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ON FINITE GROUPS IN WHICH CYCLIC SUBGROUPS OF THE SAME ORDER ARE CONJUGATE

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We consider finite groups G for which any two cyclic subgroups of the same order are conjugate in G. We prove various structure results and, in particular, that any such group has at most one non-abelian composition factor, and this is isomorphic to $PSL(2, p^m)$, with m odd if p is odd, or to $Sz(2^{2m+1})$, or to one of the sporadic groups M_{11}, M_{23} , or J_1 .

Key Words: Conjugacy classes; Cyclic subgroups.

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INTRODUCTION

In this article we shall study the class of csc-groups in the following sense.

Definition. Let π be a set of prime numbers. A finite group G is called a \csc_{π} -group if given two cyclic subgroups X, Y of G of the same order with $\pi(|X|) \subseteq \pi$, then there exists $g \in G$ such that $X = Y^g$. A finite group G is called a csc-group if G is a \csc_{π} -group for $\pi = \pi(|G|)$.

Similar kinds of problems have often been object of investigation. For instance, Fitzpatrick [6], using the classification of finite simple groups, proved that if in a finite group G any two elements of the same order are conjugate, then G is isomorphic with the symmetric group S_n , with $n \in \{1, 2, 3\}$ (see also Zhang [22], Feit and Seitz [5]). Then in Li [11] there is the classification of finite groups for which elements of the same order are conjugate or inverse-conjugate. Similar results, but concerning fusion in Aut G, have been obtained in Zhang [23], Li and Praeger [12, 13]. Stroth [18] considers finite groups G for which any two isomorphic subgroups are conjugate in G.

The main result of the present article is the following theorem.

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Address correspondence to M. Costantini, Dipartimento di Matematica Pura ed Applicata, Università di Padova, via Trieste 63, Padova 35121, Italy; E-mail: costantini@math.unipd.it **Theorem.** Let G be a finite csc-group. Then

$$F^*(G) = X_1 \times X_2 \times \cdots \times X_k,$$

where the $|X_i|$'s for $i \in \{1, 2, ..., k\}$ are pairwise coprime and one of the following holds:

- (1) X_i is a cyclic p-group;
- (2) X_i is an elementary abelian p-group;
- (3) X_i is a non-abelian 2-group such that $\Omega_1(X_i)$ and $X_i/\Omega_1(X_i)$ are elementary abelian, and either $|X_i| = |\Omega_1(X_i)|^2$ or $|X_i| = |\Omega_1(X_i)|^3$;
- (4) $X_i \simeq \text{PSL}(2, p^m)$ or $X_i \simeq \text{SL}(2, p^m)$ with $p \neq 2$, $p^m > 3$ and m odd;
- (5) $X_i \simeq \text{PSL}(2, 2^m)$ with $2^m > 2$;
- (6) X_i is one of the sporadic groups M_{11} , M_{23} , or J_1 .

Moreover, if P is a Sylow p-subgroup of $F^*(G)$ and P is not cyclic, then P is a Sylow p-subgroup of G.

We shall also determine further properties of csc-groups, giving a structure characterization in terms of certain minimal csc-subgroups.

The article is structured as follows. In Section 1, we introduce the notation and prove some preliminary results. In Section 2, we deal with solvable csc-groups and Frobenius groups. In Section 3, we classify the simple, almost-simple, and quasisimple csc-groups. In Section 4, we introduce the notion of monolithic csc-groups and determine the structure of the generalized Fitting subgroup for these groups. Finally, in Section 5, we deal with the general case.

All groups in this article are meant to be finite. We shall make use of the Classification of Finite Simple Groups.

1. NOTATION AND PRELIMINARY RESULTS

We shall denote by \mathbb{P} the set of prime numbers and by π a subset of \mathbb{P} , then we put $\pi' = \mathbb{P} \setminus \pi$. If $n \in \mathbb{N}$ with $n \ge 2$, we denote by $\pi(n)$ the set of primes dividing n.

A π -group is a group G such that $\pi(|G|) \subseteq \pi$. If G is a group, $O_{\pi}(G)$ is the largest normal subgroup of G which is a π -group. If $\pi = \{p\}$, we shall write $O_p(G)$ and $O_{p'}(G)$ instead of $O_{\pi}(G)$ and $O_{\pi'}(G)$, respectively. An element $g \in G$ is called a π -element if $\pi(|g|) \subseteq \pi$.

