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SPHERICAL ORBITS AND REPRESENTATIONS OF $\mathcal{U}_\varepsilon(\mathfrak{g})$

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Abstract. Let $\mathcal{U}_\varepsilon(\mathfrak{g})$ be the simply connected quantized enveloping algebra at roots of one associated to a finite dimensional complex simple Lie algebra $\mathfrak{g}$. The De Concini–Kac–Procesi conjecture on the dimension of the irreducible representations of $\mathcal{U}_\varepsilon(\mathfrak{g})$ is proved for the representations corresponding to the spherical conjugacy classes of the simply connected algebraic group G with Lie algebra $\mathfrak{g}$. We achieve this result by means of a new characterization of the spherical conjugacy classes of G in terms of elements of the Weyl group.

Introduction

Since their appearance in the mid 1980's quantum groups have been extensively investigated. In particular the representation theory of the quantized enveloping algebra $\mathcal{U}_\varepsilon(\mathfrak{g})$, as introduced in [DCK1], and of the quantum function algebra $F_\varepsilon[G]$ ([DCL]) has been deeply studied by many authors. Here $\mathfrak{g}$ is a simple complex Lie algebra, G is the corresponding simple simply connected algebraic group, and ε is a primitive ℓ-th root of unity, with ℓ an odd integer strictly greater than 1. However, while the irreducible representations of $F_\varepsilon[G]$ are well described ([DCP2]), the representation theory of $\mathcal{U}_\varepsilon(\mathfrak{g})$ is far from being understood. In this context there is a procedure to associate a certain conjugacy class $\mathcal{O}_V$ of G to each simple $\mathcal{U}_\varepsilon(\mathfrak{g})$-module V. The De Concini–Kac–Procesi conjecture asserts that $\ell^{\frac{1}{2}\dim \mathcal{O}_V}$ divides $\dim V$. At present the conjecture has been proved only in some cases, namely for the conjugacy classes of maximal dimension, i.e.,

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the regular orbits ([DCKP2]), for the subregular unipotent orbits in type A_n when ℓ is a power of a prime ([C3]), for all orbits in A_n when ℓ is a prime ([C1]), and for the conjugacy classes $\mathcal{O}_g$ of $g \in \mathrm{SL}_n$ when the conjugacy class of the unipotent part of g is spherical ([C2]). We recall that a conjugacy class $\mathcal{O}$ in G is called *spherical* if there exists a Borel subgroup of G with a dense orbit in $\mathcal{O}$. The proof of the conjecture in [C2] makes use of the representation theory of the quantized Borel subalgebra B_ε introduced in [DCKP3]. This method works for representations corresponding to unipotent spherical orbits and this underlines the correspondence between the geometry of the conjugacy class and the structure of the corresponding irreducible representations.

The same approach is extended in the present paper to the case of any simple Lie algebra $\mathfrak{g}$ and any spherical conjugacy class of G. For this purpose we make use of the analysis of the spherical conjugacy classes in G. In order to determine the semisimple ones, we use the classification of spherical pairs (G, H) where H is a closed connected reductive subgroup of G of the same rank (see [Kr], [Br3]). On the other hand the spherical unipotent conjugacy classes (or, equivalently, the spherical nilpotent adjoint orbits in $\mathfrak{g}$) have been classified by Panyushev in [P2] (see also [P4] for a proof which does not rely on the classification of nilpotent orbits). We finally determine the remaining spherical conjugacy classes in Section 1.3.

Our strategy in the proof of the De Concini–Kac–Procesi conjecture for representations corresponding to spherical orbits relies on a so far unknown characterization of these orbits in terms of elements of the Weyl group W of G. More precisely, let us fix a pair of opposite Borel subgroups (B, B^-). If $\mathcal{O}$ is any conjugacy class in G, there exists a unique element $z = z(\mathcal{O}) \in W$ such that $\mathcal{O} \cap B\dot{z}B$ is open dense in $\mathcal{O}$. We give a characterization of spherical conjugacy classes in the following theorem.

Theorem 1. *Let $\mathcal{O}$ be a conjugacy class in G, $z = z(\mathcal{O})$. Then $\mathcal{O}$ is spherical if and only if* $\dim \mathcal{O} = \ell(z) + \mathrm{rk}(1 - z)$.

Here $\ell(z)$ denotes the length of z and $\mathrm{rk}(1 - z)$ denotes the rank of $1 - z$ in the standard representation of W. In order to make use of the representation theory of B_ε, we show that if $\mathcal{O}$ is a spherical conjugacy class, then $\mathcal{O} \cap B\dot{z}(\mathcal{O})B \cap B^-$ is always nonempty. As a consequence of this fact we obtain our main result on the representation theory of $\mathcal{U}_\varepsilon(\mathfrak{g})$:

Theorem 2. *Assume $\mathfrak{g}$ is a finite dimensional simple complex Lie algebra and ℓ is a good integer. If V is a simple $\mathcal{U}_\varepsilon(\mathfrak{g})$-module whose associated conjugacy class $\mathcal{O}_V$ is spherical, then $\ell^{\frac{1}{2}\dim \mathcal{O}_V}$ divides* $\dim V$.

The paper is structured as follows. In the Preliminaries we introduce notation and recall the classification of the spherical nilpotent orbits of $\mathfrak{g}$. In Section 1 the spherical conjugacy classes of G are analyzed and the main theorems are proved. In establishing Theorem 1 we shall deal with the classical and the exceptional cases separately and we shall consider first the unipotent conjugacy classes of G, then the semisimple conjugacy classes and, finally, the conjugacy classes of G which are neither unipotent nor semisimple. Section 1.4 is dedicated to the analysis of the properties of the correspondence $\mathcal{O} \mapsto z(\mathcal{O})$ when $\mathcal{O}$ is a spherical conjugacy class. In Section 2 the De Concini–Kac–Procesi conjecture is proved for representations corresponding to spherical conjugacy classes. The proof is then extended, using the De Concini–Kac reduction theorem

([DCK2]), to a larger class of representations (see Corollary 32). As a consequence, the De Concini–Kac–Procesi conjecture is proved in type C_2.

As far as notation and terminology are concerned, we shall follow [DCK1] and [Hu1]. In particular for the definition of the classical groups we choose the bilinear forms associated to the following matrices with respect to the canonical bases:

$$\begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix} \text{ for } \mathsf{C}_n, \quad \begin{bmatrix} 0 & I_n \\ I_n & 0 \end{bmatrix} \text{ for } \mathsf{D}_n, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & I_n \\ 0 & I_n & 0 \end{bmatrix} \text{ for } \mathsf{B}_n.$$

In each case we fix the Borel subgroup corresponding to the set of simple roots as described in [Hu1, §12.1].

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1. Preliminaries

Let us introduce the objects of our investigation. Let A be an $n \times n$ Cartan matrix and let $\mathfrak{g}$ be the associated simple complex Lie algebra, with Cartan subalgebra $\mathfrak{h}$. Let Φ be the set of roots relative to $\mathfrak{h}$, Φ^+ a fixed set of positive roots, and $\Delta = \{\alpha_1, \dots, \alpha_n\}$ the corresponding set of simple roots. Let G be a reductive algebraic group with Lie algebra $\mathfrak{g}$, T the maximal torus with Lie algebra $\mathfrak{h}$, B the Borel subgroup determined by Φ^+, and B^- the Borel subgroup opposite to B. Let U (respectively U^-) be the unipotent radical of B (respectively B^-).

Let W be the Weyl group of $\mathfrak{g}$ and let us denote by s_α the reflection corresponding to the root α. By $\ell(w)$ we shall denote the length of the element $w \in W$, and by $\mathrm{rk}(1-w)$ we shall mean the rank of $1-w$ in the standard representation of the Weyl group. By w_0 we shall denote the longest element in W. If $N = N(T)$ is the normalizer of T in G, then $W = N/T$; given an element $w \in W$ we shall denote a representative of w in N by $\dot{w}$. For any root α of $\mathfrak{g}$, we shall denote by $x_\alpha(t)$ the elements of the corresponding root subgroup X_α of G. We shall choose the representatives $\dot{s}_\alpha \in N$ of the reflection $s_\alpha \in W$ as in [Ca1, Theorem 7.2.2]. In particular, we recall that the Weyl group of Sp_{2n} (respectively SO_{2n}) can be identified with the group of permutations σ in the symmetric group S_{2n} such that $\sigma(n+i) = \sigma(i) \pm n$ for all $1 \leqslant i \leqslant n$ (respectively $\sigma(n+i) = \sigma(i) \pm n$ and $\#\{i \leqslant n \mid \sigma(i) > n\}$ is even) and it is exactly for these elements that one can choose a monomial representative in Sp_{2n} (respectively SO_{2n}). For further details see [FH, p. 397]. In case of ambiguity we will denote the Weyl group (respectively Borel subgroups) of an algebraic group K by $W(K)$ (respectively $B(K)$, $B^-(K)$).

In order to describe the unipotent conjugacy classes of G, we will make use of their standard descriptions in terms of Young diagrams and weighted Dynkin diagrams [BC1], [BC2].

For the dimension of these classes we will refer to [Ca2, §13.1].

Definition 1. Let K be a connected algebraic group over $\mathbb{C}$ and let H be a closed subgroup of K. The homogeneous space K/H is called spherical if there exists a Borel subgroup of K with a dense orbit.

Let us recall that the sphericity of K/H depends only on the Lie algebras of K and H. By an abuse of notation, in order to lighten the presentation, we shall identify isogenous groups whenever convenient.

If $\mathfrak{g}$ is of classical type, its spherical nilpotent orbits are classified in the following theorem.

Theorem 3. [P2, §4] *The spherical nilpotent orbits in type* A_n *and* C_n *are those corresponding to Young diagrams with at most two columns. The spherical nilpotent orbits in type* B_n *and* D_n *are those corresponding to Young diagrams with at most two columns or to Young diagrams with three columns and only one row with three boxes.*

In order to deal with the exceptional Lie algebras, we shall also make use of the following theorem.

Theorem 4. [P3, Theorem 3.2] *The spherical nilpotent orbits in* $\mathfrak{g}$ *are those of type* $r\mathsf{A}_1 + s\widetilde{\mathsf{A}}_1$.

1. Spherical conjugacy classes

Definition 2. We say that an element $x \in G$ lies over an element $w \in W$, if $x \in B\dot{w}B$.

Let $\mathcal{O}$ be a conjugacy class in G. There exists a unique element $z = z(\mathcal{O}) \in W$ such that $\mathcal{O} \cap B\dot{z}B$ is open dense in $\mathcal{O}$. In particular,

$$\overline{\mathcal{O}} = \overline{\mathcal{O} \cap B\dot{z}B} \subseteq \overline{B\dot{z}B}. \tag{1.1}$$

It follows that if y is an element of $\overline{\mathcal{O}}$ and $y \in B\dot{w}B$, then $w \leqslant z$ in the Chevalley–Bruhat order of W.

Let us observe that if $\mathcal{O}$ is a spherical conjugacy class of G, and if $B.x$ is the dense B-orbit in $\mathcal{O}$, then $B.x \subseteq B\dot{z}B$.

Theorem 5. *Suppose that* $\mathcal{O}$ *contains an element* $x \in B\dot{w}B$. *Then*

$$\dim B.x \geqslant \ell(w) + \mathrm{rk}(1-w).$$

In particular $\dim \mathcal{O} \geqslant \ell(w) + \mathrm{rk}(1-w)$. *If, in addition,* $\dim \mathcal{O} \leqslant \ell(w) + \mathrm{rk}(1-w)$, *then* $\mathcal{O}$ *is spherical,* $w = z(\mathcal{O})$, *and* $B.x$ *is the dense* B*-orbit in* $\mathcal{O}$.

Proof. Let $U^w = U \cap \dot{w}U^-\dot{w}^{-1}$ and let $B^w = U^wT$. Let us estimate the dimension of the orbit $B^w.x$.

Step 1. The centralizer $C_{B^w}(x)$ is contained in a maximal torus. Let $x = \bar{u}\dot{w}b$ be the unique decomposition of x in $U^w\dot{w}B$ and let u be a unipotent element in $C_{B^w}(x)$. Then $u\bar{u}\dot{w}b = ux = xu = \bar{u}\dot{w}bu$. By the uniqueness of the decomposition it follows that $u = 1$, since $bu \in B$ and $u \in U^w$. Therefore the unipotent radical of $C_{B^w}(x)$ is trivial and, by [Hu2, Proposition 19.4(a)], $C_{B^w}(x)$ is contained in a maximal torus.

Step 2. We have: $\dim C_{B^w}(x) \leqslant n - \mathrm{rk}(1-w)$. Without loss of generality, we may assume that $C_{B^w}(x)$ is contained in T. Let $t \in C_{B^w}(x)$. Then $xtx^{-1} = t$ and, by [SS, §3.1], $\dot{w}t\dot{w}^{-1} = t$. Therefore $C_{B^w}(x) \subset T^w$, where

$$T^w = \{t \in T \mid \dot{w}t\dot{w}^{-1} = t\},$$

thus $\dim C_{B^w}(x) \leqslant \dim T^w = n - \mathrm{rk}(1-w)$.

Now let us observe that:

$$\begin{aligned}\dim B^w.x &= \dim B^w - \dim C_{B^w}(x)\\ &= \ell(w) + n - \dim C_{B^w}(x) \geqslant \ell(w) + n - n + \mathrm{rk}(1-w) = \ell(w) + \mathrm{rk}(1-w).\end{aligned}$$

It follows that if, in addition, $\ell(w) + \mathrm{rk}(1-w) \geqslant \dim \mathcal{O}$, then $\dim C_{B^w}(x) = \dim T^w$ and $\dim \mathcal{O} = \ell(w) + \mathrm{rk}(1-w)$. In particular, $B.x$ is the dense B-orbit in $\mathcal{O}$. □

Proposition 6. *Let $\mathcal{O}$ be a conjugacy class, $z = z(\mathcal{O})$. If there exists an element $w \in W$ such that $w \leqslant z$ and $\dim \mathcal{O} \leqslant \ell(w) + \mathrm{rk}(1-w)$, then $\mathcal{O}$ is spherical with $\dim \mathcal{O} = \ell(w) + \mathrm{rk}(1-w) = \ell(z) + \mathrm{rk}(1-z)$.*

Proof. From $w \leqslant z$ it follows that $\ell(w) + \mathrm{rk}(1-w) \leqslant \ell(z) + \mathrm{rk}(1-z)$. Indeed, it is enough to consider the case $\ell(z) = \ell(w) + 1$: then $\mathrm{rk}(1-z) = \mathrm{rk}(1-w) \pm 1$ so that either $\ell(z) + \mathrm{rk}(1-z) = \ell(w) + \mathrm{rk}(1-w) + 2$, or $\ell(z) + \mathrm{rk}(1-z) = \ell(w) + \mathrm{rk}(1-w)$, and the inequality follows. Therefore $\dim \mathcal{O} \leqslant \ell(w) + \mathrm{rk}(1-w) \leqslant \ell(z) + \mathrm{rk}(1-z)$. By Theorem 5 we obtain $\dim \mathcal{O} = \ell(z) + \mathrm{rk}(1-z) = \ell(w) + \mathrm{rk}(1-w)$. □

Let us observe that it may happen that $w \neq z$.

Corollary 7. *Let $\mathcal{O}$ be a conjugacy class, $z = z(\mathcal{O})$. Let $w_1, \ldots, w_k$ be elements of W such that $\mathcal{O} \cap B\dot{w}_i B \neq \varnothing$ for $i = 1, \ldots, k$, and let us consider the set X of minimal elements in*

$$\{w \in W \mid w \geqslant w_i,\ i = 1, \ldots, k\}.$$

If for every $w \in X$ we have $\dim \mathcal{O} \leqslant \ell(w) + \mathrm{rk}(1-w)$, then $\mathcal{O}$ is spherical.

Proof. Since $w_i \leqslant z$ for $i = 1, \ldots, k$, there exists $w \in X$ such that $w \leqslant z$. Then we conclude the proof by Proposition 6. □

Corollary 8. *Let $\mathcal{O}$ be a conjugacy class. Let w_1, w_2 be elements of W such that $\mathcal{O} \cap B\dot{w}_i B \neq \varnothing$ for $i = 1, 2$. If*

$$\{w \in W \mid w \geqslant w_i,\ i = 1, 2\} = \{w_0\},$$

then $z(\mathcal{O}) = w_0$. If, in addition, $\dim \mathcal{O} \leqslant \ell(w_0) + \mathrm{rk}(1-w_0)$ then $\mathcal{O}$ is spherical. □

Definition 3. Let $\mathcal{O}$ be a conjugacy class. We say that $\mathcal{O}$ is *well placed* if there exists an element $w \in W$ such that

$$\mathcal{O} \cap B^- \cap B\dot{w}B \neq \varnothing \quad \text{and} \quad \dim \mathcal{O} = \ell(w) + \mathrm{rk}(1-w).$$

It follows from Definition 3 and Theorem 5 that if a conjugacy class $\mathcal{O}$ is well placed, then it is spherical and $z(\mathcal{O}) = w$. Our aim is to show that every spherical conjugacy class is well placed.

In the sequel, we will make use of following lemma.

