipotent classes in E_6

$\mathcal{O}$	$\lambda(\mathcal{O}) = \lambda(\hat{\mathcal{O}})$
A_1	$n_2\omega_2$
$2A_1$	$n_1(\omega_1+\omega_6)+n_2\omega_2$
$3A_1$	$n_1(\omega_1+\omega_6)+n_3(\omega_3+\omega_5)+n_2\omega_2+n_4\omega_4$

4.5.2 *Semisimple classes in* E_6

Following the notation in [9], Table 2, we have

$$\begin{aligned} A_1A_5 &\longleftrightarrow \varnothing &&\longleftrightarrow w_0\\ D_5T_1 &\longleftrightarrow \{3,4,5\} &&\longleftrightarrow s_{\beta_1}s_{\beta_2}\end{aligned}$$

We obtain

Table 17 $\lambda(\mathcal{O})$ for semisimple classes in E_6

$\mathcal{O}$	H	$\lambda(\mathcal{O})$
$\exp(\pi i\check{\omega}_2)$	A_1A_5	$n_1(\omega_1+\omega_6)+n_3(\omega_3+\omega_5)+2n_2\omega_2+2n_4\omega_4$
$\exp(\zeta\check{\omega}_1)$ $\zeta\in\mathbb{C}\backslash 2\pi i\mathbb{Z}$	D_5T_1	$n_1(\omega_1+\omega_6)+n_2\omega_2$

4.6 Type E_7

Here $Z(G)=\langle h_{\alpha_2}(-1)h_{\alpha_5}(-1)h_{\alpha_7}(-1)\rangle$. We put

$$\beta_1=(2,2,3,4,3,2,1),\ \ \beta_2=(0,1,1,2,2,2,1),\ \ \beta_3=(0,1,1,2,1,0,0),$$
$$\beta_4=\alpha_7,\quad \beta_5=\alpha_5,\quad \beta_6=\alpha_3,\quad \beta_7=\alpha_2$$

4.6.1 *Unipotent classes in* E_7

$$\begin{aligned} A_1 &\longleftrightarrow \{2,3,4,5,6,7\} &&\longleftrightarrow s_{\beta_1}\\ 2A_1 &\longleftrightarrow \{2,3,4,5,7\} &&\longleftrightarrow s_{\beta_1}s_{\beta_2}\\ (3A_1)'' &\longleftrightarrow \{2,3,4,5\} &&\longleftrightarrow s_{\beta_1}s_{\beta_2}s_{\beta_4}\\ (3A_1)' &\longleftrightarrow \{2,5,7\} &&\longleftrightarrow s_{\beta_1}s_{\beta_2}s_{\beta_3}s_{\beta_6}\\ 4A_1 &\longleftrightarrow \varnothing &&\longleftrightarrow w_0=s_{\beta_1}\cdots s_{\beta_7}\end{aligned}$$

We obtain

Table 18 $\lambda(\mathcal{O})$, $\lambda(\hat{\mathcal{O}})$ for unipotent classes in E_7

$\mathcal{O}$	$\lambda(\mathcal{O})$	$\lambda(\hat{\mathcal{O}})$
A_1	$n_1\omega_1$	
$2A_1$	$n_1\omega_1 + n_6\omega_6$	
$(3A_1)''$	$n_1\omega_1 + n_6\omega_6 + 2n_7\omega_7$	$n_1\omega_1 + n_6\omega_6 + n_7\omega_7$
$(3A_1)'$	$n_1\omega_1 + n_3\omega_3 + n_4\omega_4 + n_6\omega_6$	
$4A_1$	$\sum_{i=1}^{7} n_i\omega_i$, $n_2 + n_5 + n_7$ even	$\sum_{i=1}^{7} n_i\omega_i$

In particular the simply-connected cover of $4A_1$ is a model homogeneous space, and in fact the principal one, by [28, 3.3 (8)].

Remark 4.13 From our description, it follows that $C(x)$ is connected for the classes $A_1, 2A_1$ and $(3A_1)'$, while for $(3A_1)''$ and $4A_1$ we have $C(x) = C(x)^\circ \times Z(G)$. This also follows from the tables in [1], where all unipotent classes are considered.