We denote by $Syl_p(G)$ the set of Sylow *p*-subgroups of *G*. Also E(G) denotes the subgroup of *G* generated by the quasisimple subnormal subgroups of *G*, $F^*(G) = F(G)E(G)$ is the generalized Fitting subgroup of *G* and $O_{\infty}(G)$ is the largest normal solvable subgroup of *G* (the solvable radical of *G*).

We denote by C_n the cyclic group of order *n*. For short we shall call *quaternions* the group of quaternions of order 8.

The following easy fact is essential for induction arguments on the order of G.

Lemma 1.1. Let G be a \csc_{π} -group, N a normal subgroup of G, $\overline{G} = G/N$. Let $\overline{x}, \overline{y} \in \overline{G}$ be elements of order \overline{r} with $\pi(\overline{r}) \subseteq \pi$, and let x, y be preimages of $\overline{x}, \overline{y}$ in G such that x and y are $\pi(\overline{r})$ -elements. Then $|\langle x \rangle| = |\langle y \rangle|$.

Proof. Let $r_1 \, e \, r_2$ be the orders of x and y, respectively; by hypothesis, $\pi(r_1) = \pi(r_2) = \pi(\bar{r}) \subseteq \pi$. Let $m = (r_1, r_2)$, then \bar{r} divides m, and we may write $r_1 = ms_1$ and $r_2 = ms_2$ with $(s_1, s_2) = 1$. Suppose for a contradiction that $r_1 \neq r_2$; one shoud have $s_1 \neq s_2$. The subgroups $\langle x^{s_1} \rangle$ and $\langle y^{s_2} \rangle$ have the same order and, since $\pi(m) = \pi(\bar{r}) \subseteq \pi$, they are conjugate in G. But in \overline{G} one has $|\langle \bar{x}^{s_1} \rangle| \neq |\langle \bar{y}^{s_2} \rangle|$, a contradiction.

Lemma 1.2. Let G be a \csc_{π} -group and let N be a normal subgroup of G. Then G/N is a \csc_{π} -group.

Proof. This follows immediately from Lemma 1.1.

Lemma 1.3. Let G_1 , G_2 be \csc_{π} -groups. Then $G_1 \times G_2$ is a \csc_{π} -group if and only if $\pi(|G_1|) \cap \pi(|G_2|) \cap \pi = \emptyset$.

Proof. Sufficiency is clear. To prove necessity, assume for a contradiction that there exists a prime p in $\pi(|G_1|) \cap \pi(|G_2|) \cap \pi$. Let $\langle x_1 \rangle$ be a subgroup of order p of G_1 and $\langle x_2 \rangle$ a subgroup of order p of G_2 . If $x_1 \in Z(G_1)$, then $\langle (x_1, 1) \rangle \leq Z(G_1 \times G_2)$ is not conjugate to $\langle (1, x_2) \rangle$; similarly x_2 is not in $Z(G_2)$. If we put $G = G_1 \times G_2$, we have $C_G(\langle (x_1, 1) \rangle) = C_{G_1}(x_1) \times G_2$, $C_G(\langle (1, x_2) \rangle) = G_1 \times C_{G_2}(x_2)$ and $C_G(\langle (x_1, x_2) \rangle) = C_{G_1}(x_1) \times C_{G_2}(x_2)$. In particular, $\langle (x_1, x_2) \rangle$ has order p and is conjugate neither to $\langle (x_1, 1) \rangle$ nor to $\langle (1, x_2) \rangle$, a contradiction.

Lemma 1.4. Let G be a \csc_{π} -group, and let $p \in \pi \cap \pi(|Z(G)|)$. Then the Sylow p-subgroups of G are cyclic or isomorphic to generalized quaternions.

Proof. Let P be a Sylow p-subgroup of G, and let x be an element of order p of $Z(G) \cap P$. Then $\langle x \rangle$ is the unique subgroup of order p of P, and we conclude by 5.3.6 in Robinson [15].

We denote by $Z_2(G)$ the second centre of G, i.e., the subgroup of G such that $Z_2(G)/Z(G) = Z(G/Z(G))$.

Lemma 1.5. Let G be a \csc_{π} -group. Then $O_{\pi}(Z(G))$ is cyclic and $O_{\pi}(Z_2(G)) = O_{\pi}(Z(G))$.

Proof. By Lemma 1.4, if $p \in \pi$, then $O_p(Z(G))$ is cyclic; it follows that $O_{\pi}(Z(G))$ is cyclic.