Lemma 9. *Let $\phi : G_1 \to G_2$ be an isogeny of reductive algebraic groups. Let $x_1 \in G_1$, $x_2 = \phi(x_1)$ and let $\mathcal{O}_{x_i}$ be the conjugacy class of x_i in G_i. Let $w \in W = W(G_1) = W(G_2)$ and let $\dot{w}_i$ be a representative of w in G_i. Then $B(G_1)\dot{w}_1 B(G_1) \cap B^-(G_1) \cap \mathcal{O}_{x_1} \neq \varnothing$ if and only if $B(G_2)\dot{w}_2 B(G_2) \cap B^-(G_2) \cap \mathcal{O}_{x_2} \neq \varnothing$.* □

1.1. Unipotent conjugacy classes

In view of Definition 3, we begin this section with a result concerning the intersection between U^- and the (unique) dense B-orbit in a spherical unipotent conjugacy class.

Lemma 10. *Let $\mathcal{O}$ be a unipotent spherical conjugacy class, $B.x$ the (unique) dense B-orbit in $\mathcal{O}$. Then $B.x \cap U^-$ is not empty.*

Proof. Let $g \in \mathcal{O}$ and let P be the canonical parabolic subgroup of G associated to g (see [SS] and §1.3). Then g lies in the unipotent radical P^u of P, and $H = C_G(g) \leqslant P$. Since $\mathcal{O}$ is spherical, there exists a Borel subgroup B_1 of G such that HB_1 is dense in G. In particular, PB_1 is dense in G. Without loss of generality, we may assume $P \geqslant B$, $P = P_{J_1}$ with $J_1 \subseteq \{\alpha_1, \dots, \alpha_n\}$ say, and $B_1 = \dot\tau B \dot\tau^{-1}$, with $\dot\tau \in N_{J_1,\varnothing}$, following the notation in [Ca2, §2.8] (here we have $J_2 = \varnothing$). In our case the subset K of $\{\alpha_1, \dots, \alpha_n\}$ is empty. We recall that $N_{J_1,\varnothing} = \{\dot\sigma \mid \sigma \in D_{J_1,\varnothing}\}$ and that $D_{J_1,\varnothing} = D_{J_1}^{-1}$, where $D_{J_1} = \{\sigma \in W \mid \sigma(\Phi^+_{J_1}) \subseteq \Phi^+\}$. Then $\tau^{-1}(\Phi^+_{J_1}) \subseteq \Phi^+$. We show that $P\dot\tau B\dot\tau^{-1}$ is dense in G if and only if $\tau^{-1}(\Phi^+ \setminus \Phi_{J_1}) \subseteq \Phi^-$ (which then implies that $w_0\tau$ is the longest element of W_{J_1}). We have

$$P \cap \dot\tau B \dot\tau^{-1} = (P^u \cap \dot\tau U \dot\tau^{-1})(L_{J_1} \cap \dot\tau B \dot\tau^{-1}),$$

and $L_{J_1} \cap \dot\tau B\dot\tau^{-1} = L_{J_1} \cap B_1$ is a Borel subgroup of L_{J_1} by [Ca2, Propositions 2.8.7, 2.8.9]. Let us denote by r the number of positive roots in Φ_{J_1} and by s the dimension of $P^u \cap \dot\tau U\dot\tau^{-1}$. Then $P\dot\tau B\dot\tau^{-1}$ is dense in G if and only if $\dim(P \cap \dot\tau B\dot\tau^{-1}) = \dim P + \dim B - \dim G$. Since $\dim P = \dim P^u + \dim L_{J_1} = N + n + r$, $\dim B = N + n$, $\dim L_{J_1} \cap B = n + r$, we get that $P\dot\tau B\dot\tau^{-1}$ is dense in G if and only if $s = 0$, that is $P^u \cap \dot\tau U\dot\tau^{-1} = \{1\}$. This in turn is equivalent to $(\Phi^+ \setminus \Phi_{J_1}) \cap \tau(\Phi^+) = \varnothing$, that is $\tau^{-1}(\Phi^+ \setminus \Phi_{J_1}) \subseteq \Phi^-$, as we wanted.

We are now in the position to exhibit an element in $B.x \cap U^-$. By hypothesis we have $g \in P^u = \prod_{\beta\in\Phi^+\setminus\Phi_{J_1}} X_\beta$. Then $\dot\tau^{-1}g\dot\tau$ lies in $\prod_{\beta\in\Phi^+\setminus\Phi_{J_1}} X_{\tau^{-1}\beta} \leqslant U^-$. On the other hand, from $H\,\dot\tau B\dot\tau^{-1}$ dense in G it follows that $C_G(\dot\tau^{-1}g\dot\tau)\,B$ is dense in G, hence $\dot\tau^{-1}g\dot\tau$ lies in $B.x$. □

Let us observe that we can deal directly with the minimal unipotent conjugacy class.

Proposition 11. *Let $\mathcal{O}$ be the unipotent conjugacy class of type A_1 (minimal orbit). Then $\mathcal{O}$ is well placed.*

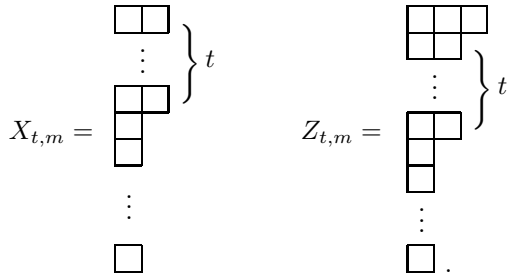
Proof. Let β_1 denote the highest root of $\mathfrak{g}$. Then $x_{-\beta_1}(1)$ is a representative of $\mathcal{O}$. For every positive root α and every $t \neq 0$ we have:

$$x_{-\alpha}(t) = x_\alpha(t^{-1})h\dot s_\alpha x_\alpha(t^{-1}) \tag{1.2}$$

for some $h \in T$ (see [Ca1, p. 106]). In particular $x_{-\beta_1}(1)$ belongs to $B\dot s_{\beta_1}B \cap B^-$. By [CMG, Lemma 4.3.5] we have $\ell(s_{\beta_1}) + \mathrm{rk}(1 - s_{\beta_1}) = \#\{\alpha \in \Phi^+ \mid \alpha \not\perp \beta_1\} + 1 = \dim \mathcal{O}$, and the statement follows. □

1.1.1. *Classical type.* This section is devoted to the analysis of the spherical unipotent conjugacy classes of G when G is of classical type. Since the case of type A_n has been treated in [C2] we shall assume that G is of type B_n, C_n or D_n.

It will be useful for our purposes to fix some notation for Young diagrams corresponding to spherical unipotent conjugacy classes. We will denote by $X_{t,m}$ and $Z_{t,m}$, respectively, the following Young diagrams with m boxes:



(1.3)

By an abuse of notation, given a unipotent element $u \in G$ and a Young diagram of fixed shape J, we will say that $u = J$ if the conjugacy class of u is described by J.

It will be convenient for our purposes to understand when an element of a classical group lies over the longest element w_0 of the Weyl group.

Remark 1. Let $G = \mathrm{Sp}_{2n}$ (respectively SO_{2n} and n even) so that $w_0 = -1$. Then the elements of B^- and B are of the form

$$\left[\begin{array}{c|c} {}^tF^{-1} & 0 \\ \hline F\Sigma & F \end{array}\right] \quad \text{and} \quad \left[\begin{array}{c|c} X & XA \\ \hline 0 & {}^tX^{-1} \end{array}\right]$$

respectively, where Σ and A are symmetric (respectively skew-symmetric), and F and X are upper triangular, invertible matrices. Therefore an element $x \in B^-$ lies over w_0 if there exist upper triangular invertible matrices X and Y, and symmetric (respectively skew-symmetric) matrices A, B such that

$$x = \left[\begin{array}{c|c} {}^tF^{-1} & 0 \\ \hline F\Sigma & F \end{array}\right] = \left[\begin{array}{c|c} X & XA \\ \hline 0 & {}^tX^{-1} \end{array}\right] \left[\begin{array}{c|c} 0 & I_n \\ \hline \mp I_n & 0 \end{array}\right] \left[\begin{array}{c|c} Y & YB \\ \hline 0 & {}^tY^{-1} \end{array}\right]. \tag{1.4}$$

A direct computation shows that (1.4) holds if and only if $F\Sigma = {}^tX^{-1}Y$, i.e., if and only if $F\Sigma$ lies in the big cell of GL_n or, equivalently, if its principal minors are different from zero (see, for example, [Hu2, Exercise 28.8]).

Similarly, if $G = \mathrm{SO}_{2n+1}$, so that $w_0 = -1$, the elements of B^- and B are of the form

$$\left[\begin{array}{c|c|c} 1 & {}^t\psi & 0 \\ \hline 0 & {}^tF^{-1} & 0 \\ \hline -F\psi & F\Sigma & F \end{array}\right] \quad \text{and} \quad \left[\begin{array}{c|c|c} 1 & 0 & {}^t\gamma \\ \hline -X\gamma & X & XA \\ \hline 0 & 0 & {}^tX^{-1} \end{array}\right]$$

respectively, where the symmetric parts of Σ and A are $-(1/2)\psi\,{}^t\psi$ and $-(1/2)\gamma\,{}^t\gamma$ respectively, and F and X are upper triangular, invertible matrices. Therefore an element

$$x = \left[\begin{array}{c|c|c} 1 & {}^t\psi & 0 \\ \hline 0 & {}^tF^{-1} & 0 \\ \hline -F\psi & F\Sigma & F \end{array}\right] \in B^-$$

lies over w_0 if and only if there exist two upper triangular invertible matrices U and X, two vectors γ and c, and two matrices A and S with symmetric part equal to $-(1/2)\gamma\,{}^t\gamma$ and $-(1/2)c\,{}^tc$, respectively, such that the following equality holds:

$$x = \left[\begin{array}{c|c|c} 1 & 0 & {}^t\gamma \\ \hline -X\gamma & X & XA \\ \hline 0 & 0 & {}^tX^{-1} \end{array}\right] \left[\begin{array}{c|c|c} (-1)^n & 0 & 0 \\ \hline 0 & 0 & I_n \\ \hline 0 & I_n & 0 \end{array}\right] \left[\begin{array}{c|c|c} 1 & 0 & {}^tc \\ \hline -Uc & U & US \\ \hline 0 & 0 & {}^tU^{-1} \end{array}\right]. \tag{1.5}$$

A tedious but straightforward computation shows that (1.5) holds if and only if $F\Sigma$ lies in the big cell of GL_n and ${}^t\psi\Sigma^{-1}\psi = (-1)^n - 1$.

Theorem 12. *Let $\mathcal{O}_g$ be a spherical unipotent conjugacy class of an element $g \in G$. Then $\mathcal{O}_g$ is well placed.*

Proof. G of type C_n. For every integer $k = 1, \dots, n$ let us consider the unipotent conjugacy class $\mathcal{O}_k$ of Sp_{2n} parametrized by a Young diagram of shape $X_{k,2n}$. We have: $\dim \mathcal{O}_k = k(2n - k + 1)$.

For every fixed k let us choose the following matrix A_k in $\mathcal{O}_k \cap B^-$:

$$A_k = \left[\begin{array}{c|c} I_n & 0_n \\ \hline I'_k & I_n \end{array}\right]$$

where I'_k is the $n \times n$ diagonal matrix $I'_k = \left[\begin{array}{c|c} I_k & 0 \\ \hline 0 & 0_{n-k} \end{array}\right]$.

In W let us consider the element w_k sending e_i to $-e_i$ for every $i = 1, \dots, k$ and fixing all the other elements of the canonical basis of $\mathbb{C}^n$. We have:

$$\mathrm{rk}(1 - w_k) + \ell(w_k) = k(2n - k + 1) = \dim \mathcal{O}_k.$$

If we choose the representative

$$\dot{w}_k = \left[\begin{array}{cc|cc} 0 & 0 & I_k & 0 \\ 0 & I_{n-k} & 0 & 0 \\ \hline -I_k & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{n-k} \end{array}\right],$$

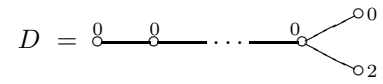
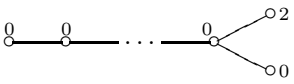
then $A_k = U_k \dot{w}_k B_k$, where

$$U_k = \left[\begin{array}{c|c} I_n & I'_k \\ \hline 0_n & I_n \end{array}\right] \quad \text{and} \quad B_k = \left[\begin{array}{cc|c} -I_k & 0 & -I'_k \\ 0 & I_{n-k} & \\ \hline & 0_n & \begin{array}{cc} -I_k & 0 \\ 0 & I_{n-k} \end{array} \end{array}\right].$$

This identity shows that A_k lies over w_k since U_k and B_k belong to B. This concludes the proof for G of type C_n.

G of type D_n. Let us consider the unipotent conjugacy classes of SO_{2n} associated to Young diagrams either of shape $X_{2k,2n}$ with $k = 1, \dots, [n/2]$, or of shape $Z_{2k,2n}$ with $k = 0, \dots, [n/2] - 1$. Let us recall that when n is even there are two distinct conjugacy

classes, $\mathcal{O}_{n/2}$ and $\mathcal{O}'_{n/2}$, associated to the Young diagram of shape $X_{n,2n}$, with weighted Dynkin diagrams

D =  and D' = 

respectively.

Moreover let $\mathcal{O}_k$ (for $k < \frac{n}{2}$) and $\widetilde{\mathcal{O}}_k$ be the unipotent conjugacy classes with Young diagrams $X_{2k,2n}$ and $Z_{2k,2n}$, respectively.

We have: $\dim \mathcal{O}_k = 2k(2n-2k-1)$, $\dim \widetilde{\mathcal{O}}_k = 4(k+1)(n-k-1)$ and $\dim \mathcal{O}'_{n/2} = n^2 - n = \dim \mathcal{O}_{n/2}$.

Now let us consider the following element w_k in the Weyl group of so_{2n}:

$$w_k : \begin{cases} e_i \mapsto -e_{i+1} & \text{if } i \text{ is odd and } 1 \leqslant i \leqslant 2k-1, \\ e_i \mapsto -e_{i-1} & \text{if } i \text{ is even and } 2 \leqslant i \leqslant 2k, \\ e_i \mapsto e_i & \text{if } i > 2k. \end{cases}$$

Then $\ell(w_k) = 4nk - 4k^2 - 3k$ and $\mathrm{rk}(1-w_k) = k$, therefore $\ell(w_k) + \mathrm{rk}(1-w_k) = \dim \mathcal{O}_k$. Let us introduce the following matrices: $S_1 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, $S_k = \mathrm{diag}(S_1, \ldots, S_1)$ of order $2k$, $J_k = \left[\begin{array}{c|c} S_k & 0 \\ \hline 0 & 0_{n-2k} \end{array}\right]$, $u_k = \left[\begin{array}{c|c} I_n & 0_n \\ \hline J_k & I_n \end{array}\right]$, and $H_k = \left[\begin{array}{c|c} I_n & -J_k \\ \hline 0_n & I_n \end{array}\right]$.

Notice that $H_k \in B$ and u_k lies in $\mathcal{O}_k \cap B^-$ for $k < n/2$. The weighted Dynkin diagram associated to $u_{n/2}$ shows that $u_{n/2} \in \mathcal{O}_{n/2} \cap B^-$. Besides, the following identity of matrices holds: $H_k \dot{w}_k H_k = u_k$ where

$$\dot{w}_k = \left[\begin{array}{cc|cc} 0_{2k} & & J_k & \\ & I_{n-2k} & & \\ \hline J_k & & 0_{2k} & \\ & & & I_{n-2k} \end{array}\right].$$

This shows that u_k lies over w_k for $k = 1, \ldots, [n/2]$.

Now let n be even and $k = n/2$, and let us consider the automorphism $\hat{\tau}$ of SO_{2n} arising from the automorphism τ of the Dynkin diagram interchanging α_{n-1} and α_n. Then $u'_{n/2} = \hat{\tau}(u_{n/2}) \in B^-$ is a representative of the conjugacy class $\mathcal{O}'_{n/2}$ associated to D'. If we apply the map $\hat{\tau}$ to the equality $u_{n/2} = H_{n/2}\dot{w}_{n/2}H_{n/2}$, we find that $u'_{n/2}$ lies over $w^\tau_{n/2} = \tau w_{n/2} \tau \in W \subset \mathrm{Aut}(\Phi)$. As τ permutes simple roots, it is clear that $\ell(w_{n/2}) = \ell(w^\tau_{n/2})$. Therefore, $\ell(w^\tau_{n/2}) + \mathrm{rk}(1 - w^\tau_{n/2}) = \ell(w_{n/2}) + \mathrm{rk}(1 - w_{n/2}) = \dim \mathcal{O}_{n/2} = \dim \mathcal{O}'_{n/2}$. This concludes the proof for G of type D_n and $\mathcal{O}$ a conjugacy class corresponding to a Young diagram of shape $X_{2k,2n}$ with $k \leq [\frac{n}{2}]$.