4.6.2 Semisimple classes in E_7

Following the notation in [9], Table 2, we have

$$\begin{array}{rcll} E_6T_1 & \longleftrightarrow & \{2,3,4,5\} & \longleftrightarrow s_{\beta_1}s_{\beta_2}s_{\beta_4} \\ D_6A_1 & \longleftrightarrow & \{2,5,7\} & \longleftrightarrow s_{\beta_1}s_{\beta_2}s_{\beta_3}s_{\beta_6} \\ A_7 & \longleftrightarrow & \varnothing & \longleftrightarrow w_0 \end{array}$$

We obtain

Table 19 $\lambda(\mathcal{O})$ for semisimple classes in E_7

$\mathcal{O}$	H	$\lambda(\mathcal{O})$
$\exp(\zeta\check{\omega}_7)$ $\zeta \in \mathbb{C}\backslash 2\pi i\mathbb{Z}$	E_6T_1	$n_1\omega_1 + n_6\omega_6 + 2n_7\omega_7$
$\exp(\pi i\check{\omega}_1)$	D_6A_1	$2n_1\omega_1 + 2n_3\omega_3 + n_4\omega_4 + n_6\omega_6$
$\exp(\pi i\check{\omega}_2)$	A_7	$\sum_{i=1}^{7} 2n_i\omega_i$

4.7 Type E_8

We put

$$\beta_1 = (2,3,4,6,5,4,3,2),\quad \beta_2 = (2,2,3,4,3,2,1,0),\quad \beta_3 = (0,1,1,2,2,2,1,0),$$
$$\beta_4 = (0,1,1,2,1,0,0,0),\quad \beta_5 = \alpha_7,\quad \beta_6 = \alpha_5,\quad \beta_7 = \alpha_3,\quad \beta_8 = \alpha_2$$

4.7.1 Unipotent classes in E_8

$$\begin{array}{rcll} A_1 & \longleftrightarrow & \{1,2,3,4,5,6,7\} & \longleftrightarrow s_{\beta_1} \\ 2A_1 & \longleftrightarrow & \{2,3,4,5,6,7\} & \longleftrightarrow s_{\beta_1}s_{\beta_2} \\ 3A_1 & \longleftrightarrow & \{2,3,4,5\} & \longleftrightarrow s_{\beta_1}s_{\beta_2}s_{\beta_3}s_{\beta_5} \\ 4A_1 & \longleftrightarrow & \varnothing & \longleftrightarrow w_0 = s_{\beta_1}\cdots s_{\beta_8} \end{array}$$

We have

$$(1-w)P = \begin{cases} \mathbb{Z}\langle\omega_8\rangle & \text{for} \quad w = s_{\beta_1} \\ \mathbb{Z}\langle\omega_1, \omega_8\rangle & \text{for} \quad w = s_{\beta_1}s_{\beta_2} \\ \mathbb{Z}\langle\omega_1, \omega_6, 2\omega_7, 2\omega_8\rangle & \text{for} \quad w = s_{\beta_1}s_{\beta_2}s_{\beta_3}s_{\beta_5} \end{cases}$$

Class $3A_1$. Here Ψ_J has basis $\{\alpha_7, \alpha_8, \beta_2, \beta_3\}$, $K = C((T^w)^\circ)'$ is of type D_4 and has centre $\langle h_{\alpha_3}(-1)h_{\alpha_5}(-1), h_{\alpha_2}(-1)h_{\alpha_3}(-1)\rangle$ which is contained in $(T^w)^\circ$. Hence $T_x = (T^w)^\circ$.
Class $4A_1$. We claim that $T_x = 1$. Suppose for a contradiction there exists an involution $\sigma \in T_x$. Then $x \in K = C(\sigma)$. From the classification of involutions of E_8, it follows that K is of type D_8 or E_7A_1. The class of x in K is spherical, and by the uniqueness of Bruhat decomposition, x lies over the longest element of the Weyl group of K, which is w_0. By comparison of weighted Dynkin diagrams, the spherical unipotent class of K over w_0 does not correspond to the class $4A_1$ of E_8, a contradiction.

We have shown that in all cases $T_x = (T^w)^\circ$, so that $C(x)$ is connected, as also follows from [12, p. 405]. We have

Table 20 $\lambda(\mathcal{O})$, $\lambda(\hat{\mathcal{O}})$ for unipotent classes in E_8

$\mathcal{O}$	$\lambda(\mathcal{O}) = \lambda(\hat{\mathcal{O}})$
A_1	$n_8\omega_8$
$2A_1$	$n_1\omega_1 + n_8\omega_8$
$3A_1$	$n_1\omega_1 + n_6\omega_6 + n_7\omega_7 + n_8\omega_8$
$4A_1$	$\sum_{i=1}^{8} n_i\omega_i$

In particular $4A_1$ is a model homogeneous space (see [2, Theorem 1.1]), and in fact the principal one, by [28, 3.3 (9)].