To prove the second statement, suppose for a contradiction that for a $p \in \pi$ there exists a *p*-element *x* of $O_p(Z_2(G))$ not lying in $O_{\pi}(Z(G))$. Let $y \in G$ with $[x, y] \neq 1$, then $x^y = xz$ for some $z \in O_p(Z(G))$. If the order of *z* is p^k , we have $x^{y^{p^k}} = xz^{p^k} = x$, so that $y^{p^k} \in C_G(x)$; without loss of generality, we may therefore assume that *y* is a *p*-element of *G*. Then $\langle x, y \rangle$ is a noncyclic *p*-subgroup of *G*. By Lemma 1.4, we must have p = 2, and the Sylow 2-subgroups of *G* are isomorphic to generalized quaternions.

Let S be a Sylow 2-subgroup of G, and let $Z(S) = \langle z \rangle$; we have $|\langle z \rangle| = 2$ and, by hypothesis, $\langle z \rangle \leq Z(G)$. In $\overline{G} = G/\langle z \rangle$, we have $O_2(Z(\overline{G})) \neq 1$ so that, by Lemma 1.4, the Sylow 2-subgroups of \overline{G} should be isomorphic to generalized quaternions. But $\overline{S} = S/\langle z \rangle$ is dihedral, a contradiction.

2. SOLVABLE csc_π-GROUPS AND FROBENIUS GROUPS

Lemma 2.1. Let G be a solvable \csc_{π} -group. Then for every $p \in \pi$, G has p-length at most 1.

Proof. Let G be a counterexample of minimal order. Since every quotient of a \csc_{π} -group is, by Lemma 1.1, a \csc_{π} -group, we have $\ell_p(G) > 1$, and every proper quotient of G has p-length less or equal 1. By Proposition 9.3.8 in Robinson [15], $N = O_{p'p}(G)$ is an elementary abelian p-subgroup of G, and there exists a subgroup H of G such that G = NH and $N \cap H = \{1\}$. In H, there is no subgroup of order p, since this then should be conjugate to every cyclic subgroup of N. Hence H is a p'-group and $G = O_{pp'}(G)$, a contradiction.

Lemma 2.2. Let G be a solvable \csc_{π} -group, $p \in \pi$, and P a Sylow p-subgroup of G. Then one of the following holds:

- (1) P is cyclic;
- (2) *P* is elementary abelian;
- (3) p = 2, P has class 2, exponent 4, and $P' = \Phi(G) = Z(P) = \Omega_1(Z(P)) = \Omega_1(P)$; moreover, $|P| = |Z(P)|^2$ or $|P| = |Z(P)|^3$.

Proof. Without loss of generality, we may assume $P \leq G$. Otherwise, we consider $G/O_{p'}(G)$ (which is \csc_{π} -group by Lemma 1.2) and use Lemma 2.1. We distinguish two cases:

(i) *P* is abelian.

Then *P* is homocyclic by Theorem VIII.5.8(b) in Huppert and Blackburn [9]. If *P* is cyclic, then we are done. Let us assume that *P* is not cyclic, and let p^k be the exponent of *P*; then there exists $n \in \mathbb{N}$ with n > 1 such that $|P| = p^{kn}$. We show that k = 1. Assume for a contradiction that k > 1. The cyclic subgroups of order p^2 in *P* are $\frac{p^{2n}-p^n}{p(p-1)} = p^{n-1}\frac{p^n-1}{p-1}$ and are permuted transitively under the action of $H = G/C_G(\Omega_2(P))$; but this number is divisible by *p* since n > 1, while *H* is a *p'*-group: a contradiction.

(ii) *P* is not abelian.

Then p = 2 by Shult [16, 17]. If *P* has only one involution, then *P* is generalized quaternions of order 2^n , say (see 5.3.6 in Robinson [15]). The condition that all subgroups of order 4 are conjugate in *G*, gives n = 3, and we are done. If *P* has more than one involution, then *P* is a Suzuki 2-group (following Definition VIII.7.1 in Huppert and Blackburn [9]), and by Theorem III.7.9 in Huppert and Blackburn [9] we conclude.

Remark 2.3. Thompson (see Theorem IX.8.6 in Huppert and Blackburn [9]) proved that if a solvable group G is such that the Sylow 2-subgroups have more

that one involution and all involutions in G are conjugate, then:

- (a) The 2-length of G is 1;
- (b) The Sylow 2-subgroups of G are homocyclic or Suzuki 2-groups.