Now we want to prove the statement for $\widetilde{\mathcal{O}}_k$. Let us first assume $n = 2m$. Let $v_{m-1} = \left[\begin{array}{c|c} {}^tF^{-1} & 0 \\ \hline F\Sigma & F \end{array}\right]$ where:

• F is the upper triangular $n \times n$ matrix with all diagonal elements equal to 1, the first upper off-diagonal equal to $(-1, 0, -1, \dots, 0, -1)$ and zero elsewhere;

• Σ is the skew symmetric matrix whose first upper off-diagonal is $(1, 0, 1, 0, \dots, 0, 1)$, all further odd upper off-diagonals are equal to $(2, 0, 2, 0, \dots, 0, 2)$ and all even off-diagonals are equal to $(0, 0, \dots, 0)$.

One can show that $v_{m-1} \in \widetilde{\mathcal{O}}_{m-1}$ and that $F\Sigma$ belongs to the big cell of GL_{2m}. Therefore, by Remark 1, $v_{m-1} \in B^-$ lies over w_0. Observe that

$$\ell(w_0) + \mathrm{rk}(1 - w_0) = n^2 = 4m^2 = \dim \widetilde{\mathcal{O}}_{m-1}$$

and that equality holds also when $n = 2$, i.e., when SO_{2n} is not simple. Hence the statement is proved for n even and $k = n/2 - 1$.

Let us consider the conjugacy class $\widetilde{\mathcal{O}}_k$ for n not necessarily even and the embedding j_{2k+2} of SO_{4k+4} into SO_{2n}:

$$j_{2k+2} : \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \longmapsto \left[\begin{array}{cc|cc} A & & B & \\ & I_{n-2k-2} & & 0_{n-2k-2} \\ \hline C & & D & \\ & 0_{n-2k-2} & & I_{n-2k-2} \end{array}\right].$$

The embedded image of v_{m-1}, for $m = k + 1$, belongs to B^-. It is a representative of $\widetilde{\mathcal{O}}_k$ and lies over $\eta_k = \begin{bmatrix} -I_{2k+2} & 0 \\ 0 & I_{n-2k-2} \end{bmatrix}$. One can check that $\ell(\eta_k) + \mathrm{rk}(1 - \eta_k) = \dim \widetilde{\mathcal{O}}_k$, so the statement is proved for G of type D_n.

G of type B_n. Let us consider the unipotent conjugacy classes $\mathcal{C}_k$ and $\widetilde{\mathcal{C}}_h$ of shape $X_{2k,2n+1}$ and $Z_{2h,2n+1}$, respectively, with $k = 1, \dots, [n/2]$ and $h = 0, \dots, [(n-1)/2]$. We have: $\dim \mathcal{C}_k = 4nk - 4k^2$ and $\dim \widetilde{\mathcal{C}}_h = 2(h+1)(2n - 2h - 1)$.

Let us consider the following embedding of SO_{2n} in SO_{2n+1}:

$$X \mapsto \left[\begin{array}{c|c} 1 & \\ \hline & X \end{array}\right].$$

Under this embedding a representative of an element $w \in W(\mathrm{SO}_{2n})$ is mapped to a representative of an element in $W(\mathrm{SO}_{2n+1})$. Through the same embedding the Borel subgroup $B(\mathrm{SO}_{2n})$ (respectively $B^-(\mathrm{SO}_{2n})$) can be seen as a subgroup of $B(\mathrm{SO}_{2n+1})$ (respectively $B^-(\mathrm{SO}_{2n+1})$). The image of the representative u_k of the class $\mathcal{O}_k \subset \mathrm{SO}_{2n}$ is a representative of the class $\mathcal{C}_k \subset \mathrm{SO}_{2n+1}$, it belongs to $B^-(\mathrm{SO}_{2n+1})$ and lies over w_k where w_k is the same as in the corresponding case of SO_{2n}. The length of w_k, viewed as an element of $W(\mathrm{SO}_{2n+1})$, is $\ell(w_k) = 4nk - 4k^2 - k$ and $\mathrm{rk}(1 - w_k) = k$, therefore $\ell(w_k) + \mathrm{rk}(1 - w_k) = \dim \mathcal{C}_k$. Hence, we have the statement for $\mathcal{C}_k$.

Similarly, if $k \leqslant [n/2] - 1$, the image of the representative v_k of the class $\widetilde{\mathcal{O}}_k \subset \mathrm{SO}_{2n}$ is a representative of the class $\widetilde{\mathcal{C}}_k \subset \mathrm{SO}_{2n+1}$. It belongs to $B^-(\mathrm{SO}_{2n+1})$ and lies over η_k where η_k is the same as in the corresponding case of SO_{2n}. If we view η_k as an element of $W(\mathrm{SO}_{2n+1})$, we obtain:

$$\ell(\eta_k) + \mathrm{rk}(1 - \eta_k) = 4nk - 4k^2 - 8k + 4n - 4 + 2k + 2 = \dim \widetilde{\mathcal{C}}_k,$$

so the statement holds for $\widetilde{\mathcal{C}}_k$ with $k \leqslant [n/2] - 1$.

Let us now prove the statement for the classes corresponding to Young diagrams with no rows consisting of only one box, i.e., for $\widetilde{\mathcal{C}}_{\frac{n-1}{2}}$ when n is odd. In this case

$$\dim \widetilde{\mathcal{C}}_{\frac{n-1}{2}} = n^2 + n = \ell(w_0) + \mathrm{rk}(1 - w_0).$$

Let us consider the element $v = \left[\begin{array}{c|c|c} 1 & {}^t\psi & 0 \\ \hline 0 & I_n & 0 \\ \hline -\psi & \Sigma & I_n \end{array}\right]$ where $\psi = {}^t(1\ 0\ \ldots\ 0)$ and Σ is the $n \times n$ matrix with diagonal equal to $(-1/2, 0, \ldots, 0)$, first upper off-diagonal equal to $(1, 1, \ldots, 1)$, first lower off-diagonal equal to $(-1, -1, \ldots, -1)$, and zero elsewhere. Then v is a representative of $\widetilde{\mathcal{C}}_{\frac{n-1}{2}}$. One can check that Σ belongs to the big cell of GL_n and that ${}^t\psi\,\Sigma^{-1}\psi = -2$. By Remark 1 we conclude the proof. $\square$

1.1.2. *Exceptional type.* This section is devoted to the analysis of the spherical unipotent conjugacy classes of G when G is of exceptional type. Let us introduce some notation: we shall denote by β_1 the highest root of $\mathfrak{g}$ and, inductively, by β_r, for $r > 1$, the highest root of the root system orthogonal to $\beta_1, \ldots, \beta_{r-1}$ when this is irreducible. Similarly, we shall denote by γ_1 the highest short root of $\mathfrak{g}$ and inductively, by γ_r, for $r > 1$, the highest short root of the root system orthogonal to $\gamma_1, \ldots, \gamma_{r-1}$ when it is irreducible.

Theorem 13. *Let $\mathcal{O}$ be a spherical unipotent conjugacy class. Then $\mathcal{O}$ is well placed.*

Proof. The unipotent spherical conjugacy classes of G are those of type $r\mathsf{A}_1 + s\widetilde{\mathsf{A}}_1$. We shall deal with the different types of orbits separately:

Type A_1. See Proposition 11.

Type $\widetilde{\mathsf{A}}_1$ ($\mathfrak{g}$ of type F_4, G_2). Let $\mathfrak{g}$ be of type G_2. The element $x_{-\gamma_1}(1)$ is a representative of the class $\mathcal{O}$ of type $\widetilde{\mathsf{A}}_1$ and lies over s_{γ_1} by (1.2). Therefore, $z(\mathcal{O}) \geqslant s_{\gamma_1}$. Besides, since $\overline{\mathcal{O}}$ contains the minimal conjugacy class, by (1.1) it follows that $z(\mathcal{O}) \geqslant s_{\beta_1}$, hence, by Corollary 8, $z(\mathcal{O}) = w_0$. We now conclude using Lemma 10 and noticing that $\dim \mathcal{O} = 8 = \ell(w_0) + \mathrm{rk}(1 - w_0)$.

Let $\mathfrak{g}$ be of type F_4. The element $x = x_{-\beta_1}(1)x_{-\beta_2}(1)$ is a representative of the class of type $\widetilde{\mathsf{A}}_1$, as the calculation of its weighted Dynkin diagram shows. By (1.2) x belongs to $B\dot{s}_{\beta_1}\dot{s}_{\beta_2}B$, and one can check that $\ell(s_{\beta_1}s_{\beta_2}) + \mathrm{rk}(1 - s_{\beta_1}s_{\beta_2}) = 22 = \dim \mathcal{O}$.

Type $2\mathsf{A}_1$ ($\mathfrak{g}$ of type E_6, E_7, E_8). The element $x_{-\beta_1}(1)x_{-\beta_2}(1)$ is a representative of this class. By construction and by (1.2) $x_{-\beta_1}(1)x_{-\beta_2}(1)$ lies over $s_{\beta_1}s_{\beta_2}$. One can check that $\ell(s_{\beta_1}s_{\beta_2}) + \mathrm{rk}(1 - s_{\beta_1}s_{\beta_2}) = \dim \mathcal{O}$.

Type $3\mathsf{A}_1$ ($\mathfrak{g}$ of type E_6, E_7, E_8). If $\mathfrak{g}$ is of type E_7, there are two conjugacy classes of type $3\mathsf{A}_1$ that, following [BC2], we shall denote by $(3\mathsf{A}_1)'$, $(3\mathsf{A}_1)''$. A representative of the class $(3\mathsf{A}_1)''$ is $x_{-\beta_1}(1)x_{-\beta_2}(1)x_{-\alpha_7}(1)$, as one can verify by computing its weighted Dynkin diagram. Relation (1.2) implies that $x_{-\beta_1}(1)x_{-\beta_2}(1)x_{-\alpha_7}(1)$ lies over $s_{\beta_1}s_{\beta_2}s_{\alpha_7}$, since α_7 is orthogonal to β_1 and β_2. One can verify that $\ell(s_{\beta_1}s_{\beta_2}s_{\alpha_7}) + \mathrm{rk}(1 - s_{\beta_1}s_{\beta_2}s_{\alpha_7}) = 54 = \dim \mathcal{O}$.

In order to handle the remaining classes of type $3\mathsf{A}_1$, we consider subalgebras of type D_4 in $\mathfrak{g}$ and the corresponding immersions of algebraic groups. We realize the root systems of these subalgebras as the sets of roots orthogonal to $\ker(1-w)$ where $w \in W$ is chosen as follows:

- $w_0 = s_{\beta_1} s_{\beta_2} s_{\beta_3} s_{\alpha_4}$ if $\mathfrak{g}$ is of type E_6;
- $s_{\beta_1} s_{\beta_2} s_{\alpha_2+\alpha_3+2\alpha_4+\alpha_5} s_{\alpha_3}$ if $\mathfrak{g}$ is of type E_7;
- $s_{\beta_1} s_{\beta_2} s_{\beta_3} s_{\alpha_7}$ if $\mathfrak{g}$ is of type E_8.

In Theorem 12 we proved that if $\mathcal{O}'$ is the class of type $3\mathsf{A}_1$ of D_4, then $z(\mathcal{O}')$ is the longest element of the Weyl group of D_4. By construction, in each case w is the longest element of the Weyl group of the corresponding copy of D_4. One can verify that $\ell(w) + \mathrm{rk}(1-w)$ is equal to the dimension of the unipotent orbit of type $3\mathsf{A}_1$ if $\mathfrak{g}$ is of type E_6 or E_8 and $(3\mathsf{A}_1)'$, if $\mathfrak{g}$ is of type E_7. In the latter case, Theorem 5 implies that a representative of the class of type $3\mathsf{A}_1$ in D_4 is a representative of the class of type $(3\mathsf{A}_1)'$.

Type $\mathsf{A}_1 + \widetilde{\mathsf{A}}_1$ ($\mathfrak{g}$ of type F_4). Let us consider the subgroup of G of type B_4 generated by $X_{\pm\alpha}$ for $\alpha \in \{\alpha_2 + 2\alpha_3 + 2\alpha_4,\, \alpha_1,\, \alpha_2,\, \alpha_3\}$. By Theorem 12, if $\mathcal{O}'$ is the conjugacy class of type $\mathsf{A}_1 + \widetilde{\mathsf{A}}_1$ in B_4, then $z(\mathcal{O}')$ is the longest element of the Weyl group of B_4 and coincides with the longest element of W. Therefore there is a representative of the conjugacy class of type $\mathsf{A}_1 + \widetilde{\mathsf{A}}_1$ in F_4 in $B\dot{w}_0 B$. We have: $\dim \mathcal{O} = 28 = \ell(w_0) + \mathrm{rk}(1-w_0)$.

Type $4\mathsf{A}_1$ ($\mathfrak{g}$ of type E_7, E_8). We observe that $\dim \mathcal{O} = \dim B = \ell(w_0) + \mathrm{rk}(1-w_0)$ therefore we need to prove that $z(\mathcal{O}) = w_0$. In order to do so we shall apply Corollary 8.

Let us consider the following subalgebras of type D_6 in $\mathfrak{g}$ and the corresponding immersions of algebraic groups: as above, we realize the root systems of these subalgebras as the sets of roots orthogonal to $\ker(1-w_i)$ where the w_i's in W are chosen as follows:

- if $\mathfrak{g}$ is of type E_7:
  $w_1 = s_{\beta_1} s_{\beta_2} s_{\alpha_2+\alpha_3+2\alpha_4+\alpha_5} s_{\alpha_3} s_{\alpha_2} s_{\alpha_5} = w_0 s_{\alpha_7}$;
  $w_2 = s_{\beta_1} s_{\beta_2} s_{\alpha_2+\alpha_3+2\alpha_4+\alpha_5} s_{\alpha_3} s_{\alpha_2} s_{\alpha_7} = w_0 s_{\alpha_5}$;
- if $\mathfrak{g}$ is of type E_8:
  $w_1 = s_{\beta_1} s_{\beta_2} s_{\beta_3} s_{\alpha_2+\alpha_3+2\alpha_4+\alpha_5} s_{\alpha_2} s_{\alpha_5} = w_0 s_{\alpha_3} s_{\alpha_7}$;
  $w_2 = s_{\beta_1} s_{\beta_2} s_{\beta_3} s_{\alpha_2+\alpha_3+2\alpha_4+\alpha_5} s_{\alpha_3} s_{\alpha_7} = w_0 s_{\alpha_2} s_{\alpha_5}$.

It is shown in Theorem 12 that if $\mathcal{O}'$ is the conjugacy class of type $4\mathsf{A}_1$ in D_6 then $z(\mathcal{O}')$ is the longest element of the Weyl group of D_6 which coincides with w_i in each case. The only element in W which is greater than or equal to both w_1 and w_2 is w_0, hence the statement. □

1.2. Semisimple conjugacy classes

As for spherical unipotent conjugacy classes, we establish a result concerning the intersections $B^- \cap \mathcal{O} \cap B\dot{w}B$, with $w \in W$, when $\mathcal{O}$ is a semisimple conjugacy class.

Lemma 14. *Let t be a semisimple element of G such that $\mathcal{O}_t \cap B\dot{w}B \neq \varnothing$ for some $w \in W$. Then $B^- \cap \mathcal{O}_t \cap B\dot{w}B \neq \varnothing$.*

Proof. Without loss of generality, we may assume that t lies in T. Let $g \in G$ be such that $g^{-1}tg \in B\dot{w}B$, and let $g = u_\sigma \dot{\sigma} b$ be the unique decomposition of g in $U^\sigma \dot{\sigma} B$. Then $\dot{\sigma}^{-1}u_\sigma^{-1}tu_\sigma\dot{\sigma}$ belongs to $\mathcal{O}_t \cap B\dot{w}B \cap B^-$ since $\dot{\sigma}^{-1}u_\sigma\dot{\sigma}$ lies in U^-. $\square$

1.2.1. *Classical type.* In this section we shall analyze the spherical semisimple conjugacy classes of G when G is of classical type. Using [Br3, Remarque §0], we list the spherical semisimple conjugacy classes (up to a certain element of the torus) in Table 1, where we indicate a representative g of each semisimple class $\mathcal{O}_g$, the dimension of $\mathcal{O}_g$, and the structure of the Lie algebra of the centralizer of g. By ζ we shall denote a primitive $2n$-th root of 1. Note that D_1 must be interpreted as a 1-dimensional torus T_1 wherever it occurs and that A_0 and B_0 denote the trivial Lie algebra.

Remark 2. It is well known that $X, Y \in \mathrm{Sp}_{2n}$ are conjugated in Sp_{2n} if and only if they are conjugated in GL_{2n}. The same holds for X, Y in the orthogonal group O_m (see, for example, [SS, Ex. 2.15 (ii)]). It follows that if $X, Y \in \mathrm{SO}_m$ are conjugated in GL_m and $C_{\mathrm{O}_m}(X) \not\subset \mathrm{SO}_m$, then X and Y are conjugated in SO_m. In contrast, if $C_{\mathrm{O}_m}(X) \subset \mathrm{SO}_m$, then the conjugacy class of X in O_m splits into two distinct conjugacy classes in SO_m of the same dimension.