4.7.2 Semisimple classes in E_8

Following the notation in [9], Table 2, we have

$$\begin{array}{rcl} A_1E_7 \longleftrightarrow \{2,3,4,5\} \longleftrightarrow s_{\beta_1}s_{\beta_2}s_{\beta_3}s_{\beta_5} \\ D_8 \longleftrightarrow \varnothing \qquad\quad \longleftrightarrow w_0 \end{array}$$

We obtain

Table 21 $\lambda(\mathcal{O})$ for semisimple classes in E_8

$\mathcal{O}$	H	$\lambda(\mathcal{O})$
$\exp(\pi i\check{\omega}_8)$	A_1E_7	$n_1\omega_1 + n_6\omega_6 + 2n_7\omega_7 + 2n_8\omega_8$
$\exp(\pi i\check{\omega}_1)$	D_8	$\sum_{i=1}^{8} 2n_i\omega_i$

4.8 Type F_4

We put

$$\beta_1 = (2,3,4,2),\ \beta_2 = (0,1,2,2),$$
$$\beta_3 = (0,1,2,0),\ \beta_4 = (0,1,0,0)$$

4.8.1 Unipotent classes in F_4

$$\begin{array}{ccccc} A_1 & \longleftrightarrow & \{2,3,4\} & \longleftrightarrow & s_{\beta_1} \\ \tilde{A}_1 & \longleftrightarrow & \{2,3\} & \longleftrightarrow & s_{\beta_1}s_{\beta_2} \\ A_1+\tilde{A}_1 & \longleftrightarrow & \varnothing & \longleftrightarrow & w_0 = s_{\beta_1}\cdots s_{\beta_4} \end{array}$$

We obtain

Table 22 $\lambda(\mathcal{O})$, $\lambda(\hat{\mathcal{O}})$ for unipotent classes in F_4

$\mathcal{O}$	$\lambda(\mathcal{O})$	$\lambda(\hat{\mathcal{O}})$
A_1	$n_1\omega_1$	
$\tilde{A}_1$	$n_1\omega_1 + 2n_4\omega_4$	$n_1\omega_1 + n_4\omega_4$
$A_1+\tilde{A}_1$	$n_1\omega_1 + n_2\omega_2 + 2n_3\omega_3 + 2n_4\omega_4$	

4.8.2 Semisimple classes in F_4

Following the notation in [9], Table 2, we have

$$\begin{array}{ccccc} A_1C_3 & \longleftrightarrow & \varnothing & \longleftrightarrow & w_0 \\ B_4 & \longleftrightarrow & \{1,2,3\} & \longleftrightarrow & s_{\gamma_1} \end{array}$$

where γ_1 is the highest short root $(1,2,3,2)$.

We obtain

Table 23 $\lambda(\mathcal{O})$ for semisimple classes in F_4

$\mathcal{O}$	H	$\lambda(\mathcal{O})$
$\exp(\pi i\check{\omega}_1)$	A_1C_3	$\sum_{i=1}^{4} 2n_i\omega_i$
$\exp(\pi i\check{\omega}_4)$	B_4	$n_4\omega_4$

4.8.3 Mixed class in F_4

We put $f_2 = \exp(\pi i\check{\omega}_4)$. Then following [9], Table 4

$$\mathcal{O}_{f_2 x_{\beta_1}(1)} \longleftrightarrow \varnothing \longleftrightarrow w_0$$

Assuming the existence of an involution in T_x we get a contradiction, proving therefore that $T_x = 1$. Hence

Table 24 $\lambda(\mathcal{O})$ for the mixed class in F_4

$\mathcal{O}$	$\lambda(\mathcal{O}) = \lambda(\hat{\mathcal{O}})$
$f_2 x_{\beta_1}(1)$	$\sum_{i=1}^{4} n_i \omega_i$

In particular $\mathcal{O}_{f_2 x_{\beta_1}(1)}$ is a model homogeneous space, and in fact the principal one, by [28, 3.3 (6)], see also [28, p. 300].

4.9 Type G_2

We put $\beta_1 = (3, 2),\ \beta_2 = \alpha_1$.

4.9.1 Unipotent classes in G_2

$$\begin{array}{ccccc} A_1 & \longleftrightarrow & \{1\} & \longleftrightarrow & s_{\beta_1} \\ \tilde{A}_1 & \longleftrightarrow & \varnothing & \longleftrightarrow & w_0 = s_{\beta_1} s_{\beta_2} \end{array}$$

We get

Table 25 $\lambda(\mathcal{O}), \lambda(\hat{\mathcal{O}})$ for unipotent classes in G_2

$\mathcal{O}$	$\lambda(\mathcal{O}) = \lambda(\hat{\mathcal{O}})$
A_1	$n_2\omega_2$
$\tilde{A}_1$	$n_1\omega_1 + n_2\omega_2$

In particular $\tilde{A}_1$ is a model homogeneous space, and in fact the principal one, by [28, 3.3 (5)].