On the other hand, Gaschütz and Yen (see Theorem IX.8.7 in Huppert and Blackburn [9]) proved that if G is a p-solvable group, where p is an odd prime divisor of |G| and if the subgroups of order p of G are permuted transitively under the action of Aut G, then the p-length of G is 1.

Lemma 2.1 could be obtained by the above mentioned results. We have given a direct short proof to make the article as self contained as possible.

Remark 2.4. The Sylow 2-subgroups of $S_Z(2^d)$ and of $PSU(3, 2^n)$ admit a solvable group of automorphisms which permutes transitively their involutions (see Remark XI.3.7.c in Huppert and Blackburn [10]). Moreover:

- (a) Let S be a Sylow 2-subgroup of $Sz(2^d)$; then $|S| = 2^{2d}$ and $|\Omega_1(S)| = 2^d$, and there is an automorphism $\alpha \in Aut(S)$ of order $2^d 1$ which permutes transitively the involutions of S. The semidirect product $G = S\langle \alpha \rangle$ is a csc-group.
- (b) Let S be a Sylow 2-subgroup of PSU(3, 2ⁿ); then |S| = 2³ⁿ and |Ω₁(S)| = 2ⁿ, and there is an automorphism α ∈ Aut(S) of order 2ⁿ − 1 which permutes transitively the involutions of S. The semidirect product G = S⟨α⟩ is not a csc-group since G/Ω₁(S) is not a csc-group. However, there exists β ∈ Aut(S) of order 2ⁿ + 1 such that [α, β] = 1, and the semidirect product of S with the cyclic group ⟨α, β⟩ is a csc-group.

We also observe that the Suzuki 2-groups S such that $|S| = |\Omega_1(S)|^2$ are classified in Huppert and Blackburn [9]: they are the groups $A(2^n, \theta)$ of matrices of the form

$$\begin{pmatrix} 1 & a & b \\ 0 & 1 & a^{\theta} \\ 0 & 0 & 1 \end{pmatrix}$$

with $a, b \in GF(2^n)$ and θ a nontrivial automorphism of odd order of $GF(2^n)$. In particular, not all these groups are Sylow subgroups of a simple Suzuki group.

Further information on the structure of 2-groups with automorphism group acting transitively on the set of involutions may be found in Bryukhanova [4] and in Wilkens [20].

Let us consider the Galois group \mathcal{G} of the field extension $GF(p^m)/GF(p)$; we have $\mathcal{G} = \langle \sigma | \sigma : GF(p^m) \to GF(p^m), x \mapsto x^p \rangle$, and \mathcal{G} is cyclic of order *m*. We may consider the following transformation groups of $GF(p^m)$:

(i) $A(p^m) = \{x \mapsto x + b \mid b \in V\}$, the *translation* group, isomorphic to the additive group of $GF(p^m)$;

FINITE csc GROUPS

(ii) The semilinear group $\Gamma(p^m) = \{x \mapsto ax^{\tau} \mid a \in GF(p^m)^{\sharp}, \tau \in \mathcal{G}\};$

(iii) The subgroup $\Gamma_0(p^m) = \{x \mapsto ax \mid a \in GF(p^m)^{\sharp}\}$, normal in $\Gamma(p^m)$;

(iv) The semilinear affine group

$$A\Gamma(p^m) = \{ x \mapsto ax^{\tau} + b \, | \, a \in GF(p^m)^{\sharp}, \, \tau \in \mathcal{G}, \, b \in GF(p^m) \}.$$

We note that the group $\Gamma(p^m)$ is metacyclic, since $\Gamma_0(p^m) \simeq GF(p^m)^{\sharp}$ and $\Gamma(p^m)/\Gamma_0(p^m) \simeq \mathcal{G}$ are cyclic of order $p^m - 1$ and *m*, respectively.

Proposition 2.5. Let G be a solvable \csc_p -group with $O_{p'}(G) = \{1\}$. Let P be a Sylow p-subgroup of G, and suppose $P/\Phi(P)$ has order p^m . If $p^m \notin \{5^2, 7^2, 11^2, 23^2, 3^4\}$, then $G/\Phi(P)$ is isomorphic to a subgroup of $A\Gamma(p^m)$. Moreover, if $p \neq 2$ and m > 1, then $\Phi(P) = \{1\}$.