Theorem 15. *Let $\mathcal{O}_g$ be a spherical semisimple conjugacy class of G. Then $\mathcal{O}_g$ is well placed.*

Table 1

	$\dim \mathcal{O}_g$	$\mathrm{Lie}(C_G(g))$
A_{n-1}		
$g_k = \mathrm{diag}(-I_k, I_{n-k})$ k even and $1 \leqslant k \leqslant \left[\frac{n}{2}\right]$	$2k(n-k)$	$\mathbb{C} + \mathsf{A}_{k-1} + \mathsf{A}_{n-k-1}$
$g_{\zeta,k} = \mathrm{diag}(-\zeta I_k, \zeta I_{n-k})$ k odd and $1 \leqslant k \leqslant \left[\frac{n}{2}\right]$	$2k(n-k)$	$\mathbb{C} + \mathsf{A}_{k-1} + \mathsf{A}_{n-k-1}$
B_n		
$\rho_k = \mathrm{diag}(1, -I_k, I_{n-k}, -I_k, I_{n-k})$ $1 \leqslant k \leqslant n$	$2k(2n-2k+1)$	$\mathsf{D}_k + \mathsf{B}_{n-k}$
$b_\lambda = \mathrm{diag}(1, \lambda I_n, \lambda^{-1} I_n)$ $\lambda \in \mathbb{C} \setminus \{0, \pm 1\}$	$n^2 + n$	$\mathbb{C} + \mathsf{A}_{n-1}$
C_n		
$\sigma_k = \mathrm{diag}(-I_k, I_{n-k}, -I_k, I_{n-k})$ $1 \leqslant k \leqslant \left[\frac{n}{2}\right]$	$4k(n-k)$	$\mathsf{C}_k + \mathsf{C}_{n-k}$
$c_\lambda = \mathrm{diag}(\lambda, I_{n-1}, \lambda^{-1}, I_{n-1})$ $\lambda \in \mathbb{C} \setminus \{0, \pm 1\}$	$4n-2$	$\mathbb{C} + \mathsf{C}_{n-1}$
$c = \mathrm{diag}(i\, I_n, -i\, I_n)$	$n^2 + n$	$\mathbb{C} + \widetilde{A}_{n-1}$
D_n		
$\sigma_k = \mathrm{diag}(-I_k, I_{n-k}, -I_k, I_{n-k})$ $1 \leqslant k \leqslant \left[\frac{n}{2}\right]$	$4k(n-k)$	$\mathsf{D}_k + \mathsf{D}_{n-k}$
$c = \mathrm{diag}(i\, I_n, -i\, I_n)$	$n^2 - n$	$\mathbb{C} + \mathsf{A}_{n-1}$
$d = \mathrm{diag}(i\, I_{n-1}, -i, -i\, I_{n-1}, i)$	$n^2 - n$	$\mathbb{C} + \mathsf{A}_{n-1}$

Proof. For each class $\mathcal{O}_g$ we shall exhibit an element w of the Weyl group such that $\dim \mathcal{O}_g = \ell(w) + \mathrm{rk}(1-w)$, and a representative of $\mathcal{O}_g$ in $B\dot{w}B$. The proof will follow from Lemma 14.

Type A_{n-1}. Let $t_k = \begin{bmatrix} -\eta I_k & 0 & 0 \\ 0 & \eta I_{n-2k} & 0 \\ -2\eta Y_k & 0 & \eta I_k \end{bmatrix}$, where $\eta = \begin{cases} \zeta & \text{if } k \text{ is odd,} \\ 1 & \text{if } k \text{ is even,} \end{cases}$

and Y_k is the $k \times k$ matrix $\begin{bmatrix} 0 & & 1 \\ & \iddots & \\ 1 & & 0 \end{bmatrix}$. Then $t_k \in O_{g_k} \cap B^-$ if k is even and $t_k \in \mathcal{O}_{g_{\zeta,k}} \cap B^-$ if k is odd. As in the proof of [C2, Theorem 3.4], $t_k \in B\dot{w}_k B$ where

$$w_k = (n, n-1, \dots, n-k+1, k+1, k+2, \dots, n-k, k, \dots, 1)$$

and $\ell(w_k) + \mathrm{rk}(1-w_k) = 2k(n-k)$.

Type C_n. Let us consider the conjugacy class $\mathcal{O}_{\sigma_k}$. The following element $\dot{v}_k$ lies in $N \cap \mathcal{O}_{\sigma_k}$:

$$\dot{v}_k = \left[\begin{array}{cc|cc} 0_{2k} & 0 & S_k & 0 \\ 0 & I_{n-2k} & 0 & 0_{n-2k} \\ \hline -S_k & 0 & 0_{2k} & 0 \\ 0 & 0_{n-2k} & 0 & I_{n-2k} \end{array}\right]$$

where S_k is the $2k \times 2k$ matrix introduced in the proof of Theorem 12. Let v_k be the image of $\dot{v}_k$ in W. Then $\ell(v_k) + \mathrm{rk}(1-v_k) = 4nk - 4k^2 = 4k(n-k) = \dim \mathcal{O}_{\sigma_k}$.

Let us now consider the class $\mathcal{O}_{c_\lambda}$. Let us first assume $n = 2$. With the help of Remark 1, one can check that the element $x = \left[\begin{array}{c|c} A & 0 \\ \hline C & D \end{array}\right] = \left[\begin{array}{cc|cc} \lambda & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 1 & 1 & \lambda^{-1} & 0 \\ \lambda & 0 & 0 & 1 \end{array}\right]$ is contained in $\mathcal{O}_{c_\lambda} \cap B^- \cap B\dot{w}_0 B$. Let us now suppose $n > 2$. Then the element $\left[\begin{array}{cc|cc} A & & & \\ & I_{n-2} & & 0_n \\ \hline C & & D & \\ & 0_{n-2} & & I_{n-2} \end{array}\right]$ is a representative of $\mathcal{O}_{c_\lambda}$ lying over the following element w of W:

$$w(e_i) = \begin{cases} -e_i & \text{if } i = 1, 2, \\ e_i & \text{if } i \neq 1, 2. \end{cases}$$

We have: $\ell(w) + \mathrm{rk}(1-w) = 4n - 2 = \dim \mathcal{O}_{c_\lambda}$.

Let us now consider the class $\mathcal{O}_c$. One can check that the element $\left[\begin{array}{c|c} 0 & I_n \\ \hline -I_n & 0 \end{array}\right]$ lies in $\mathcal{O}_c \cap B\dot{w}_0 B$ and that $\dim \mathcal{O}_c = n^2 + n = \ell(w_0) + \mathrm{rk}(1-w_0)$.

Type D_n. Let us notice that the centralizer of σ_k in O_{2n} contains the element

$$\left[\begin{array}{cc|cc} 0 & & 1 & \\ & I_{n-1} & & 0_{n-1} \\ \hline 1 & & 0 & \\ & 0_{n-1} & & I_{n-1} \end{array}\right],$$

which does not lie in SO_{2n}. By Remark 2, an element $x \in \mathrm{SO}_{2n}$ belongs to $\mathcal{O}_{\sigma_k}$ if and only if it is conjugated to σ_k in GL_{2n}.

Let us consider the element $\widetilde{w}_k$ of $W(\mathrm{SO}_{2n})$ represented in N by:

$$\dot{\widetilde{w}}_k = \left[\begin{array}{cc|cc} 0_{2k} & & I_{2k} & \\ & I_{n-2k} & & 0_{n-2k} \\ \hline I_{2k} & & 0_{2k} & \\ & 0_{n-2k} & & I_{n-2k} \end{array}\right].$$

Then $\dot{\widetilde{w}}_k$ lies in $\mathcal{O}_{\sigma_k}$ and $\ell(\widetilde{w}_k) + \mathrm{rk}(1-\widetilde{w}_k) = 4k(n-k) = \dim \mathcal{O}_{\sigma_k}$.

Let us consider the conjugacy class $\mathcal{O}_c$ of c. In this case $C_{\mathrm{O}_{2n}}(c) \subset \mathrm{SO}_{2n}$ so the conjugacy class of c in O_{2n} splits into the conjugacy classes $\mathcal{O}_c$ and $\mathcal{O}_d$ in SO_{2n}. Let J_k be the $n \times n$ matrices introduced in the proof of Theorem 12. If n is even, then the element $\dot{w} = \left[\begin{array}{c|c} 0 & J_{n/2} \\ \hline J_{n/2} & 0 \end{array}\right] \in N$ is a representative of $\mathcal{O}_c$. Besides, if $w \in W$ is the image of $\dot{w}$ in W, then $\ell(w)+\mathrm{rk}(1-w) = n^2-n = \dim \mathcal{O}_c$. If n is odd, then the element $\dot{w}' = \left[\begin{array}{cc|cc} 0_{n-1} & & J_{(n-1)/2} & \\ & i & & \\ \hline & J_{(n-1)/2} & 0_{n-1} & \\ & & & -i \end{array}\right]$ lies in $\mathcal{O}_c \cap N$ and $\ell(w') + \mathrm{rk}(1-w') = n^2 - n = \dim \mathcal{O}_c$ where w' is the image of $\dot{w}'$ in W.

Let us now consider the class $\mathcal{O}_d$. If n is odd, then $-d \in \mathcal{O}_c$ so that $z(\mathcal{O}_d) = w = z(\mathcal{O}_c)$. If n is even, then $d = \hat{\tau}(c)$, where $\hat{\tau}$ is the automorphism of SO_{2n} introduced in the proof of Theorem 12. Therefore $\hat{\tau}(\dot{w}) \in N$ is a representative of $\mathcal{O}_d$ and its projection w^τ is such that $\ell(w^\tau) + \mathrm{rk}(1-w^\tau) = n^2 - n = \dim \mathcal{O}_d$.

Type B_n. Let ρ_k with $k = 1, \ldots, n$ be the semisimple elements of SO_{2n+1} introduced in Table 1. The following cases need to be analysed separately.

Case I: $1 \leqslant k \leqslant [\frac{n}{2}]$. We already proved that, under these hypotheses, the conjugacy class $\mathcal{O}_{\sigma_k}$ of σ_k in SO_{2n} contains the element

$$\dot{\widetilde{w}}_k = \left[\begin{array}{cc|cc} 0 & & I_{2k} & \\ & I_{n-2k} & & 0_{n-2k} \\ \hline I_{2k} & & 0 & \\ & 0_{n-2k} & & I_{n-2k} \end{array}\right].$$

Then $\dot{v}_k = \left[\begin{array}{cc} 1 & \\ & \dot{\widetilde{w}}_k \end{array}\right]$ lies in $\mathcal{O}_{\rho_k} \cap N$. Let v_k be the element in $W(\mathrm{SO}_{2n+1})$ represented by $\dot{v}_k$. Then $\ell(v_k) + \mathrm{rk}(1-v_k) = 2k(2n-2k+1) = \dim \mathcal{O}_{\rho_k}$.

Case II: $[\frac{n}{2}] < k \leqslant n$. Let us consider the following element of N:

$$\dot{Z}_{n-k} = \left[\begin{array}{ccc|ccc} -1 & & & & & \\ & 0_{2(n-k)+1} & & & -I_{2(n-k)+1} & \\ & & -I_{2k-n-1} & & & 0_{2k-n-1} \\ \hline & -I_{2(n-k)+1} & & & 0_{2(n-k)+1} & \\ & & 0_{2k-n-1} & & & -I_{2k-n-1} \end{array}\right].$$

Since the element $\mathrm{diag}(-1, I_{2n})$ belongs to the centralizer $C_{\mathrm{O}_{2n+1}}(\rho_k)$, it follows from Remark 2 that $\dot{Z}_{n-k}$ lies in $\mathcal{O}_{\rho_k}$. Besides, $\ell(Z_{n-k}) + \mathrm{rk}(1 - Z_{n-k}) = 2k(2n-2k+1) = \dim \mathcal{O}_{\rho_k}$.

Finally, let $\mathcal{O}_{b_\lambda}$ be the conjugacy class of b_λ. Then $\dim \mathcal{O}_{b_\lambda} = n^2 + n = \ell(w_0) + \mathrm{rk}(1 - w_0)$. Let

$$v = \left[\begin{array}{c|c|c} 1 & {}^t\psi & 0 \\ \hline 0 & \lambda^{-1} I_n & 0 \\ \hline -\lambda\psi & \lambda\Sigma & \lambda I_n \end{array}\right]$$

where $\psi = {}^t(1\ 0\ \dots\ 0)$ and Σ is the $n \times n$ matrix with diagonal $(-1/2, 0, \dots, 0)$, first upper off-diagonal $(1, 1, \dots, 1)$, first lower off-diagonal $(-1, -1, \dots, -1)$, and with zero elsewhere. Since the element $\mathrm{diag}(-1, I_{2n})$ belongs to the centralizer in O_{2n+1} of b_λ, v lies in $\mathcal{O}_{b_\lambda}$ and, by Remark 1, lies over w_0. □

1.2.2. *Exceptional type.* In this section we shall analyze the spherical semisimple conjugacy classes of G when G is of exceptional type. Using [Br3, Remarque §0], we are able to list the spherical semisimple conjugacy classes up to a certain element of the torus. The results are collected in Table 2, where we indicate a representative g of each semisimple class $\mathcal{O}_g$, the dimension of $\mathcal{O}_g$, and the structure of the Lie algebra of the centralizer of g. If $\mathfrak{g}$ has rank n, we shall denote by $\check{\omega}_i$, for $i = 1, \dots, n$, the elements in $\mathfrak{h}$ defined by

$$\langle \alpha_j, \check{\omega}_i \rangle = \delta_{ji}, \quad j = 1, \dots, n.$$

Theorem 16. *Let $\mathcal{O}_g$ be a spherical semisimple conjugacy class. Then $\mathcal{O}_g$ is well placed.*

Table 2

	$\dim \mathcal{O}_g$	$\mathrm{Lie}(C_G(g))$	w
E_6			
$p_1 = \exp(\pi i \check{\omega}_2)$	40	$\mathsf{A}_1 + \mathsf{A}_5$	$w_0 = s_{\beta_1} s_{\beta_2} s_{\beta_3} s_{\alpha_4}$
$p_2 = \exp(\pi i \check{\omega}_1)$	32	$\mathbb{C} + \mathsf{D}_5$	$s_{\beta_1} s_{\beta_2}$
E_7			
$q_1 = \exp(\pi i \check{\omega}_2)$	70	A_7	w_0
$q_2 = \exp(\pi i \check{\omega}_1)$	64	$\mathsf{A}_1 + \mathsf{D}_6$	$s_{\beta_1} s_{\beta_2} s_{\alpha_2+\alpha_3+2\alpha_4+\alpha_5} s_{\alpha_3}$
$q_3 = \exp(\pi i \check{\omega}_7)$	54	$\mathbb{C} + \mathsf{E}_6$	$s_{\beta_1} s_{\beta_2} s_{\alpha_7}$
E_8			
$r_1 = \exp(\pi i \check{\omega}_1)$	128	D_8	w_0
$r_2 = \exp(\pi i \check{\omega}_8)$	112	$\mathsf{A}_1 + \mathsf{E}_7$	$s_{\beta_1} s_{\beta_2} s_{\beta_3} s_{\alpha_7}$
F_4			
$f_1 = \exp(\pi i \check{\omega}_1)$	28	$\mathsf{A}_1 + \mathsf{C}_3$	w_0
$f_2 = \exp(\pi i \check{\omega}_4)$	16	B_4	s_{γ_1}
G_2			
$e_1 = \exp(\pi i \check{\omega}_2)$	8	$\mathsf{A}_1 + \tilde{A}_1$	w_0
$e_2 = \exp((2\pi i \check{\omega}_1)/3)$	6	A_2	s_{γ_1}

Proof. Let us consider the conjugacy class $\mathcal{O}_{p_2}$ in E_6 and the element

$$z = \dot{s}_{\beta_1}\dot{s}_{\beta_2}x_{\beta_1}(1)x_{\beta_2}(1)\exp(\pi i\check{\omega}_1)x_{\beta_2}(-1)x_{\beta_1}(-1)\dot{s}_{\beta_2}^{-1}\dot{s}_{\beta_1}^{-1}.$$

Then $z = x_{-\beta_1}(t_1)x_{-\beta_2}(t_2)\dot{s}_{\beta_1}\dot{s}_{\beta_2}x_{\beta_2}(1)x_{\beta_1}(1)\exp(\pi i\check{\omega}_1)\dot{s}_{\beta_2}^{-1}\dot{s}_{\beta_1}^{-1}$ for some t_1 and t_2 different from zero so that $z = x_{-\beta_1}(2t_1)x_{-\beta_2}(2t_2)h$ for some $h \in T$. Hence z lies in $\mathcal{O}_{p_2} \cap B\dot{s}_{\beta_1}\dot{s}_{\beta_2}B \cap B^-$. Besides, $\ell(s_{\beta_1}s_{\beta_2}) + \mathrm{rk}(1 - s_{\beta_1}s_{\beta_2}) = 32 = \dim \mathcal{O}_{p_2}$.