Using the embedding of G into $SO(7)$, one can determine explicitly an $x \in \mathcal{O} \cap w_0 B$, where $\mathcal{O} = \tilde{A}_1$. Then one can check that both α_1 and α_2 occur in x (see the Discussion before Proposition 3.12). This fact will be used in Sect. 5 to determine $\mathbb{C}[\overline{\mathcal{O}}]$.

4.9.2 Semisimple classes in G_2

Following the notation in [9], Table 2, we have

$$\begin{array}{ccccc} A_1\tilde{A}_1 & \longleftrightarrow & \varnothing & \longleftrightarrow & w_0 \\ A_2 & \longleftrightarrow & \{2\} & \longleftrightarrow & s_{\gamma_1} \end{array}$$

where γ_1 is the highest short root (2, 1).

The group G_2 has one class of involutions. However there is also a class of elements of order 3 which is spherical. We obtain

Table 26 $\lambda(\mathcal{O})$ for semisimple classes in G_2

$\mathcal{O}$	H	$\lambda(\mathcal{O})$
$\exp(\pi i \check{\omega}_2)$	$A_1\tilde{A}_1$	$\sum_{i=1}^{2} 2n_i\omega_i$
$\exp(\frac{2\pi i}{3}\check{\omega}_1)$	A_2	$n_1\omega_1$

5 The coordinate ring of $\overline{\mathcal{O}}$

In this section we determine the decomposition of $k[\overline{\mathcal{O}}]$ into simple G-modules, where $\overline{\mathcal{O}}$ is the closure of a spherical conjugacy class. Normality of conjugacy classes' closures has been deeply investigated. For a survey on this topic, see [23, Sect. 8], [8, 7.9], Remark (iii). The first observation is that the problem is reduced to unipotent conjugacy classes in G [23, 8.1]. In the following we are interested only in spherical conjugacy classes, and I recall the facts in this context. It is known that the closure of the minimal nilpotent orbit is always normal [45, Theorem 2].

In [17], Hesselink proved normality for several small nilpotent orbits: actually Hesselink's results cover all nilpotent orbits of height 2, which in particular includes all spherical unipotent conjugacy classes for A_n and C_n and, for the exceptional cases, following the notation in [12], A_1 and $2A_1$ in E_6, A_1, $2A_1$ and $(3A_1)''$ in E_7, A_1 and $2A_1$ in E_8, A_1 and $\tilde{A}_1$ in F_4, A_1 in G_2, therefore leaving out only one orbit in each G_2, F_4 , E_6 and two orbits in E_7 or E_8.

The classical groups have been considered in [24,25]: for the special linear groups the closure of every conjugacy class is normal. For the symplectic and orthogonal groups there exist conjugacy classes with non-normal closure. However every spherical conjugacy class in the symplectic group has normal closure, since from the classification obtained in [33] we know that the unipotent spherical conjugacy classes have only 2 columns (see also [17], Sect. 5, Criterion 2). For special orthogonal groups the results in [25] left open a portion of the very even unipotent classes. Sommers proved that these have normal closure in [39]. Taking into account the results in [25] (note that the spherical very even classes are dealt with in [25], Theorem 17.3 b)) it follows that every unipotent spherical conjugacy class in type D_n and B_n has normal closure except for the maximal class Z_{m+1} in B_n, when $n = 2m + 1$, $m \geq 1$. From this and the classification of spherical conjugacy classes, it follows that every spherical conjugacy class has normal closure, except for the above mentioned class in B_{2m+1}.

For the exceptional groups, besides the results on the minimal orbit and Hesselink's results, in [27] it is shown that the orbit $\tilde{A}_1$ in G_2 has a non-normal closure (see also [23]): here there is bijective normalization, contrary to the case of Z_{m+1} in B_{2m+1} where the closure is branched in codimension 2. In [7] the case of type F_4 is completely handled, and it follows that every spherical conjugacy class has normal closure. The same holds for E_6, as follows from [38] where every nilpotent orbit is considered. For the remaining nilpotent orbits in E_7 and E_8, in [8], 7.9, Remark (iii), Broer gives a list of orbits with normal closure. Among these there are all spherical nilpotent orbits in E_7 and E_8. We may therefore state