Proof. Let P be a Sylow p-subgroup of G. Since $O_{p'}(G) = \{1\}$, by Lemma 2.1, we have $P \leq G$ and F(G) = P. If P is cyclic, then G/P is isomorphic to a subgroup of Aut P, and we are done.

Otherwise, by eventually considering the quotient $G/\Phi(P)$, we may assume that P is elementary abelian. Then $C_G(P) = P$ and G = PH with (|P|, |H|) = 1; if $|P| = p^m$, we may consider H as a subgroup of GL(m, p). Let Z be the centre of GL(m, p), and let $\hat{H} = HZ$. Since H permutes transitively the subgroups of order p of P, it follows that \hat{H} permutes transitively the elements of order p of P. Therefore, the group $\hat{G} = P\hat{H}$ is a solvable 2-transitive group. Such groups have been classified by Huppert (see Theorem XII.7.3 in Huppert and Blackburn [10]), and we conclude that \hat{G} is either a subgroup of the semilinear affine group $A\Gamma(p^m)$, or p^m lies in $\{3^2, 5^2, 7^2, 11^2, 23^2, 3^4\}$. If $p^m = 3^2$, then $|Aut P| = 2^4 \cdot 3$, and since P is a Sylow 3-subgroup of G, the order of H is a divisor of 16. But then G is isomorphic to a subgroup of $A\Gamma(3^2)$.

The last statement follows from Lemma 2.2.

The following examples explain the structure of the *exceptional* solvable csc-groups appearing in the statement of Proposition 2.5.

Example 1. Let *P* be an elementary abelian group of order 5^2 . There exists a subgroup *H* of GL(2, 5) with $H \simeq SL(2, 3)$ such that the semidirect product G = PH is a csc₅-group. Such a *G* is a Frobenius group and turns out to be a csc-group.

Example 2. Let *P* be an elementary abelian group of order 7^2 . There exists a subgroup *H* of GL(2, 7) with $H \simeq$ GL(2, 3) such that the semidirect product G = PH is a csc₇-group. Such a *G* is a Frobenius group, but *G* is not a csc₂-group since *H* has a subgroup *K* of index 2 isomorphic to SL(2, 3), and in $H \setminus K$ there are elements of order 2.

Example 3. Let *P* be an elementary abelian group of order 11². There exist subgroups H_1 and H_2 of GL(2, 11) with $H_1 \simeq SL(2, 3)$ and $H_2 \simeq SL(2, 3) \times C_5$ such that the semidirect products $G_1 = PH_1$ and $G_2 = PH_2$ are \csc_{11} -groups. Such groups are Frobenius groups, and are both csc-groups.

Example 4. Let *P* be an elementary abelian group of order 23². There exist subgroups H_1 and H_2 of GL(2, 23) with $H_1 \simeq$ GL(2, 3) and $H_2 \simeq$ GL(2, 3) × C_{11} such that the semidirect products $G_1 = PH_1$ and $G_2 = PH_2$ are csc₂₃-groups. Such groups are Frobenius groups, but are not csc₂-groups.

Example 5. Let *P* be an elementary abelian group of order 3^4 . There exist subgroups H_1 , H_2 , and H_3 of GL(4, 3) of order $2^5 \cdot 5$, $2^6 \cdot 5$, and $2^7 \cdot 5$, respectively (such groups are explicitly described in Example XII.7.4 in Huppert and Blackburn [10]) such that the semidirect products $G_1 = PH_1$, $G_2 = PH_2$, and $G_3 = PH_3$ are csc₃-groups. The structure of the Sylow 2-subgroups of H_1 , H_2 , and H_3 shows that G_1 , G_2 , and G_3 are neither csc₂-groups nor Frobenius groups.

Corollary 2.6. Let G be a solvable csc-group such that $O_{p'}(G) = \{1\}$, and let P be a Sylow p-subgroup of G. If $|P/\Phi(P)| \notin \{5^2, 11^2\}$, then G/P is isomorphic to a subgroup of $\Gamma(p^m)$, where $p^m = |P/\Phi(P)|$.

Proof. This follows directly from Proposition 2.5 and the discussion in the above examples. \Box

Remark 2.7. Let G be a (Frobenius) sharply 2-transitive (here G is not necessarily assumed to be solvable), and let $|F(G)| = p^m$. If $p^m \notin \{7^2, 23^2\}$, then G is a csc-group.