In a similar way, for the conjugacy class $\mathcal{O}_{q_3}$ in E_7, let us consider the element

$$\begin{aligned} y &= \dot{s}_{\beta_1}\dot{s}_{\beta_2}\dot{s}_{\alpha_7}x_{\beta_1}(1)x_{\beta_2}(1)x_{\alpha_7}(1)\exp(\pi i\check{\omega}_7)x_{\alpha_7}(-1)x_{\beta_2}(-1)x_{\beta_1}(-1)\dot{s}_{\alpha_7}^{-1}\dot{s}_{\beta_2}^{-1}\dot{s}_{\beta_1}^{-1} \\ &= x_{-\beta_1}(t_1)x_{-\beta_2}(t_2)x_{-\alpha_7}(t_3)\dot{s}_{\beta_1}\dot{s}_{\beta_2}\dot{s}_{\alpha_7}x_{\alpha_7}(1)x_{\beta_2}(1)x_{\beta_1}(1)\exp(\pi i\check{\omega}_7)\dot{s}_{\alpha_7}^{-1}\dot{s}_{\beta_2}^{-1}\dot{s}_{\beta_1}^{-1} \end{aligned}$$

for some t_1, t_2, and t_3 different from zero. Then $y = x_{-\beta_1}(2t_1)x_{-\beta_2}(2t_2)x_{-\alpha_7}(2t_3)h$ for some $h \in T$. Hence, y lies in $\mathcal{O}_{q_3} \cap B\dot{s}_{\beta_1}\dot{s}_{\beta_2}\dot{s}_{\alpha_7}B \cap B^-$. Besides, $\ell(s_{\beta_1}s_{\beta_2}s_{\alpha_7}) + \mathrm{rk}(1 - s_{\beta_1}s_{\beta_2}s_{\alpha_7}) = 54 = \dim \mathcal{O}_{q_3}$.

Now let $\mathfrak{g}$ be of type F_4 and let us consider the conjugacy class $\mathcal{O}_{f_2}$. Let us fix a short root γ which does not belong to the root system of $C_G(f_2)$ (which is of type B_4), and let $w \in W$ be such that $w(\gamma) = -\gamma_1$. Then we have:

$$x := \dot{w}x_{\gamma}(-1)f_2x_{\gamma}(1)\dot{w}^{-1} = x_{-\gamma_1}(t)h$$

for some $t \neq 0$ and some $h \in T$. Therefore x lies in $\mathcal{O}_{f_2} \cap B\dot{s}_{\gamma_1}B \cap B^-$. Since the root system of type F_4 is self-dual, we have:

$$\ell(s_{\gamma_1}) + \mathrm{rk}(1 - s_{\gamma_1}) = \ell(s_{\beta_1}) + \mathrm{rk}(1 - s_{\beta_1}) = 16 = \dim \mathcal{O}_{f_2}.$$

Now let $\mathfrak{g}$ be of type G_2. Then the element

$$\dot{s}_{\gamma_1}^{-1}x_{\gamma_1}(-1)e_2x_{\gamma_1}(1)\dot{s}_{\gamma_1} = \dot{s}_{\gamma_1}^{-1}x_{\gamma_1}(t)e_2\dot{s}_{\gamma_1},$$

for some $t \neq 0$, lies in $B\dot{s}_{\gamma_1}B \cap \mathcal{O}_{e_2} \cap B^-$. As in type F_4 we have:

$$\ell(s_{\gamma_1}) + \mathrm{rk}(1 - s_{\gamma_1}) = \ell(s_{\beta_1}) + \mathrm{rk}(1 - s_{\beta_1}) = 6 = \dim \mathcal{O}_{e_2}.$$

For the remaining spherical semisimple conjugacy classes we shall assume $G = G_{\mathrm{ad}}$ and use Lemma 9. For each of these classes $\mathcal{O}_g$ we shall prove the statement by exhibiting an element $\dot{w} \in N \cap \mathcal{O}_g$ such that $\ell(w) + \mathrm{rk}(1 - w) = \dim \mathcal{O}_g$ and by using Lemma 14. The elements w's are listed in Table 2. Let us observe that for every element w in Table 2 corresponding to these classes, we can choose a representative $\dot{w} \in N$ of order two in G_{ad}. For $w = w_0$, when $w_0 = -1$, this fact was observed in [Vö, Lemma 2]. In general this can be seen using the expression of w as a product of reflections with respect to mutually orthogonal roots as in Table 2 and [Ca1, Lemma 7.2.1]. From the analysis of the conjugacy classes of the involutions of G_{ad} in [K1] (see also [H, §X.5]), we deduce $\dim \mathcal{O}_{\dot{w}} \leqslant \ell(w_0) + \mathrm{rk}(1 - w_0)$. If $w = w_0$, by Theorem 5, $\dim \mathcal{O}_{\dot{w}_0} = \ell(w_0) + \mathrm{rk}(1 - w_0)$. By [H, §X.5, Tables II, III] there is only one conjugacy class of involutions in G_{ad} whose dimension is equal to $\ell(w_0) + \mathrm{rk}(1 - w_0)$. Therefore $\dot{w}_0$ lies in the spherical semisimple conjugacy class of maximal dimension.

Finally we are left with the conjugacy classes $\mathcal{O}_{q_2}$ and $\mathcal{O}_{r_2}$. In order to prove that the element $\dot{w}$ lies in the corresponding orbit $\mathcal{O}_g$ when g is either q_2 or r_2, it is sufficient to use [H, §X.5, Tables II, III] and estimate the dimension of the centralizer of $\dot{w}$. One can perform this computation in the Lie algebra of G, namely, calculating the dimension of $\mathrm{Lie}(C_G(\dot{w})) = \{x \in \mathfrak{g} \mid \mathrm{Ad}(\dot{w})(x) = x\}$. This can be done analyzing the eigenspaces of $\mathrm{Ad}(\dot{w})$ in the stable subspaces of the form $\mathfrak{g}_\alpha + \mathfrak{g}_{w(\alpha)}$, with the use of [Ca1, Lemma 7.2.1]. □

1.3. The remaining conjugacy classes

In this section we shall investigate the spherical conjugacy classes $\mathcal{O}_g$ of elements $g \in G$ that are neither semisimple nor unipotent. If the conjugacy class $\mathcal{O}_g$ of an element g with Jordan decomposition su is spherical, then both $\mathcal{O}_s$ and $\mathcal{O}_u$ are spherical. Indeed, if $BC_G(g)$ is dense in G, then also $BC_G(s) \supset BC_G(g)$ and $BC_G(u) \supset BC_G(g)$ are dense in G. Therefore the semisimple parts of the elements we shall consider in this section are those occurring in § 1.2. Let us notice that when the identity component of the centralizer of such a semisimple element is not simple, it is isomorphic either to an almost direct product G_1G_2 or to an almost direct product $G_1G_2T_1$, where T_1 is a one-dimensional torus. When this is the case, we will identify G_j with a subgroup of G and write a unipotent element commuting with s as a pair (u_1, u_2) or, equivalently, as a product u_1u_2 with $u_j \in G_j$ unipotent. If the conjugacy class of $su = su_1u_2$ is spherical, then the conjugacy class of $u_j \in G_j$ is necessarily spherical.

In the sequel we will need the following definition and results.

Definition 4. Let $\bar{G}$ be a reductive connected algebraic group. Let H be a closed subgroup of $\bar{G}$. We say that $H = H^uK$ is a Levi decomposition of H if H^u is the unipotent radical of H and K is a maximal reductive subgroup of H.

In characteristic zero such a decomposition always exists.

Proposition 17. [Br3, Proposition I.1] *Let $\bar{G}$ be a reductive connected algebraic group over an algebraically closed field of characteristic zero. Let H be a closed subgroup of $\bar{G}$ with Levi decomposition $H = H^uK$. Let P be a parabolic subgroup of $\bar{G}$ with a Levi decomposition $P = P^uL$ such that $H^u \subset P^u$ and $K \subset L$. Then the following conditions are equivalent:*

1. *$\bar{G}/H$ is spherical;*
2. *K has an open orbit in P^u/H^u and the generic K-stabilizer of P^u/H^u is spherical in L.*

When H is the centralizer $C_{\bar{G}}(u)$ of a unipotent element in a semisimple algebraic group $\bar{G}$, a construction of the subgroups P, K and L from Proposition 17 is given in [E, Lemma 5.3], using key results of [SS]. Let us recall this construction. Let e be the nilpotent element of $\bar{\mathfrak{g}} = \mathrm{Lie}(\bar{G})$ corresponding to u and let (e, h, f) be an SL_2-triple in $\bar{\mathfrak{g}}$. The semisimple element h determines a natural $\mathbb{Z}$-grading on $\bar{\mathfrak{g}}$ by $\bar{\mathfrak{g}}_j := \{z \in \bar{\mathfrak{g}} \mid [h, z] = jz\}$. The subalgebra $\mathfrak{p} := \bigoplus_{j\geq 0} \bar{\mathfrak{g}}_j$ is parabolic and $\mathfrak{p}^u := \bigoplus_{j>0} \bar{\mathfrak{g}}_j$ is its nilpotent radical. The subalgebra $\mathfrak{p}$ is called the canonical parabolic subalgebra associated to e and it is independent of the choice of the SL_2-triple. Let P be the parabolic subgroup of $\bar{G}$ whose Lie algebra is $\mathfrak{p}$ and let L be the connected, reductive subgroup of $\bar{G}$ whose Lie algebra is $\bar{\mathfrak{g}}_0$, i.e., $L = \{g \in \bar{G} \mid \mathrm{Ad}(g)h = h\}^\circ$. The group P is called the *canonical parabolic* associated to u and $P = P^uL$ is a Levi decomposition of P. It turns out that $C_{\bar{G}}(u) \subset P$, $C_{\bar{G}}(u)^u \subset P^u$ and that $C_{\bar{G}}(u) = (P \cap C_{\bar{G}}(u))(C_{\bar{G}}(u) \cap L)$ is a Levi decomposition of $C_{\bar{G}}(u)$.

A similar construction works in the case of nonsemisimple elements as follows.

Lemma 18. *Let $\bar{G}$ be a connected reductive algebraic group with Lie algebra $\bar{\mathfrak{g}}$, let $g \in \bar{G}$ be an element with Jordan decomposition $g = su$, $u \neq 1$, and let $H = C_{\bar{G}}(g)$. Then the Levi decomposition $P = P^uL$ of the canonical parabolic P associated to u induces a Levi decomposition $H = H^uK$ of H with $K = L \cap H$.*

Proof. The semisimple element s lies in $C_{\bar{G}}(u)$ and u lies in $C_{\bar{G}}(s)^\circ$, which is a reductive subgroup. Hence, there exists an SL_2-triple $(e,\, h,\, f)$ of elements of $\mathrm{Lie}(C_{\bar{G}}(s)^\circ) = \{x \in \bar{\mathfrak{g}} \mid \mathrm{Ad}(s)x = x\}$ where e is the nilpotent element associated to u. It follows that $s \in C_{\bar{G}}(h) = \{y \in \bar{G} \mid \mathrm{Ad}(y)h = h\} = \widetilde{L}$ where $\widetilde{L}^\circ = L$. The canonical parabolic P associated to u contains $H = C_{\bar{G}}(u) \cap C_{\bar{G}}(s)$. The subgroup $K = L \cap H$ is reductive because it is the centralizer of a semisimple element $s \in \widetilde{L} \cap C_{\bar{G}}(u)$ (see [St2, Corollary 9.4]). The subgroup

$$V = P^u \cap H = C_{\bar{G}}(u)^u \cap C_{\bar{G}}(s)$$

is a unipotent normal subgroup of H. In order to prove that $H = KV$ is a Levi decomposition of H and, in particular, that $H^u = V$, it is enough to show that $H \subseteq KV$ because $K \cap V = 1$ follows from the Levi decomposition of $C_{\bar{G}}(u)$. Let $z \in H$. As $H \subset C_{\bar{G}}(u)$, there exist unique $v \in C_{\bar{G}}(u)^u$ and $t \in C_{\bar{G}}(u) \cap L$ such that $z = vt$. Then $svs^{-1} \in V$ because V is normal in H, and $sts^{-1} \in C_{\bar{G}}(u) \cap L$ because both $t,\, s \in C_{\bar{G}}(u) \cap \widetilde{L}$ and L is normal in $\widetilde{L}$. Besides, $z = szs^{-1}$. By the uniqueness of the decomposition in $C_{\bar{G}}(u)$, we necessarily get $sts^{-1} = t$ and $svs^{-1} = v$, i.e., $t \in K$ and $v \in V$. $\square$

Corollary 19. *Let $\bar{G}$ be a connected reductive algebraic group with Lie algebra $\bar{\mathfrak{g}}$, let $g \in \bar{G}$ be an element with Jordan decomposition $g = su$, $u \neq 1$, and let $H = C_{\bar{G}}(g)$. Then the Levi decomposition $P = P^u L$ of the canonical parabolic P associated to u induces a Levi decomposition $H^\circ = H^u K^\circ$ of H°, with $K = L \cap H$.*

Proof. The corollary follows from $H^u \subseteq H^\circ$. $\square$

As we have already observed, the sphericity of $\bar{G}/H$ depends only on the Lie algebras of $\bar{G}$ and H. In particular, for the analysis of the conjugacy class of an element $g \in G$, it does not matter whether we consider $C_G(g)$ or its identity component.

Remark 3. Let $G_1 \subset G_2$ be reductive algebraic groups and let u be a unipotent element in G_1. Suppose that the conjugacy class of u in G_2 is spherical. Then the conjugacy class of u in G_1 is spherical by [P2, Corollary 2.3, Theorem 3.1].

Again we shall handle the classical and the exceptional cases separately.

1.3.1. *Classical type.* In this section we shall assume that G is of classical type.

Proposition 20. *Let $g = su \in G$ with $s \neq 1$ and $u \neq 1$. If the conjugacy class of g is spherical, then only the following possibilities may occur:*

- *G is of type C_n and, up to a central element, $g = \sigma_k u$ with $u = X_{1,2n}$;*
- *G is of type B_n and, up to a central element, $g = \rho_n u$ where $u = X_{2t,2n+1}$ with $t = 1, \ldots, \left[\frac{n}{2}\right]$.*

Proof. We shall use Proposition 17 in order to show that if g is not as in the statement, then $\mathcal{O}_g$ cannot be spherical. With notation as in Lemma 18, we shall describe K° and its action on $P^u/H^u \cong \mathfrak{p}^u/\mathfrak{h}^u$.

Type A_{n-1}. Since $\mathcal{O}_g$ is spherical, s is conjugated, up to a central element, to one of the g_k's or of the $g_{\zeta,k}$'s (see Table 1). We shall show that necessarily $u = 1$, leading to a contradiction. As u is a unipotent element of the centralizer of g_k (respectively $g_{\zeta,k}$), it can be identified with a pair $(u_1,\, u_2) \in \mathrm{SL}_k \times \mathrm{SL}_{n-k}$. It is enough to prove that if

one of the $u_j \neq 1$ and the other is equal to 1, then $\mathcal{O}_g$ is not spherical. Suppose that $u = (u_1, 1)$ with u_1 spherical with Young diagram of shape $X_{t,k}$ for $1 \leqslant t \leqslant \left[\frac{k}{2}\right]$. We have:

- $\mathfrak{p}^u/\mathfrak{h}^u \cong \mathrm{Mat}_{t,n-k} \times \mathrm{Mat}_{n-k,t}$;
- $K \cong \{(A, B, C) \in \mathrm{GL}_t \times \mathrm{GL}_{k-2t} \times \mathrm{GL}_{n-k} \mid \det A^2 \det B \det C = 1\}$;
- action of K on $\mathfrak{p}^u/\mathfrak{h}^u$:

$$(A, B, C).(P, Q) = (Y_t A Y_t P C^{-1}, CQA^{-1})$$

where Y_t is a symmetric $t \times t$ matrix such that $Y_t^2 = 1$, depending on the choice of u_1.

Since $\mathrm{Tr}(QY_tP)$ is a nontrivial polynomial invariant of the action of K on $\mathfrak{p}^u/\mathfrak{h}^u$, $\mathcal{O}_g$ is not spherical. The case $u_2 \neq 1$ is similar and left to the reader.

Now let G be orthogonal or symplectic. Then, if the conjugacy class of $g = su_1u_2$ is spherical, then u_1 and u_2 are either of shape $X_{t,m}$ or of shape $Z_{2t,m}$, with u_j of shape $Z_{2t,m}$ only if $G = \mathrm{SO}_m$.

Type C_n. Let us distinguish the following possibilities for s:

(i) $s = \sigma_k$. If $u_1 = X_{t,2k}$ with $t \geqslant 1$ and $u_2 = 1$, we have:

- $\mathfrak{p}^u/\mathfrak{h}^u \cong \mathrm{Mat}_{2n-2k,t}$;
- $K^\circ \cong \mathrm{Sp}_{2n-2k} \times \mathrm{SO}_t \times \mathrm{Sp}_{2k-2t}$;
- action of K°: orthosymplectic of $\mathrm{Sp}_{2n-2k} \times \mathrm{SO}_t$.

If $t \geqslant 2$, the orthosymplectic action of $\mathrm{Sp}_{2n-2k} \times \mathrm{SO}_t$ on $\mathrm{Mat}_{2n-2k,t}$ cannot have a dense orbit because it has a nontrivial invariant. Indeed, if $X \in \mathrm{Mat}_{2n-2k,t}$, E is the matrix of the form with respect to which SO_t is orthogonal, and if J is the matrix of the form with respect to which Sp_{2n-2k} is symplectic, then $\mathrm{Tr}((E\,{}^tXJX)^2)$ is a nontrivial invariant for the $\mathrm{Sp}_{2n-2k} \times \mathrm{SO}_t$-action. Then, if u_1 is of shape $X_{t,2k}$ and $t \geqslant 2$, $\mathcal{O}_{\sigma_k u}$ is not spherical. By the symmetry in the roles of u_1 and u_2, the same holds if u_2 is of shape $X_{t,2n-2k}$ with $t \geqslant 2$.