Theorem 5.1 *Let $\mathcal{O}$ be a spherical conjugacy class. Then $\overline{\mathcal{O}}$ is normal except for the class Z_{m+1} in B_{2m+1} $(m \geq 1)$ and the class $\tilde{A}_1$ in G_2.* □

Remark 5.2 As already observed, Hesselink's approach exploits a resolution of singularities $G \times^Q V \to \overline{\mathcal{O}}$ with a completely reducible Q-module V. In [13, Example 4.4, Proposition 4.5], the authors provide an alternative proof for nilpotent orbits of height 2. In this context we observe that also from (3.9) and Corollary 3.17 it is possible to prove normality of $\overline{\mathcal{O}}$ in certain cases. For instance in type C_n from Table 3 we get $\lambda(X_\ell) = 2P_w^+$ for every unipotent class X_ℓ. From (3.9) it follows that $\lambda(\overline{\mathcal{O}}) = \lambda(\mathcal{O})$, so that $\overline{\mathcal{O}}$ is normal.

Remark 5.3 In [35, 6.1], normality of $\mathcal{N}^{\mathrm{sph}}$ (the union of all spherical nilpotent orbits, which is in fact the closure of the unique maximal spherical nilpotent orbit) is discussed.

We recall that in general $k[\mathcal{O}]$ is the integral closure of $k[\overline{\mathcal{O}}]$ in its field of fractions and that $k[\overline{\mathcal{O}}] = k[\mathcal{O}]$ if and only if $\overline{\mathcal{O}}$ is normal [22, Proposition and Corollary in 8.3]. By

Theorem 5.1, to describe the decomposition of $k[\overline{\mathcal{O}}]$ we are left to deal with Z_{m+1} in B_{2m+1} and with $\tilde{A}_1$ in G_2. We use the notation and the tables from Sect. 4 for the cases B_{2m+1} and G_2.

Theorem 5.4 *Let $\mathcal{O} = Z_{m+1}$ in B_n, $n = 2m+1$, $m \geq 1$. Then*

$$\lambda(\overline{\mathcal{O}}) = \left\{\sum_{i=1}^{2m} n_i\omega_i \mid \sum_{i=1}^{m} n_{2i-1} \text{ even}\right\} \cup \left\{\sum_{i=1}^{n} n_i\omega_i \mid n_n \text{ even}, n_n \geq 2\right\}$$

Proof. Considering the (G-equivariant) restriction $r : k[\overline{\mathcal{O}}] \to k[\overline{Z_m}] = k[Z_m]$, we obtain $\left\{\sum_{i=1}^{2m} n_i\omega_i \mid \sum_{i=1}^{m} n_{2i-1} \text{ even}\right\} \leq \lambda(\overline{\mathcal{O}})$. In particular for every even j, $\omega_j \in \lambda(\overline{\mathcal{O}})$, and for every pair of odd j, k, with $1 \leq j \leq k < n$, $\omega_j + \omega_k \in \lambda(\overline{\mathcal{O}})$. By Corollary 3.13, we have $2\omega_n \in \lambda(\overline{\mathcal{O}})$. We show that $\omega_j + 2\omega_n \in \lambda(\overline{\mathcal{O}})$ for every odd j, $j < n$. We have $2\omega_{n-1} - \alpha_{n-1} = \omega_{n-2} + 2\omega_n$ and since α_{n-1} occurs in $x \in w_0B \cap \mathcal{O}$, by Corollary 3.17, we get $\omega_{n-2} + 2\omega_n \in \lambda(\overline{\mathcal{O}})$. Let j be odd, $j < n-2$. Then $\omega_j + 2\omega_n + 2\omega_{n-2} \in \lambda(\overline{\mathcal{O}})$ since $\omega_{n-2} + 2\omega_n$ and $\omega_j + \omega_{n-2}$ are in $\lambda(\overline{\mathcal{O}})$.

There exists B-eigenvectors F, H in $k[\overline{\mathcal{O}}]$ of weights $\omega_j + 2\omega_n + 2\omega_{n-2}$, $2\omega_{n-2}$ respectively. Then F/H is a rational function on $\overline{\mathcal{O}}$ of weight $\omega_j + 2\omega_n$ defined at least on $\mathcal{O}$. However, $2\omega_{n-2}$ is also a weight in $\lambda(Z_m)$, so that H is non-zero on the dense B-orbit v in Z_m. Hence F/H is defined on v, and it is zero on v, since F is zero on Z_m, $\omega_j + 2\omega_n + 2\omega_{n-2}$ not being in $\lambda(Z_m)$. It follows that F/H is defined on Z_m, so that it is a regular function on $\mathcal{O} \cup Z_m$. By [25, Theorem 16.2, (iii)], F/H extends to $\overline{\mathcal{O}}$, and $\omega_j + 2\omega_n$ lies in $\lambda(\overline{\mathcal{O}})$. We have shown that

$$\lambda(\overline{\mathcal{O}}) \geq \left\{\sum_{i=1}^{2m} n_i\omega_i \mid \sum_{i=1}^{m} n_{2i-1} \text{ even}\right\} \cup \left\{\sum_{i=1}^{n} n_i\omega_i \mid n_n \text{ even}, n_n \geq 2\right\}$$