Proof. Sharply 2-transitive groups have been classified by Zassenhaus (see Theorem XII.9.1, XII.9.4 in Huppert and Blackburn [10]). They are Frobenius groups, whose kernel P is an elementary abelian p-group and the action of G on P permutes transitively the elements of P^{\sharp} . The Frobenius complement in such groups is metacyclic, with 7 exceptions, 4 of which give rise to solvable groups (described in Examples 1–4), and the remaining are described in the following examples.

Example 6. Let *P* be an elementary abelian group of order 11^2 . There exists a subgroup *H* of GL(2, 11) with $H \simeq SL(2, 5)$ such that the semidirect product G = PH is a sharply 2-transitive group. One may check that *G* is a csc-group.

Example 7. Let P be an elementary abelian group of order 29². There exist subgroups H_1 and H_2 of GL(2, 23) with $H_1 \simeq SL(2, 5) \times C_7$ and $H_2 \simeq SL(2, 5)$ such that the semidirect products $G_1 = PH_1$ and $G_2 = PH_2$ are Frobenius groups (G_1 is sharply 2-transitive) One may check that G_1 are G_2 are csc-groups.

Example 8. Let *P* be an elementary abelian group of order 59². There exist subgroups H_1 and H_2 of GL(2, 59) with $H_1 \simeq SL(2, 5) \times C_{29}$ and $H_2 \simeq SL(2, 5)$ such that the semidirect products $G_1 = PH_1$ and $G_2 = PH_2$ are Frobenius groups (G_1 is sharply 2-transitive) One may check that G_1 are G_2 are csc-groups.

There is a further *exceptional* case which is not a sharply 2-transitive group, and which therefore does not appear in Zassenhaus' list. Such a group comes from Hering's list in Hering [7], classifying 2-transitive groups of affine type (see also Remark XII.7.5 in Huppert and Blackburn [10]).

Example 9. Let *P* be an elementary abelian group of order 19². There exist subgroups H_1 , H_2 , and H_3 of GL(2, 19) with $H_1 \simeq SL(2, 5)$, $H_2 \simeq SL(2, 5) \times C_3$, and $H_3 \simeq SL(2, 5) \times C_9$, such that $G_1 = PH_1$, $G_2 = PH_2$, and $G_3 = PH_3$ are \csc_{19} -groups. One may check that G_1 is a Frobenius group and a csc-group. On the other hand, from the structure of H_2 and H_3 it follows that G_2 and G_3 are neither \csc_3 -groups nor Frobenius groups. We observe that G_3 is a 2-transitive group.

A set of generators for Frobenius complements of the groups described in Examples 1–4 and 6–8 in terms of matrices of $M_2(GF(p))$ ($p \in \{5, 7, 11, 29, 59\}$) is given in Remark XII.9.5 in Huppert and Blackburn [10]. For completeness, we observe that the matrices

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 \\ 16 & 17 \end{pmatrix}$$

with coefficients in GF(19) generate the complement H_1 of the csc-group G_1 of Example 9.

Corollary 2.8. Let G be a solvable csc-group, $p \in \pi(|G|)$. If P is a Sylow p-subgroup of G and P is neither cyclic, nor quaternions, then P is normal in G.

Proof. We argue by induction on the order of G. We distinguish two cases:

(i) $O_p(G) \neq \{1\}.$

If in $\overline{G} = G/O_p(G)$, \overline{P} is neither cyclic nor quaternions, by induction $\overline{P} \leq \overline{G}$, so that $P \leq G$. Suppose P is elementary abelian; then if $P/O_p(G)$ is cyclic, there exists $x \in G \setminus O_p(G)$ of order p, and for every $y \in O_p(G)$ with $y \neq 1$, the subgroups $\langle x \rangle$ and $\langle y \rangle$ cannot be conjugate in G, a contradiction.

Hence p = 2 and suppose P has the structure as in Lemma 2.2(3). If $P/O_p(G)$ is cyclic of order 4 or quaternions, there would be involutions in both $O_2(G)$ and $G \setminus O_2(G)$, a contradiction. If $P/O_2(G)$ is cyclic of order 2, then there are elements of order 4 both in $O_2(G)$ and in $G \setminus O_2(G)$, again a contradiction. Hence in this case $P \leq G$.

(ii) $O_p(G) = \{1\}$, so that (|F(G)|, p) = 1.