If $u_1 = X_{1,2k}$ and $u_2 = X_{1,2n-2k}$, we have:

- $\mathfrak{p}^u/\mathfrak{h}^u \cong \mathbb{C}^{2n-2k} \oplus \mathbb{C}^{2k-2} \oplus \mathbb{C}$;
- $K^\circ \cong \mathrm{Sp}_{2n-2k} \times \mathrm{Sp}_{2k-2}$;
- action of K°: standard of $\mathrm{Sp}_{2n-2k} \oplus$ standard of $\mathrm{Sp}_{2k-2} \oplus$ trivial.

It is clear that the action of K° on $\mathfrak{p}^u/\mathfrak{h}^u$ cannot have an open orbit.

(ii) $s = c_\lambda$. Since $u \in C_G(c_\lambda)$,

$$u = \left[\begin{array}{c|c} \begin{matrix} 1 & \\ & U_1 \end{matrix} & \begin{matrix} & U_2 \\ 1 & \end{matrix} \\ \hline \begin{matrix} & U_3 \end{matrix} & \begin{matrix} & U_4 \end{matrix} \end{array}\right]$$

where $\left[\begin{array}{c|c} U_1 & U_2 \\ \hline U_3 & U_4 \end{array}\right]$ is a spherical unipotent element of Sp_{2n-2}. In particular, the Young diagram of u has shape $X_{k,2n}$ with $1 \leqslant k \leqslant n-1$. We have:

- $\mathfrak{p}^u/\mathfrak{h}^u \cong \mathbb{C}^k \oplus \mathbb{C}^k$;

- $K^\circ \cong \mathbb{C}^* \times \mathrm{SO}_k \times \mathrm{Sp}_{2n-2k-2}$;
- K° acts as follows: $(a, A, B).(v, w) = (aAv, a^{-1}Aw)$.

This action never has a dense orbit since the product $({}^tvEv)({}^twEw)$ is invariant.

(iii) $s = c$. Then necessarily $u = \left[\begin{array}{c|c} A & \\ \hline & {}^tA^{-1} \end{array}\right]$ where A is a spherical unipotent element of SL_n. In particular, the Young diagram of u has shape $X_{2k,2n}$ with $1 \leqslant k \leqslant [\frac{n}{2}]$. We have:

- $\mathfrak{p}^u/\mathfrak{h}^u \cong \mathrm{Sym}_k \oplus \mathrm{Sym}_k \oplus \mathrm{Mat}_{k,n-2k} \oplus \mathrm{Mat}_{n-2k,k}$ where Sym_k is the space of $k \times k$ symmetric matrices;
- $K^\circ \cong \mathrm{GL}_k \times \mathrm{GL}_{n-2k}$;
- K° acts as follows:

$$(A, B).(Z, L, M, N) = (Y_kAY_kZY_k\ {}^tAY_k,\ {}^tA^{-1}LA^{-1},\ {}^tA^{-1}MB^{-1}, BNY_k\ {}^tAY_k).$$

This action never has a dense orbit since $\mathrm{Tr}(Y_kZY_kL)$ is a nonzero polynomial invariant.

Type D_n. Let us distinguish the following possibilities for s:

(i) $s = c$. This case can be treated as for G of type C_n. In the computations, Sym_k is replaced by Ant_k, the space of skew-symmetric $k \times k$ matrices. When $k = 1$, the product MN is a nontrivial invariant.

(ii) $s = d$. If n is odd, the proof follows by noticing that $\mathcal{O}_c = \mathcal{O}_{-d}$. If n is even, the conjugacy class of cu is spherical if and only if the conjugacy class of $\hat{\tau}(cu) = d\hat{\tau}(u)$ is spherical. Then the proof follows from (i).

(iii) $s = \sigma_k$. If $u_1 = X_{2t,2k}$ and $u_2 = 1$, we have:

- $\mathfrak{p}^u/\mathfrak{h}^u \cong \mathrm{Mat}_{2n-2k,2t}$;
- $K^\circ \cong \mathrm{SO}_{2n-2k} \times \mathrm{Sp}_{2t} \times \mathrm{SO}_{2k-4t}$;
- the action of K° is the orthosymplectic of $\mathrm{SO}_{2n-2k} \times \mathrm{Sp}_{2t}$.

If $u_1 = Z_{2t,2k}$ and $u_2 = 1$, we have:

- $\mathfrak{p}^u/\mathfrak{h}^u \cong \mathrm{Mat}_{2n-2k,2t} \oplus \mathbb{C}^{2n-2k} \oplus \mathbb{C}^{2t}$;
- $K^\circ \cong \mathrm{SO}_{2n-2k} \times \mathrm{Sp}_{2t} \times \mathrm{SO}_{2k-4t-3}$;
- action of K°: orthosymplectic of $\mathrm{SO}_{2n-2k} \times \mathrm{Sp}_{2t}$ $\oplus$ standard of SO_{2n-2k} $\oplus$ standard of Sp_{2t}.

The orthosymplectic action of $\mathrm{SO}_{2n-2k} \times \mathrm{Sp}_{2t}$ on $\mathrm{Mat}_{2n-2k,2t}$ has a nontrivial invariant, namely $\mathrm{Tr}((J\,{}^tXEX)^2)$, unless $t = 0$ which occurs only if $u_1 = Z_{2t,2k}$.

If $u_1 = Z_{0,2k}$, the standard action of SO_{2n-2k} has no dense orbit because if $v \in \mathbb{C}^{2n-2k}$, then ${}^tvEv$ is a nontrivial invariant. Therefore when $u_2 = 1$, the conjugacy class of g is not spherical unless $g = u_1$ or $g = \sigma_k$, leading to a contradiction. By the symmetry in the roles of u_1 and u_2, the result follows for G of type D_n.

Type B_n. Let us distinguish the following possibilities for s:

(i) $s = \rho_k$. Let $u = X_{2t,r}$, with $r = 2k$ if $u_2 = 1$, and $r = 2n - 2k + 1$ if $u_1 = 1$. We have:

- $\mathfrak{p}^u/\mathfrak{h}^u \cong \mathrm{Mat}_{2n+1-r,2t}$;
- $K^\circ \cong \mathrm{SO}_{2n+1-r} \times \mathrm{Sp}_{2t} \times \mathrm{SO}_{r-4t}$;
- the action of K° is the orthosymplectic of $\mathrm{SO}_{2n+1-r} \times \mathrm{Sp}_{2t}$.

Let $u = Z_{2t,r}$, with $r = 2k$ if $u_2 = 1$, and $r = 2n - 2k + 1$ if $u_1 = 1$. We have:

- $\mathfrak{p}^u/\mathfrak{h}^u \cong \mathrm{Mat}_{2n+1-r,2t} \oplus \mathbb{C}^{2n+1-r} \oplus \mathbb{C}^{2t}$;
- $K^\circ \cong \mathrm{SO}_{2n+1-r} \times \mathrm{Sp}_{2t} \times \mathrm{SO}_{r-4t-3}$;
- the action of K° is orthosymplectic of $\mathrm{SO}_{2n+1-r} \times \mathrm{Sp}_{2t}$ $\oplus$ standard of SO_{2n+1-r} $\oplus$ standard of Sp_{2t}.

In both cases, by arguments similar to the previous ones, the action of K° on $\mathfrak{p}^u/\mathfrak{h}^u$ can never have a dense orbit unless $g = \rho_n u$ where $u = X_{2t,2n+1}$.

(ii) $s = b_\lambda$. Then, necessarily, $u = \left[\begin{array}{c|c} 1 & \\ \hline & A \\ \hline & \end{array}\right.$ $\left.\begin{array}{c|} \\ \hline \\ \hline {}^tA^{-1} \end{array}\right]$ where A is a spherical unipotent element of SL_n. In particular the Young diagram of u has shape $X_{2k,2n+1}$ with $1 \leqslant k \leqslant [\frac{n}{2}]$. We have:

- $\mathfrak{p}^u/\mathfrak{h}^u \cong \mathbb{C}^k \oplus \mathbb{C}^k \oplus \mathrm{Ant}_k \oplus \mathrm{Ant}_k \oplus \mathrm{Mat}_{k,n-2k} \oplus \mathrm{Mat}_{n-2k,k}$;
- $K^\circ \cong \mathrm{GL}_k \times \mathrm{GL}_{n-2k}$;
- K° acts on $\mathfrak{p}^u/\mathfrak{h}^u$ as follows:

$$\begin{aligned}&(A,B).(v,w,Z,L,M,N)\\ &= ({}^tA^{-1}v, Y_kAY_kw, Y_kAY_kZY_k\,{}^tAY_k,\ {}^tA^{-1}LA^{-1},\ {}^tA^{-1}MB^{-1},\ BNY_k\,{}^tAY_k)\end{aligned}$$

where Y_k is as above.

This action never has a dense orbit since ${}^twY_kv$ is a nonzero polynomial invariant. The statement of Proposition 20 now follows. □

Let us now analyze the remaining possibilities.

Theorem 21. *Let $g = su$ be an element of G such that:*

- *either G is of type C_n, $s = \sigma_k$ and $u = X_{1,2n}$;*
- *or G is of type B_n, $s = \rho_n$ and u is a spherical unipotent element associated with a Young diagram with two columns.*

Then $\mathcal{O}_g$ is spherical and well placed.

Proof. We shall show that $\mathcal{O}_g$ is well placed and hence spherical by exhibiting an element $x \in \mathcal{O}_g \cap B\dot{w}B \cap B^-$ for some w such that $\ell(w) + \mathrm{rk}(1-w) = \dim \mathcal{O}_g$.

Type C_n. Let $u = (u_1, u_2) \in C_{\mathrm{Sp}_{2n}}(\sigma_k) \cong \mathrm{Sp}_{2k} \times \mathrm{Sp}_{2n-2k}$, where $1 \leqslant k \leqslant [\frac{n}{2}]$, and let us distinguish the following cases:

(1) $u_1 = 1$, $u_2 = X_{1,2(n-k)}$. In this case $\dim \mathcal{O}_g = (4k+2)(n-k)$.

(1i) Let us assume $k = [\frac{n}{2}]$. Then $\dim \mathcal{O}_g = n^2 + n = \ell(w_0) + \mathrm{rk}(1 - w_0)$. Let us choose the following element $M \in B^-$:

$$M = \left[\begin{array}{ccc|ccc} & & S & & & 0 \\ \hline 1 & 1 & 0 & & & \\ -1 & 0 & 1 & & & \\ 0 & -1 & 0 & 1 & & \\ & & & -1 & \ddots & \ddots \\ & & & & \ddots & \ddots \end{array}\right.$$
$$\left.\begin{array}{c} \\ \\ \\ \\ \\ \\ \end{array}\right.$$
$$\left.\begin{array}{c} \\ \\ \\ S \\ \\ \\ \end{array}\right]$$

where $S = \mathrm{diag}(1, -1, 1, -1, 1, \dots)$ is an $n \times n$ matrix. Then one can verify that M lies over w_0 and that $M \in \mathcal{O}_g$.

(1ii) Now let us suppose $k < [\frac{n}{2}]$. Notice that in this case $n - k \geqslant k + 2$. Let i_{2k+1} be the following embedding of Sp_{4k+2} into Sp_{2n}:

$$\left[\begin{array}{c|c} A_{2k+1} & B_{2k+1} \\ \hline C_{2k+1} & D_{2k+1} \end{array}\right] \overset{i_{2k+1}}{\longmapsto} \left[\begin{array}{cc|cc} A_{2k+1} & & B_{2k+1} & \\ & I_{n-2k-1} & & 0_{n-2k-1} \\ \hline C_{2k+1} & & D_{2k+1} & \\ & 0_{n-2k-1} & & I_{n-2k-1} \end{array}\right].$$

Case (1i) shows that if $G = \mathrm{Sp}_{4k+2}$, $g' = \sigma_k u$ where $u = (u_1, u_2)$, $u_1 = 1$, $u_2 = X_{1,2(k+1)}$, then $\mathcal{O}_{g'}$ contains a matrix $M \in B^-(\mathrm{Sp}_{4k+2})$ lying over w_0. In particular this implies that $i_{2k+1}(M)$ lies in $B^- \cap B\dot{w}_{2k+1}B$ where

$$\dot{w}_{2k+1} = \left[\begin{array}{cc|cc} 0_{2k+1} & 0 & -I_{2k+1} & 0 \\ 0 & I_{n-2k-1} & 0 & 0_{n-2k-1} \\ \hline I_{2k+1} & 0 & 0_{2k+1} & 0 \\ 0 & 0_{n-2k-1} & 0 & I_{n-2k-1} \end{array}\right].$$

The thesis follows by noticing that $i_{2k+1}(M)$ belongs to $\mathcal{O}_g$ and that

$$\ell(w_{2k+1}) + \mathrm{rk}(1 - w_{2k+1}) = (4k + 2)(n - k) = \dim \mathcal{O}_g.$$

(2) $u_1 = X_{1,2k}$, $u_2 = 1$. In this case $\dim \mathcal{O}_g = 2k(2n - 2k + 1)$.

(2i) Let us first suppose that n is even and let $k = \frac{n}{2}$ so that $\dim \mathcal{O}_g = n^2 + n$. Let us choose the following element $\bar{M} \in B^- \cap \mathcal{O}_g$:

$$\bar{M} = \left[\begin{array}{cccccccc|c} & & & & \multicolumn{3}{c}{\bar{S}} & & 0 \\ \hline 1 & 1 & & & & & & & \\ -1 & 0 & 1 & & & 0 & & & \\ & -1 & 0 & 1 & & & & & \\ & & \ddots & \ddots & \ddots & & & & \bar{S} \\ & & & \ddots & \ddots & 1 & & & \\ & 0 & & & -1 & 0 & -1 & & \\ & & & & & 1 & 0 & & \end{array}\right]$$

where $\bar{S} = \mathrm{diag}(-1, 1, -1, 1, \dots)$ is an $n \times n$ matrix. Since $\dim \mathcal{O}_g = \ell(w_0) + \mathrm{rk}(1 - w_0)$, it is enough to show that $\bar{M}$ lies over w_0, and this follows, using Remark 1, from a straightforward calculation.

(2ii) Now let us suppose $k < \frac{n}{2}$. Case (2i) shows that if $G = \mathrm{Sp}_{4k}$, $g' = \sigma_k u$ where $u = (u_1, u_2)$, $u_1 = X_{1,2k}$, $u_2 = 1$, then $\mathcal{O}_{g'}$ contains a matrix $\bar{M} \in B^-(\mathrm{Sp}_{4k})$ lying over w_0. Using the embedding i_{2k} of Sp_{4k} into Sp_{2n}, it is immediately seen that $i_{2k}(\bar{M})$ belongs to $B^- \cap B\dot{\sigma}B$ where

$$\dot{\sigma} = \left[\begin{array}{cc|cc} 0_{2k} & 0 & -I_{2k} & 0 \\ 0 & I_{n-2k} & 0 & 0_{n-2k} \\ \hline I_{2k} & 0 & 0_{2k} & 0 \\ 0 & 0_{n-2k} & 0 & I_{n-2k} \end{array}\right].$$

Finally, let us notice that $i_{2k}(\bar{M})$ is conjugated to g and that if σ is the projection of $\dot{\sigma}$ in W, then $\ell(\sigma) + \mathrm{rk}(1-\sigma) = 4kn - 4k^2 + 2k = \dim \mathcal{O}_g$.

Type B_n. Let $g = \rho_n u$ where $u = (u_1, 1)$, u_1 is of shape $X_{2k,2n}$ and $1 \leqslant k \leqslant [n/2]$. If $k < \frac{n}{2}$, the class $\mathcal{O}_g$ is completely determined by the diagram $X_{2k,2n}$. If n is even, let $t_n \in \mathrm{SO}_{2n+1}$ be a representative of $s_{\alpha_n} \in W(\mathrm{SO}_{2n+1})$. Conjugation by t_n fixes ρ_n and induces the automorphism $\hat{\tau}$ of SO_{2n} (see the proof of Theorem 12). Therefore if u_1 and u_1' are representatives of the two distinct unipotent conjugacy classes of SO_{2n} associated to $X_{n,2n}$, and if $u = (u_1, 1)$ and $u' = (u_1', 1)$, then $u' \in \mathcal{O}_u$ and $\rho_n u' \in \mathcal{O}_{\rho_n u}$. Therefore also for $k = \frac{n}{2}$, the class $\mathcal{O}_g$ is completely determined by the diagram $X_{2k,2n}$. Thus let us denote by $\mathcal{O}_k$ the conjugacy class of $g = \rho_n u$ with u_1 of shape $X_{2k,2n}$. Then $\dim \mathcal{O}_k = 4nk - 4k^2 + 2n - 2k$. Let us first assume that k is maximal, i.e., $k = k_{\max} = [\frac{n}{2}]$. Then $\dim \mathcal{O}_{k_{\max}} = n^2 + n = \dim B(\mathrm{SO}_{2n+1})$.