We prove that also the opposite inclusion holds. Assume $\lambda = \sum_{i=1}^{n} n_i\omega_i \in \lambda(\overline{\mathcal{O}})$. Since $\lambda(\overline{\mathcal{O}}) \leq \lambda(\mathcal{O})$, we have n_n even. If $n_n \neq 0$ we are done. So assume $n_n = 0$. Let $y \in Z_{m+1} \cap U^- \cap Bw_0B$. We observe that $y_1 := \lim_{z\to 0} h_{\alpha_n}(z)^{-1} y h_{\alpha_n}(z)$ exists, and lies in $Z_m \cap U^- \cap BwB$, where $w = w(Z_m)$ (in [9] we give representatives for both classes in $SO(2n+1)$, so that this may be checked directly). Now let $F : \overline{\mathcal{O}} \to k$ be a highest weight vector of weight λ, with $F(y) = 1$. Then $F(y_1) = 1$, since $\lambda(h_{\alpha_n}(z)) = 1$ for every $z \in k^*$. Since $x_1 \in Z_m \cap wB$ lies in the B-orbit of y_1, we have $F(x_1) \neq 0$. But $\sigma = \prod_{i=1}^{m} h_{\alpha_{2i-1}}(-1) \in C(x_1)$, so that $F(x_1) = F(\sigma x_1 \sigma) = \lambda(\sigma)F(x_1)$ implies $\lambda(\sigma) = 1$, and we are done. □

Theorem 5.5 *Let $\mathcal{O} = \tilde{A}_1$ in G_2. Then $\lambda(\overline{\mathcal{O}})$ is the submonoid of $\lambda(\mathcal{O})$ generated by* $2\omega_1, 3\omega_1, \omega_2$.

Proof. We know that $\omega_1 \in \lambda(\mathcal{O})$ and it follows from the proof of [27], Theorem 3.13, that $\omega_1 \notin \lambda(\overline{\mathcal{O}})$. We have

$$2\omega_1 - \alpha_1 = \omega_2, \quad 2\omega_2 - \alpha_2 = 3\omega_1$$

hence, by Corollaries 3.13 and 3.17, we get $2\omega_1, 3\omega_1, \omega_2 \in \lambda(\overline{\mathcal{O}})$, since both α_1, α_2 occur in $x \in w_0B \cap \mathcal{O}$. Suppose for a contradiction that $\omega_1 + n\omega_2 \in \lambda(\overline{\mathcal{O}})$ for a certain $n \in \mathbb{N}$. There exists B-eigenvectors F, H in $k[\overline{\mathcal{O}}]$ of weights $\omega_1 + n\omega_2$, $n\omega_2$ respectively. Then F/H is a rational function on $\overline{\mathcal{O}}$ of weight ω_1 defined at least on $\mathcal{O}$. However $n\omega_2$ is also a weight in $\lambda(A_1)$, so that H is non-zero on the dense B-orbit v in A_1. Hence F/H is defined on v, and it is zero on v, since F is zero on A_1, because $\omega_1 + n\omega_2$ is not in $\lambda(A_1)$. It follows that F/H is defined on A_1. But A_1 has normal closure, so that F/H is defined on the closure of A_1, and then on $\overline{\mathcal{O}}$, so that there is in $k[\overline{\mathcal{O}}]$ a B-eigenvector of weight ω_1, a contradiction. $\square$

6 The general case

Let G be as usual simply-connected, $D \leq Z(G)$, $\overline{G} = G/D$, $\pi : G \to \overline{G}$ the canonical projection. For $g \in G$ we put $\overline{g} = \pi(g)$. We give a procedure to describe the coordinate ring of $\mathcal{O}_{\overline{p}}$, where $\mathcal{O}_{\overline{p}}$ is a spherical conjugacy class of $\overline{G}$. Passing to G, we have to consider the quotient $G/\pi^{-1}(C_{\overline{G}}(\overline{p}))$. Let $p = sv$ be the Jordan-Chevalley decomposition of p, $w = w(\mathcal{O}_p)$. We may assume $s \in T$. Let $W_{s,D} = \{w \in W \mid wsw^{-1} = zs, z \in D\}$, and $N_{s,D} \leq N$ such that $N_{s,D}/T = W_{s,D}$. Then $\pi^{-1}(C_{\overline{G}}(\overline{p})) = C(v) \cap N_{s,D}C(s)$. Reasoning as in [42, Corollary II, 4.4], we have a homomorphism $\pi^{-1}(C_{\overline{G}}(\overline{p})) \to D$, $g \mapsto [g, p]$ with kernel $C(p)$.