If F(G) is cyclic, then G/F(G) is abelian, hence cyclic being a csc-group. If $F(G) = Q_8 \times C_n$ (*n* odd), then, since G is a csc-group, G/F(G) must be a 2'-group, and again, from the structure of $\operatorname{Aut}(Q_8 \times C_n)$ (and since in a solvable group $C_G(F(G)) \leq F(G)$), G/F(G) is abelian, and then cyclic. In both cases we have shown that G/F(G) is cyclic, and this is not possible by the hypothesis.

So let us suppose that a Sylow q-subgroup Q of F(G) is neither cyclic nor quaternions. Then by induction and the above reasoning, we deduce $Q \in Syl_q(G)$. We may assume, up to considering the quotient $G/\Phi(Q)$, that Q is elementary abelian. Then $G/C_G(Q)$ has the structure described in Proposition 2.5; in particular, P centralizes Q. We may, therefore, consider G/Q; proceeding in this way, after a finite number of steps we are reduced to the case when F(G) is cyclic. This is again a contradiction.

Hence p divides |F(G)| and $P \trianglelefteq G$.

Corollary 2.9. Let G be a solvable csc-group. Then the derived length of G is at most 4.

Proof. We argue by induction on the order of G. If G has at least 2 minimal normal subgroups N_1 and N_2 , the induction hypothesis applied to G/N_1 and to G/N_2 gives that both G/N_1 and G/N_2 have derived length at most 4, and the same holds for G, since $N_1 \cap N_2 = \{1\}$.

Let us suppose that G has a unique minimal normal subgroup N. Then N is a p-group for a certain prime p, so that F(G) is also a p-group. Let P be a Sylow p-subgroup of G; then $F(G) \leq P$ and since F(G) is neither cyclic nor quaternions, also P is neither cyclic nor quaternions. By Corollary 2.8, we have $P \leq G$, so that F(G) = P. By Proposition 2.5 and Corollary 2.9, we have the following cases:

- (i) F(G) is elementary abelian of order p^m , and G is isomorphic to a subgroup of $A\Gamma(p^m)$. In this case, we conclude by observing that $A\Gamma(p^m)$ has derived length 3.
- (ii) F(G) is a 2-group with structure as in Lemma 2.2, and $G/\Phi(F(G))$ is isomorphic to a subgroup of $A\Gamma(2^m)$, where $|F(G)/\Phi(F(G))| = 2^m$. Then $G^{(3)} \leq \Phi(F(G))$, and since $\Phi(F(G))$ is abelian, in this case we are also done.
- (iii) $|F(G)| = 5^2$ and $G/F(G) \simeq SL(2, 3)$. Then G satisfies the thesis (see Example 1).
- (iv) $|F(G)| = 11^2$ and $G/F(G) \simeq SL(2, 3)$ or $G/F(G) \simeq SL(2, 3) \times C_5$. Then G satisfies the thesis (see Example 3).

3. SIMPLE, ALMOST-SIMPLE, AND QUASISIMPLE csc-GROUPS

We observe that if G is a csc-group, then G has at most $\phi(n)$ conjugacy classes of elements of order n, where ϕ denotes Euler's function. In particular, G has a unique class of involutions; we shall also use the fact that $\phi(3) = 2$ and $\phi(4) = 2$. We start by giving the list of the simple groups with only one class of involutions. This may be found in Yamaki [21], here we present a more detailed statement.

Proposition 3.1. The non-abelian simple groups with precisely one class of involutions are those in the following List (A):

(a) Groups of Lie type in odd characteristic:

(a1) PSL(2, q), q > 3;(a2) PSL(3, q);(a3) $PSL(4, q), q \equiv 5 \mod 8;$ (a4) PSU(3, q);(a5) $PSU(4, q), q \equiv 3 \mod 8;$ (a6) ${}^{3}D_{4}(q);$ (a7) $G_{2}(q);$ (a8) ${}^{2}G_{2}(q), q = 3^{2m+1}, m \geq 1.$

(b) Groups of Lie type in characteristic 2:

(b1) *PSL*(2, q), q > 2;
(b2) *PSL*(3, q);

```
(b3) PSU(3, q), q > 2;
(b4) Sz(q) = {}^{2}B_{2}(q), q = 2^{2m+1}, m \ge 1.
```

(c) Alternating groups \mathbb{A}_n , $5 \le n \le 7$.