Let $g_n = \left[\begin{array}{c|c|c} 1 & {}^t\psi & 0 \\ \hline 0 & -I_n & 0 \\ \hline \psi & \Sigma & -I_n \end{array}\right]$ where $\psi = {}^t(1, 0, \ldots, 0)$, Σ is the $n \times n$ matrix with diagonal $(1/2, 0, \ldots, 0)$, first upper off-diagonal $(1, 1, \ldots, 1)$, first lower off-diagonal $(-1, -1, \ldots, -1)$, and 0 elsewhere. By Remark 1 the element g_n lies over w_0. As $\mathrm{diag}(-1, 1, \ldots, 1) \in C_{\mathrm{O}_{2n+1}}(\rho_n u)$, it follows from Remark 2 that g_n belongs to $\mathcal{O}_{k_{\max}}$, so the assertion is proved for k maximal.

Let us now assume that $2k < n - 1$, i.e., that there are strictly more than two rows with one box in $X_{2k,2n}$. We consider the following embedding of $\mathrm{SO}_{4k+3} \times \mathrm{SO}_{2n-4k-2}$ in SO_{2n+1}:

$$\left(\left[\begin{array}{c|c|c} a & {}^t\alpha & {}^t\beta \\ \hline \gamma & A & B \\ \hline \delta & C & D \end{array}\right], \left[\begin{array}{c|c} A' & B' \\ \hline C' & D' \end{array}\right]\right) \mapsto \left[\begin{array}{c|c|c|c|c} a & {}^t\alpha & 0 & {}^t\beta & 0 \\ \hline \gamma & A & 0 & B & 0 \\ 0 & 0 & A' & 0 & B' \\ \hline \delta & C & 0 & D & 0 \\ 0 & 0 & C' & 0 & D' \end{array}\right].$$

Let g_{2k+1} be the representative of the conjugacy class of $\mathcal{O}_{k_{\max}}$ in SO_{4k+3}. One can check that the embedded image of $(g_{2k+1}, -1)$ is a representative of $\mathcal{O}_k$ in $B^-(\mathrm{SO}_{2n+1})$ and that it lies over $\omega_k = \left[\begin{array}{c|c} -I_{2k+1} & 0 \\ \hline 0 & I_{n-2k-1} \end{array}\right] \in W(\mathrm{SO}_{2n+1})$. As $\mathrm{rk}(1-\omega_k) + \ell(\omega_k) = (2k+1)^2 + (2k+1) + 2(n-2k-1)(2k+1) = \dim \mathcal{O}_k$, we have the statement for $k = 1, \ldots, [n/2]$. $\square$

1.3.2. *Exceptional type.* In this section we shall assume that G is of exceptional type. We already recalled that if the conjugacy class $\mathcal{O}_g$ of an element g with Jordan decomposition su is spherical, then both $\mathcal{O}_s$ and $\mathcal{O}_u$ are spherical. Besides, as $\mathcal{O}_g$ is spherical, $\dim \mathcal{O}_g \leqslant \dim B$. Therefore a dimensional argument rules out all the possibilities except the following:

- $g = p_1 x_{\beta_1}(1)$ if $\mathfrak{g}$ is of type E_6;
- $g = q_2 x_{\beta_1}(1)$ if $\mathfrak{g}$ is of type E_7;
- $g = r_2 x_{\beta_1}(1)$ if $\mathfrak{g}$ is of type E_8;
- $g = f_2 x_{\beta_1}(1)$ if $\mathfrak{g}$ is of type F_4.

The following result excludes the first three cases.

Proposition 22. *If $\mathfrak{g}$ is of type E_6, E_7, or E_8, any spherical conjugacy class of G is either semisimple or unipotent.*

Proof. By the discussion above it is enough to prove that the class of $sx_{\beta_1}(1)$, with $s = p_1, q_2, r_2$, is not spherical. Let H be the centralizer of $sx_{\beta_1}(1)$ in G. We shall use the same notation as in Lemma 18. Let S be a stabilizer in general position for the action of K on $\mathfrak{l}/\mathfrak{k}$, where $\mathfrak{l} = \mathrm{Lie}(L)$ and $\mathfrak{k} = \mathrm{Lie}(K)$. Let $c_M(X)$ denote the complexity of the action of a reductive algebraic group M, with Borel subgroup B_M, on the variety X, i.e., $c_M(X) = \min_{x\in X} \mathrm{codim} B_M.x$. Then, by [P2, Theorem 1.2 (i)],

$$c_G(G/H) = c_L(L/K) + c_S(\mathfrak{p}^u/\mathfrak{h}^u). \tag{1.6}$$

We see that in all cases $\mathfrak{l} = \mathfrak{k} \oplus \mathbb{C}h_{\beta_1}$ so that $c_L(L/K) = 0$ and $S = K$. In particular, if $\mathfrak{g}$ is of type E_6, E_7, E_8, then K is of type A_5, D_6, E_7, respectively. By [Br1, Théorème 1.4] (see also [L, Theorem 1.4]) D_6 and E_7 have no linear multiplicity free representations, hence E_7 and E_8 have no spherical exceptional conjugacy classes that are neither semisimple nor unipotent.

As far as E_6 is concerned, one can check that

$$\mathfrak{p}^u \cong \bigoplus_{\substack{\alpha>0,\\ \alpha\not\perp\beta_1}} \mathfrak{g}_\alpha, \quad \mathfrak{h}^u = \mathfrak{g}_{\beta_1},$$

therefore $\dim(\mathfrak{p}^u/\mathfrak{h}^u) = 20$. By [Br1, Théorème 1.4] there are no multiplicity free representations of a group of type A_5 on a vector space of dimension 20, hence the statement. □

Theorem 23. *Let $\mathfrak{g}$ be of type F_4 and let $\mathcal{O}$ be the conjugacy class of $f_2x_{\beta_1}$. Then $\mathcal{O}$ is spherical and well placed.*

Proof. We have: $\dim \mathcal{O} = 28 = \ell(w_0) + \mathrm{rk}(1 - w_0)$. We shall show that $z(\mathcal{O}) = w_0$, which implies by Theorem 5 that $\mathcal{O}$ is spherical.

The element f_2 lies in $T \subset C_G(f_1)$. Besides, $C = C_G(f_1)$ is the subgroup of G of type $\mathsf{C}_3 \times \mathsf{A}_1$ with simple roots $\{\alpha_2, \alpha_3, \alpha_4\}$ and β_1. Since $(f_2)^2 = 1$, it follows that f_2 is of the form $(s, t) \in \mathsf{C}_3 \times \mathsf{A}_1$ with t central and $s^2 = 1$. Hence, f_2 is conjugated (up to a central element) in C to an element of the form (σ_1, t). By Theorem 16 f_2 is conjugated, up to a central element in C, by an element in the component of type C_3 to $\dot{s}_{\alpha_4}\dot{s}_{\alpha_2+2\alpha_3+\alpha_4}h$ for some $h \in T$. Hence $f_2x_{-\beta_1}(1)$ is conjugated to

$$\dot{s}_{\alpha_4}\dot{s}_{\alpha_2+2\alpha_3+\alpha_4}hx_{-\beta_1}(1) \in B\dot{s}_{\alpha_4}\dot{s}_{\alpha_2+2\alpha_3+\alpha_4}\dot{s}_{\beta_1}B = B\dot{w}_0\dot{s}_{\alpha_2}B$$

for some $h \in T$.

On the other hand, the involution $\rho_4 = h_{\alpha_2}(-1)h_{\alpha_2+2\alpha_3+2\alpha_4}(-1)$ (notation as in [St1, Lemma 28]) is conjugated to f_2, since its centralizer is the subgroup of type B_4 with simple roots $\{\alpha_2 + 2\alpha_3, \alpha_1, \alpha_2, \alpha_3 + \alpha_4\}$. Therefore the element $\rho_4x_{\beta_1}(1) \in C_G(\rho_4)$ is a representative of the class $\mathcal{O}$. By Theorem 21 there exists a representative of the conjugacy class $\mathcal{O}_{\rho_4x_{\beta_1}(1)}$ in $C_G(\rho_4)$ lying over the element $w_0s_{\alpha_3+\alpha_4}$. By Corollary 8 $z(\mathcal{O}) = w_0$. Let us finally show that $\mathcal{O} \cap B\dot{w}_0B \cap B^- \neq \varnothing$. Let $g \in G$ be such that

$g^{-1}f_2x_{\beta_1}(1)g \in B\dot{w}_0B$ and let $g = u_\sigma\dot{\sigma}b$ be its unique decomposition in $U^\sigma\dot{\sigma}B$. Then $g^{-1}f_2x_{\beta_1}(1)g$ lies in $B\dot{w}_0B$ if and only if

$$\dot{\sigma}^{-1}u_\sigma^{-1}f_2x_{\beta_1}(1)u_\sigma\dot{\sigma} = \dot{\sigma}^{-1}u_\sigma^{-1}f_2u_\sigma\dot{\sigma}x_{\sigma^{-1}(\beta_1)}(t),$$

with $t \in \mathbb{C}^*$, lies in $B\dot{w}_0B$. Notice that u_σ and $x_{\beta_1}(1)$ commute because β_1 is the highest root of $\mathfrak{g}$. The root $\sigma^{-1}(\beta_1)$ is negative, otherwise $z(\mathcal{O}_{f_2})$ would be w_0, which is impossible by Theorem 5. Then, as in Lemma 14, $\dot{\sigma}^{-1}u_\sigma^{-1}f_2x_{\beta_1}(1)u_\sigma\dot{\sigma}$ lies in $B\dot{w}_0B \cap B^-$. □

1.4. Classification and remarks

The results of the previous sections can be summarized in the following theorem.

Theorem 24. *A conjugacy class $\mathcal{O}$ is spherical if and only if it is well placed.*

In fact, our results lead also to the following characterization of spherical conjugacy classes.

Theorem 25. *Let $\mathcal{O}$ be a conjugacy class in G, $z = z(\mathcal{O})$. Then $\mathcal{O}$ is spherical if and only if* $\dim \mathcal{O} = \ell(z) + \mathrm{rk}(1 - z)$.

Corollary 26. *Let $\mathcal{O}$ be a spherical conjugacy class of G and let $z = z(\mathcal{O})$. Let $x \in \mathcal{O}$ be an element such that $B.x$ is dense in $\mathcal{O}$. Then $B.x = B^z.x = \mathcal{O} \cap B\dot{z}B$.*

Proof. Theorem 24 and Theorem 5 show that if y lies in $\mathcal{O} \cap B\dot{z}B$, then $B.y$ is dense in $\mathcal{O}$. It follows that y belongs to $B.x$, hence $B.x = \mathcal{O} \cap B\dot{z}B$. Besides, $U^z.x = U.x$ since they are irreducible, closed, and have the same dimension. Therefore $B^z.x = TU^z.x = TU.x = B.x$. □

Let us introduce the map

$$\tau : \{\text{Spherical conjugacy classes of } G\} \longrightarrow W, \quad \mathcal{O} \mapsto z(\mathcal{O})$$

and let us analyze some of its properties. A description of the image of τ is given in Tables 2, 3, 4, and 5. In the tables, we use the notation introduced in §1.1.2. When G is of type B (respectively D), the root system orthogonal to β_1 is no longer irreducible: it consists of three components of type A_1 if G is of type D_4, and of one component of type A_1 and one component of type B (respectively D) in the other cases. When G is of type D_4, we shall define $\mu_1 = \alpha_1$. When G is not of type D_4, we shall denote by μ_1 the positive root of the component of type A_1 and by ν_1 the highest root of the component of type B (respectively D). Inductively, for $r > 1$, we shall denote by μ_r the positive root of the component of type A_1 and by ν_r the highest root of the component of type B (respectively D) of the root system orthogonal to β_1, μ_j, ν_j for every $j = 1, \dots, r-1$.

In a similar way, when G is of type C, the root system orthogonal to γ_1 consists of one component of type A_1 and one component of type C. We shall denote by γ_2' the highest short root of the component of type C. Inductively, for $r > 1$, we shall denote by γ_r', the highest short root of the component of type C of the root system orthogonal to γ_1, γ_j' for every $j = 1, \dots, r-1$.

Remark 4. We note that if $\mathcal{O}$ is a spherical conjugacy class, then $z(\mathcal{O})$ is an involution. The reason for this is that if G is of adjoint type, then each spherical conjugacy class $\mathcal{O}$ coincides with its inverse. For unipotent classes this follows from [Co1, Lemma 1.16], and [Co2, Lemma 2.3]. For the semisimple classes in almost all cases, we are dealing with involutions in G. In the remaining cases we always have $w_0 = -1$, and in this case every semisimple element is conjugate to its inverse. Finally, for the classes $\mathcal{O}_{su}$ where $s \neq 1 \neq u$, the result follows from the fact that s is an involution and that u is conjugate to its inverse in $C_G(s)$.

Remark 5. Let $\pi_1 : G \longrightarrow G/U$ and $\pi_2 : G/U \longrightarrow G/B$ be canonical projections. Let B act on G by conjugation, on G/B by left multiplication, and on G/U as follows:

$$b(gU) = bgb^{-1}U.$$

Then π_1 and π_2 are B-equivariant maps. In particular, $\pi_2 \circ \pi_1$ maps every B-orbit of G to a B-orbit of G/B, i.e., a Schubert cell $C_\sigma = B\dot{\sigma}B/B$, for some $\sigma \in W$.

Let $\mathcal{O}$ be a spherical conjugacy class and let $z = z(\mathcal{O})$. Let $B.x$ be the dense B-orbit in $\mathcal{O}$. Then $\dim \mathcal{O} = \dim B.x = \ell(z) + \mathrm{rk}(1 - z)$. Besides, $\pi_2 \circ \pi_1(B.x) = C_z$, and by [DCP1, Proposition 16.4] $\dim \pi_1(\mathcal{O}) = \ell(z) + \mathrm{rk}(1 - z)$. It follows that the map $\rho = \pi_1|_{\mathcal{O}} : \mathcal{O} \to G/U$ has finite fibers. We think that the map ρ could give a relation between $\mathcal{O}$ and the symplective leaves of B^- coming from the quantization of B^- (see [DCP2]).

Table 3: Unipotent spherical conjugacy classes

$\mathfrak{g}$	$\mathcal{O}$	$z(\mathcal{O})$	$\mathcal{O}$	$z(\mathcal{O})$
A_{n-1}	$X_{k,n}$	$s_{\beta_1} \dots s_{\beta_k}$		
B_n	$X_{2k,2n+1}$	$s_{\beta_1} s_{\nu_1} \dots s_{\nu_{k-1}}$	$Z_{2k,2n+1}$ $k < \frac{(n-1)}{2}$	$s_{\gamma_1} \dots s_{\gamma_{2k+2}}$
	$Z_{n-1,2n+1}$	w_0		
C_n	$X_{2k,2n}$	$s_{\beta_1} \dots s_{\beta_k}$		
D_n	$X_{2k,2n}$ $k < \frac{n}{2}$	$s_{\beta_1} s_{\nu_1} \dots s_{\nu_{k-1}}$	$Z_{2k,2n}$	$s_{\beta_1} s_{\mu_1} s_{\nu_1} s_{\mu_2} \dots s_{\nu_k} s_{\mu_{k+1}}$
	$X_{n,2n}$	$s_{\beta_1} s_{\nu_1} \dots s_{\nu_{n/2-2}} s_{\alpha_n}$	$X'_{n,2n}$	$s_{\beta_1} s_{\nu_1} \dots s_{\nu_{n/2-2}} s_{\alpha_{n-1}}$
E_6	A_1	s_{β_1}	$2\mathsf{A}_1$	$s_{\beta_1} s_{\beta_2}$
	$3\mathsf{A}_1$	w_0		
E_7	A_1	s_{β_1}	$2\mathsf{A}_1$	$s_{\beta_1} s_{\beta_2}$
	$(3\mathsf{A}_1)'$	$s_{\beta_1} s_{\beta_2} s_{\alpha_2+\alpha_3+2\alpha_4+\alpha_5} s_{\alpha_3}$	$(3\mathsf{A}_1)''$	$s_{\beta_1} s_{\beta_2} s_{\alpha_7}$
	$4\mathsf{A}_1$	w_0		
E_8	A_1	s_{β_1}	$2\mathsf{A}_1$	$s_{\beta_1} s_{\beta_2}$
	$3\mathsf{A}_1$	$s_{\beta_1} s_{\beta_2} s_{\beta_3} s_{\alpha_7}$	$4\mathsf{A}_1$	w_0
F_4	A_1	s_{β_1}	$\widetilde{A}_1$	$s_{\beta_1} s_{\beta_2}$
	$\mathsf{A}_1 + \widetilde{A}_1$	w_0		
G_2	A_1	s_{β_1}	$\widetilde{A}_1$	w_0

Table 4: Spherical conjugacy classes which are neither semisimple nor unipotent

$\mathfrak{g}$	$\mathcal{O}_{su}$	$z(\mathcal{O}_{su})$
B_n	$s=\rho_n,\ u=(X_{2k,2n},1),\ k=[n/2]$	w_0
	$s=\rho_n,\ u=(X_{2k,2n},1),\ k<[n/2]$	$s_{\gamma_1}\dots s_{\gamma_{2k+1}}$
C_n	$s=\sigma_k,\ u=(1,X_{1,2n-2k})$	$s_{\beta_1}\dots s_{\beta_{2k+1}}$
	$s=\sigma_k,\ u=(X_{1,2k},1)$	$s_{\beta_1}\dots s_{\beta_{2k}}$
F_4	$f_2 x_{\beta_1}(1)$	w_0

Table 5: Spherical semisimple conjugacy classes, $\mathfrak{g}$ of classical type

$\mathfrak{g}$	$\mathcal{O}$	$z(\mathcal{O})$
A_{n-1}	$\mathcal{O}_{g_k}$	$s_{\beta_1}\dots s_{\beta_k}$
	$\mathcal{O}_{g_{\zeta,k}}$	$s_{\beta_1}\dots s_{\beta_k}$
B_n	$\mathcal{O}_{\rho_k}$ $1\leqslant k\leqslant[n/2]$	$s_{\gamma_1}\dots s_{\gamma_{2k}}$
	$\mathcal{O}_{\rho_k}$ $[n/2]<k\leqslant n$	$s_{\gamma_1}\dots s_{\gamma_{2(n-k)+1}}$
	$\mathcal{O}_{b_\lambda}$	w_0
C_n	$\mathcal{O}_{\sigma_k}$	$s_{\gamma_1}s_{\gamma'_2}\dots s_{\gamma'_k}$
	$\mathcal{O}_{c_\lambda}$	$s_{\beta_1}s_{\beta_2}$
	$\mathcal{O}_c$	w_0
D_n	$\mathcal{O}_c$ (n even)	$s_{\beta_1}s_{\nu_1}\dots s_{\nu_{n/2-2}}s_{\alpha_n}$
	$\mathcal{O}_d$ (n even)	$s_{\beta_1}s_{\nu_1}\dots s_{\nu_{n/2-2}}s_{\alpha_{n-1}}$
	$\mathcal{O}_c$ (n odd)	$s_{\beta_1}s_{\nu_1}\dots s_{\nu_{(n-3)/2}}$
	$\mathcal{O}_d$ (n odd)	$s_{\beta_1}s_{\nu_1}\dots s_{\nu_{(n-3)/2}}$
	$\mathcal{O}_{\sigma_k}$ $k<n/2$	$s_{\beta_1}s_{\mu_1}s_{\nu_1}s_{\mu_2}\dots s_{\mu_{k-1}}s_{\nu_{k-1}}s_{\mu_k}$
	$\mathcal{O}_{\sigma_{n/2}}$	$s_{\beta_1}s_{\mu_1}s_{\nu_1}s_{\mu_2}\dots s_{\nu_{n/2-2}}s_{\mu_{n/2-1}}s_{\alpha_{n-1}}s_{\alpha_n}$

Let us recall that w_0 can also be decomposed as a product of reflections corresponding to mutually orthogonal roots.