Let $y \in \mathcal{O}_p \cap BwB$ be such that $L = L_J$ is adapted to $C(y)$. If $H = \pi^{-1}(C_{\overline{G}}(\overline{y}))$, then $\lambda(\mathcal{O}_{\overline{p}}) = \lambda(G/H) = \{\lambda \in P_w^+ \mid \lambda(T \cap H) = 1\}$ by Corollary 3.19. Let $x \in \mathcal{O}_p \cap wB$, $x = \dot{w}u$, with $u \in U$ and let $T_{x,D} = T \cap \pi^{-1}(C_{\overline{G}}(\overline{x}))$. By Proposition 3.5, we get $T \cap H = T_{x,D}$, hence

$$\lambda(\mathcal{O}_{\overline{x}}) = \{\lambda \in P_w^+ \mid \lambda(T_{x,D}) = 1\} \tag{6.12}$$

Let $T_D^w = \{t \in T \mid wtw^{-1} = zt, z \in D\}$. From the Bruhat decomposition, we get $T_{x,D} \leq T_D^w$. Moreover since w is an involution, for $t \in T_D^w$ we have $t = w^2tw^{-2} = z^2t$, so that $z^2 = 1$. In particular $\pi^{-1}(C_{\overline{G}}(\overline{s})) = N_{s,D_2}C(s)$, $T_D^w = T_{D_2}^w$, where $D_2 = D \cap T_2$.

Let $t \in T$ and write $t = ab$, with $a \in (T^w)^\circ$, $b \in (S^w)^\circ$. Then $wtw^{-1} = tz$ with $z \in D_2$ if and only if $z = b^2$. Since $(S^w)^\circ$ is connected, we get $T_D^w = T^w_{D_2 \cap (S^w)^\circ}$ and

$$\frac{\pi^{-1}(C_{\overline{G}}(\overline{x}))}{C(x)} \cong \frac{T_{x,D}}{T_x} \hookrightarrow \frac{T_D^w}{T^w} \cong D_2 \cap (S^w)^\circ$$

with $T_x = T^w \cap C(u)$, $T_{x,D} = T_D^w \cap C(u)$. In particular, if $D_2 \cap (S^w)^\circ = 1$, then $\lambda(\mathcal{O}_{\overline{x}}) = \lambda(\mathcal{O}_x)$. This equality means that x is not conjugate to zx for any $z \in D_2$, $z \neq 1$, and this may be directly checked in many cases, for instance in type A_n or C_n (and of course always holds for x unipotent). However, to deal with orthogonal groups and E_7, we determined explicitly the cases when $D_2 \cap (S^w)^\circ$ is non-trivial, and in each case we determined $T_{x,D}$ and therefore $\lambda(\mathcal{O}_{\overline{x}})$.

Here we just observe that if $D_2 \cap (S^w)^\circ \neq 1$, then $D_2 \cap (S^w)^\circ \cong \mathbb{Z}/2\mathbb{Z}$, except possibly for $D = Z(G)$ in type D_n, $n = 2m$. Assuming for convenience $k = \mathbb{C}$, it turns out that in this case for $\exp(\pi i \check{\omega}_m)$, we have $T_x = T_2$ and $T_{x,Z(G)}/T_x \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. More precisely

$$T_{x,Z(G)} = T_{Z(G)}^{w_0} = T_2\, \langle h_{\alpha_{n-1}}(i)h_{\alpha_n}(i), \quad \prod_{i=1}^{m} h_{\alpha_{2i-1}}(i)\rangle$$

so that in $G/Z(G) = PSO(2n)$, $n = 2m$,

$$\lambda(\mathcal{O}_{\overline{\exp(\pi i \check{\omega}_m)}}) = \left\{ \sum_{k=1}^{n} 2m_k\omega_k \mid m_k \in \mathbb{N}, \quad m_{n-1} + m_n \text{ and } \sum_{i=1}^{m} m_{2i-1} \text{ even} \right\}$$

We add that for $SO(2n+1)$, $n \geq 1$ and $b_\lambda = \mathrm{diag}(1, \lambda I_n, \lambda^{-1} I_n)$, $\lambda \neq \pm 1$, $\mathcal{O}_{b_\lambda}$ is a model orbit, and in fact the principal one by [28, 3.3 (2′)].