Remark 6. We recall that for a B-variety X, the following objects are defined:

$$\mathcal{P}=\{f\in k(X)\setminus\{0\}\mid b.f=\lambda_f(b)f,\ \text{for all } b\in B\}$$

where λ_f is an element of $\chi(B)$, the character group of B;

$$\begin{gathered}\psi:\mathcal{P}\to\chi(B),\ f\mapsto\lambda_f;\\ \Gamma(X)=\psi(\mathcal{P});\\ r(X):=\operatorname{rank}(\Gamma(X));\\ u(X)=\max_{x\in X}\dim U.x.\end{gathered}$$

Here we note that if $\mathcal{O}$ is a spherical conjugacy class, then $r(\mathcal{O}) = \mathrm{rk}(1 - z(\mathcal{O}))$ and $u(\mathcal{O}) = \ell(z(\mathcal{O}))$. Indeed, this follows from [P1, Corollary 1, Corollary 2(ii)], Theorem 5, and [Kn, Lemma 2.1].

Remark 7. Let us recall that a nilpotent orbit $\mathfrak{O}$ in $\mathfrak{g}$ is called a *model orbit* if $\mathbb{C}[\mathfrak{O}]$ consists exactly of the self-dual representations of G with highest weights in the root lattice, each occurring once (see [MG2, p. 229]). In this case the corresponding unipotent conjugacy class $\mathcal{O}$ in G is spherical ([Br2], [Po], [V]) and, by Remark 6, $\mathrm{rk}(1 - z(\mathcal{O})) = \mathrm{rk}(1 - w_0)$. It follows from the proofs of Theorems 12 and 13 that $z(\mathcal{O}) = w_0$ (cf. [MG1, Table 4] and [AHV]).

2. The proof of the DKP conjecture

In this section we prove the De Concini–Kac–Procesi conjecture for representations corresponding to spherical conjugacy classes.

Let ℓ be a positive odd integer greater than one. We will assume that ℓ is a good integer, i.e., that ℓ is coprime with the bad primes (for the definition of the bad primes see [B]) and that G is simply connected.

2.1. Strategy of the proof

Let ε be a primitive ℓ-th root of unity and let $\mathcal{U}_\varepsilon(\mathfrak{g})$ be the simply connected quantum group associated to $\mathfrak{g}$, as defined in [DCKP1], with generators E_i, F_i, K_β with β in the weight lattice P and $i = 1, \dots, n$. For our purposes it is convenient to introduce the subalgebra B_ε of $\mathcal{U}_\varepsilon(\mathfrak{g})$ generated by $E_1, \dots, E_n$ and K_β with $\beta \in P$. The representation theory of this algebra has been deeply investigated in [DCP2], where $B_\varepsilon \cong F_\varepsilon[B^-]$.

The center of $\mathcal{U}_\varepsilon(\mathfrak{g})$ contains a proper, finitely generated subalgebra Z_0 such that $\mathcal{U}_\varepsilon(\mathfrak{g})$ is a finite Z_0-module (in particular, it follows that every irreducible $\mathcal{U}_\varepsilon(\mathfrak{g})$-module has finite dimension).

For any associative algebra A, let us denote by Spec A the set of the equivalence classes of the irreducible representations of A. It is worth noticing that Spec $Z_0 = \{(t^{-1}u^-, tu) \mid u \in U,\ t \in T,\ u^- \in U^-\}$ ([DCKP2, §4.4]). In [DCKP1], the map $\pi : \mathrm{Spec}\ Z_0 \longrightarrow G$, $\pi(t^{-1}u^-, tu) = (u^-)^{-1}t^2u$, which is an unramified covering of the big cell $\Omega = B^-B$ of G, is considered. Let φ be the map obtained by composing π with the central character $\chi : \mathrm{Spec}\ \mathcal{U}_\varepsilon(\mathfrak{g}) \longrightarrow \mathrm{Spec}\ Z_0$. It follows that, for every $g \in \Omega$, one can define a certain finite-dimensional quotient $\mathcal{U}^g$ of $\mathcal{U}_\varepsilon(\mathfrak{g})$ such that if $g = \varphi(V)$, then V is a $\mathcal{U}^g$-module.

In [DCKP1, §6.1, Proposition (a)] the following crucial result is established:

$$\begin{aligned}&\text{if } g, h \in \Omega \text{ are conjugated in } G \text{ up to a central element, then}\\ &\mathcal{U}^g \text{ and } \mathcal{U}^h \text{ are isomorphic.}\end{aligned} \tag{2.1}$$

In [DCKP1, §6.8] the following conjecture is formulated.

Conjecture *If $\sigma \in$ Spec $\mathcal{U}_\varepsilon(\mathfrak{g})$ is an irreducible representation of $\mathcal{U}_\varepsilon(\mathfrak{g})$ on a vector space V such that $\varphi(\sigma)$ belongs to a conjugacy class $\mathcal{O}_V$ in G, then* $\dim V$ *is divisible by* $\ell^{\frac{1}{2}\dim \mathcal{O}_V}$.

The De Concini–Kac–Procesi conjecture has been proved in the following cases:

(i) $\mathcal{O}$ is a regular conjugacy class ([DCKP2, Theorem 5.1]);

(ii) G is of type A_n and $\ell = p$ is a prime ([C1]);
(iii) G is of type A_n, $\ell = p^k$ and $\mathcal{O}$ is a subregular unipotent conjugacy class ([C2]);
(iv) G is of type A_n and $\mathcal{O}$ is a spherical unipotent conjugacy class ([C3]).

We recall that the subalgebra B_ε contains a copy of the coordinate ring $\mathbb{C}[B^-]$ of B^-. Given $b \in B^-$, let us denote by m_b the corresponding maximal ideal of $\mathbb{C}[B^-]$ and let us consider the algebra $A_b := B_\varepsilon / m_b B_\varepsilon$. This is a finite-dimensional algebra with the following properties.

Theorem 27. [DCKP3] *If p, $q \in B^-$ lie over the same element $w \in W$, then the algebras A_p and A_q are isomorphic.*

Theorem 28. [DCKP3] *Let $p \in B^- \cap B\dot{w}B$ be a point over $w \in W$ and let A_p be the corresponding algebra. Assume that ℓ is a good integer. Then the dimension of each irreducible representation of A_p is equal to $\ell^{(\ell(w)+\mathrm{rk}(1-w))/2}$.*

Corollary 29. *If $p \in B^-$ lies over $w \in W$ and σ is an irreducible representation of $\mathcal{U}_\varepsilon(\mathfrak{g})$ on a vector space V such that $\varphi(\sigma)$ is conjugated to p, then* $\dim(V)$ *is divisible by $\ell^{(\ell(w)+\mathrm{rk}(1-w))/2}$.*

Proof. See [C2, Corollary 2.9]. □

Theorem 24 and Corollary 29 lead to the following result.

Theorem 30. *Let $\mathfrak{g}$ be a simple complex Lie algebra, and let ℓ be a good integer. If V is a simple $\mathcal{U}_\varepsilon(\mathfrak{g})$-module whose associated conjugacy class $\mathcal{O}_V$ is spherical, then $\ell^{\frac{1}{2}\dim \mathcal{O}_V}$ divides* $\dim V$. □

It was shown in [DCK2, §8] that in order to prove the conjecture it is enough to consider the exceptional conjugacy classes, that is, the conjugacy classes of exceptional elements. For convenience of the reader we recall that a semisimple element $g \in G$ is *exceptional* if its centralizer in G has a finite center. An element $g \in G$ is called exceptional if its semisimple part is exceptional. From the classification of the semisimple exceptional elements ([K2, Lemma 7.1], [DCK2, §7]) it follows that when $\mathfrak{g}$ is of classical type or of type G_2, all the semisimple exceptional elements are spherical. The elements σ_k and ρ_k in Table 1, with $k = 1, \dots, \left[\frac{n}{2}\right]$ for $\mathfrak{g}$ of type C_n, $k = 2, \dots, \left[\frac{n}{2}\right]$ for $\mathfrak{g}$ of type D_n, and $k = 1, \dots, n$ for $\mathfrak{g}$ of type B_n, are, up to central elements, representatives of all semisimple, exceptional conjugacy classes. The elements appearing in Table 2, except p_2 and q_3, are, up to central elements, representatives of all spherical, semisimple, exceptional conjugacy classes for $\mathfrak{g}$ of exceptional type.

Using the De Concini–Kac reduction theorem ([DCK2, §8]) we can go a bit further in the proof of the conjecture.

Corollary 31. *Let $\mathfrak{g}$ be of classical type or of type G_2, and let s be a semisimple element of G. Then any irreducible representation V of $\mathcal{U}_\varepsilon(\mathfrak{g})$ lying over $\mathcal{O}_s$ has dimension divisible by $\ell^{\frac{1}{2}\dim \mathcal{O}_s}$.*

Proof. An irreducible representation of $\mathcal{U}_\varepsilon(\mathfrak{g})$ lying over a semisimple element of G is either exceptional or induced by an exceptional semisimple representation of $\mathcal{U}_\varepsilon(\mathfrak{g}')$ ([DCK2, §8]). By Theorem 15 the De Concini–Kac–Procesi conjecture follows for all irreducible representations lying over semisimple elements. □

Corollary 32. *Let g be a nonexceptional element of G with Jordan decomposition $g = su$ such that $\mathcal{O}_s$ and $\mathcal{O}_u$ are spherical. Then any irreducible representation V of $\mathcal{U}_\varepsilon(\mathfrak{g})$ lying over $\mathcal{O}_g$ has dimension divisible by $\ell^{\frac{1}{2}\dim \mathcal{O}_g}$.*

Proof. Since $\mathcal{O}_s$ is spherical, s can be chosen among the nonexceptional elements in Tables 1 and 2. The case of G of type A_n was dealt with in [C2]. Using [DCK2, §8] we have:

(1) G is of type C_n and $s = c_\lambda$. Then V is induced by an irreducible $\mathcal{U}_\varepsilon(sp_{2n-2})$-module V' lying over the spherical, unipotent conjugacy class of the element $\left[\begin{array}{c|c} U_1 & U_2 \\ \hline U_3 & U_4 \end{array}\right] \in \mathrm{Sp}_{2n-2}$.

(2) G is of type C_n (respectively D_n) and $s = c$. Then V is induced by an irreducible $\mathcal{U}_\varepsilon(sl_n)$-module V' lying over the unipotent spherical conjugacy class of the element $A \in \mathrm{SL}_n$ where A is as in the proof of Proposition 20.

(3) G is of type D_n and $s = d = \hat{\tau}(c)$. Then, since $C_G(c)$ is generated by the root subgroups corresponding to the simple roots $\alpha_1, \ldots, \alpha_{n-1}$, the centralizer of d in G is generated by the root subgroups corresponding to the simple roots $\alpha_1, \ldots, \alpha_{n-2}, \alpha_n$, and V is induced by an irreducible $\mathcal{U}_\varepsilon(sl_n)$-module V' lying over a spherical unipotent conjugacy class of SL_n.

(4) G is of type D_n and $s = \sigma_1$. Then $u = \left[\begin{array}{cc|cc} 1 & & & \\ & U_1 & & U_2 \\ \hline & & 1 & \\ & U_3 & & U_4 \end{array}\right]$ where $u' = \left[\begin{array}{c|c} U_1 & U_2 \\ \hline U_3 & U_4 \end{array}\right]$ is a spherical unipotent element of SO_{2n-2}. Then V is induced by an irreducible $\mathcal{U}_\varepsilon(so_{2n-2})$-module V' lying over u'.

(5) G is of type B_n and there are two possibilities:

(i) $s = \rho_1$ and, as in the previous case, V is induced by an irreducible $\mathcal{U}_\varepsilon(so_{2n-1})$-module V' lying over a spherical unipotent element;

(ii) $s = b_\lambda$, and V is induced by an irreducible $\mathcal{U}_\varepsilon(sl_n)$-module V' lying over the unipotent spherical conjugacy class of the element $A \in \mathrm{SL}_n$ where A is as in the proof of Proposition 20.

(6) G is of type E_6. In this case $s = \exp(\pi i \check{\omega}_1)$ and, since u commutes with s, u belongs to the subgroup of type D_5 with simple roots $\alpha_2, \ldots, \alpha_6$. By [DCK2, §8] V is induced by an irreducible $\mathcal{U}_\varepsilon(so_{10})$-module V' corresponding to the conjugacy class of the element u. Besides, the conjugacy class of u in D_5 is again spherical by Remark 3.

(7) G is of type E_7. In this case $s = \exp(\pi i \check{\omega}_7)$ and u belongs to the subgroup of type E_6 with simple roots $\alpha_1, \ldots, \alpha_6$. By [DCK2, §8] V is induced by an irreducible $\mathcal{U}_\varepsilon(e_6)$-module V' corresponding to the unipotent spherical conjugacy class of the element u. The conjugacy class of u in E_6 is again spherical by Remark 3.

By Theorem 30 the proof is concluded. □

Remark 8. We point out that Corollary 32 can be generalized to a larger class of representations by making use of the De Concini–Kac reduction theorem. In particular the conjecture follows whenever the following conditions are satisfied:

(1) s lies in the identity component of $Z(C_G(s))$;
(2) $\mathcal{O}_u$ is spherical.

When $\mathfrak{g}$ is of classical type Condition (1) is equivalent to the following conditions in the corresponding matrix groups:

- $G = \mathrm{SO}_{2n+1}$: $C_G(s)^\circ$ contains no copy of type D_k with $k \geqslant 2$, i.e., if s is diagonal, no submatrix of s is conjugated to ρ_k with $k \geqslant 2$;
- $G = \mathrm{Sp}_{2n}$: $C_G(s)$ contains at most one copy of type C_k with $k \geqslant 1$, i.e., if s is diagonal, no submatrix of s is conjugated to σ_k with $k \geqslant 1$;
- $G = \mathrm{SO}_{2n}$: $C_G(s)^\circ$ contains at most one copy of type D_k with $k \geqslant 2$, i.e., if s is diagonal, no submatrix of s is conjugated to σ_k with $k \geqslant 2$.

Let us notice that when $\mathfrak{g}$ is of type A_n Condition (1) is always satisfied ([C2, Theorem 3.4]).

Corollary 33. *Any irreducible representation V of $\mathcal{U}_\varepsilon(sp_4)$ has dimension divisible by $\ell^{\frac{1}{2}\dim \mathcal{O}_V}$.*

Proof. Thanks to the De Concini–Kac reduction theorem, it is enough to consider the exceptional representations of $\mathcal{U}_\varepsilon(sp_4)$. Since an exceptional element of Sp_4 is either spherical or regular, the De Concini–Kac–Procesi conjecture follows from Theorem 30 and [DCKP2, Theorem 5.1]. □

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