We conclude by presenting the results for E_7.

6.1 Type E_7, $D = Z(G)$

In this case $Z(G) = \langle z \rangle$, where $z = h_{\alpha_2}(-1)h_{\alpha_5}(-1)h_{\alpha_7}(-1) = \exp(2\pi i \check{\omega}_2) = \exp(2\pi i \check{\omega}_7)$.

There are three elements of the Weyl group to be considered and only for $w = s_{\beta_1}s_{\beta_2}s_{\beta_4}$ and $w = w_0$ we have $z \in (S^w)^\circ$.

Class of type A_7, $w = w_0$. Here $x = n_{\beta_1} \cdots n_{\beta_7}$,

$$T^{w_0}_{Z(G)} = T_2 \langle \exp(\pi i \check{\omega}_2) \rangle = T_2 \langle h_{\alpha_2}(i) h_{\alpha_5}(i) h_{\alpha_7}(i) \rangle$$

since $\exp(\pi i \check{\omega}_2) \in (S^{w_0})^\circ = T$ and $\exp(\pi i \check{\omega}_2)^2 = z$.

Proposition 6.1 *Let G be of type E_7, $D = Z(G)$, then*

$$\lambda(\mathcal{O}_{\overline{\exp(\pi i \check{\omega}_2)}}) = \left\{ \sum_{i=1}^{7} 2n_i \omega_i \mid n_2 + n_5 + n_7 \ \textit{even} \right\}$$

Proof. This follows from the fact that $T_{x,Z(G)} = T^{w_0}_{Z(G)}$. □

Classes of type E_6T_1, $w = s_{\beta_1}s_{\beta_2}s_{\beta_4}$, $T^w = (T^w)^\circ \times \langle h_{\alpha_7}(-1) \rangle = (T^w)^\circ \times Z(G)$.

We have $T^w_{Z(G)} = T^w \langle \exp(\pi i \check{\omega}_7) \rangle = T^w \langle h_{\alpha_1}(-1) h_{\alpha_7}(i) \rangle$. If $\zeta \in \mathbb{C} \backslash 2\pi i \mathbb{Z}$, then

$$x_\zeta = n_{\beta_1} n_{\beta_2} n_{\alpha_7} h x_{\beta_1}(\xi) x_{\beta_2}(\xi) x_{\alpha_7}(\xi) \in \mathcal{O}_{\exp(\zeta \check{\omega}_7)} \cap n_{\beta_1} n_{\beta_2} n_{\alpha_7} B$$

for a certain $h \in T$, with $\xi = \frac{1+e^\zeta}{1-e^\zeta}$, so that

$$T_{x_\zeta, Z(G)} = \begin{cases} T^w_{Z(G)} & \text{if } \zeta \in \pi i \mathbb{Z} \backslash 2\pi i \mathbb{Z} \\ T^w & \text{if } \zeta \in \mathbb{C} \backslash \pi i \mathbb{Z} \end{cases}$$

since $\alpha_7(\exp(\pi i \check{\omega}_7)) = -1$.

Proposition 6.2 *Let G be of type E_7, $D = Z(G)$, then*

$$\lambda(\mathcal{O}_{\overline{\exp(\zeta \check{\omega}_7)}}) = \begin{cases} \{n_1\omega_1 + n_6\omega_6 + 2n_7\omega_7 \mid n_1 + n_7 \ \textit{even}\} & \textit{if} \ \ \zeta \in \pi i \mathbb{Z} \backslash 2\pi i \mathbb{Z} \\ \{n_1\omega_1 + n_6\omega_6 + 2n_7\omega_7\} & \textit{if} \ \ \zeta \in \mathbb{C} \backslash \pi i \mathbb{Z} \end{cases}$$

□

Addendum In [9, Remark 5], we stated that if $\pi_1 : G \to G/U$ is the canonical projection, and $\mathcal{O}$ is a spherical conjugacy class, then $\pi_{1|\mathcal{O}} : \mathcal{O} \to G/U$ has finite fibers. This is not correct, and one can only say that $\pi_{1|\mathcal{O}}$ has generically finite fibers (if $w = w(\mathcal{O})$, and $g \in \mathcal{O} \cap BwB$, then $\pi_1^{-1}(gU)$ has $|T^w/T_x|$ elements, where $x \in \mathcal{O} \cap wB$).

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