(d) Sporadic groups:

$$M_{11}, M_{22}, M_{23}, J_1, J_3, McL, Ly, O'N, Th = F_3.$$

In the next two lemmas, we determine the simple and almost-simple groups which are csc-groups. We shall prove them at the same time. Let us introduce the following list of simple groups List (B):

(a) Groups of Lie type in odd characteristic:

(a1)
$$PSL(2, q), q > 3, q = p^m, m \text{ odd};$$

- (b) Groups of Lie type in characteristic 2:
 - (b1) PSL(2, q), q > 2;(b2) PSL(3, 2);(b3) $Sz(q) = {}^{2}B_{2}(q), q = 2^{2m+1}, m \ge 1;$
- (c) Alternating group \mathbb{A}_5 ;
- (d) Sporadic groups M_{11} , M_{23} , J_1 .

Note that $PSL(3, 2) \simeq PSL(2, 7)$, $\mathbb{A}_5 \simeq PSL(2, 4) \simeq PSL(2, 5)$. Therefore, a simple group is in the List (B) if and only if it is isomorphic to one in the following List (C):

(i) $PSL(2, q), q \ge 4, q = p^m, m \text{ odd if } p \text{ odd};$ (ii) $Sz(q), q = 2^{2m+1}, m \ge 1;$ (iii) $M_{11}, M_{23}, J_1.$

Lemma 3.2. The finite simple groups which are csc-groups are those in List (B).

Lemma 3.3. Let $S < G \le \text{Aut } S$, with S simple non-abelian. Then G is csc-group if and only if G is of the form $G = S : \langle \psi \rangle$, where S is isomorphic to PSL(2, q), $q \ge 4$, $q = p^m$, m odd if p odd, or to $Sz(2^{2m+1})$, $m \ge 1$, and ψ is a field automorphism of S of order coprime to |S|.

Note that in particular if G is almost simple with socle S, and G is a csc-group, then S is a csc-group. To prove Lemma 3.3, we shall use the following result.

Lemma 3.4 (Yamaki [21], Lemma 2). Let S be a simple group with at least 2 conjugacy classes of involutions. Then not all involutions in S are conjugate in Aut S.

Proof of Lemmas 3.2, 3.3. Assume $S \le G \le \text{Aut } S$, with S simple non-abelian. If G is a csc-group, then, by Lemma 3.4, S has only one class of involutions hence, by Proposition 3.1, S is in List (A). For every S in List (A), we determine in which cases an almost-simple group G with socle S is a csc-group. For root subgroups we use the notation in Carter [2].

Groups of Lie type in odd characteristic:

(i) $S = G_2(q)$, $q = p^f$ odd. Here S has two subgroups of order p which are not conjugate in Aut S. In fact, if α and β are orthogonal roots, with β long, then $x_{\alpha}(1)x_{\beta}(1)$ and $x_{\beta}(1)$ have centralizers of different order in S, hence $\langle x_{\alpha}(1)x_{\beta}(1)\rangle$ and $\langle x_{\beta}(1)\rangle$, which have order p, are not conjugate in Aut S.

(ii) $S = {}^{2}G_{2}(q)$, $q = 3^{2m+1}$, $m \ge 1$. If P, P_{1} are distinct Sylow 3-subgroups of G, then $P \cap P_{1} = 1$ (Ward [19, Theorem (2)]). Moreover, there are subgroups X, Y of P of order 3 such that $X \le Z(P)$ and $Y \le Z(P)$. Hence X and Y are not conjugate in Aut S.

(iii) $S = {}^{3}D_{4}(q)$, $q = p^{m}$, p odd (note that $S \leq P\Omega_{8}^{+}(q^{3})$). We have Aut $S = S : \langle \varphi \rangle$, where φ is a field automorphism of order 3m. The elements $x_{\alpha_{1}}(1)x_{\alpha_{3}}(1)x_{\alpha_{4}}(1)$ and $x_{\alpha_{2}}(k)$ for $k \in GF(q^{3})^{\sharp}$ are not conjugate in Aut S (since they are not conjugate in $P\Omega^{+}(q^{3})$ and φ is a field automorphism), hence $\langle x_{\alpha_{1}}(1)x_{\alpha_{3}}(1)x_{\alpha_{4}}(1) \rangle$ and $\langle x_{\alpha_{2}}(1) \rangle$, which have order p, are not conjugate in Aut S.

(iv) $S = PSU(4, q), q \equiv 3 \mod 8, q = p^m, p \text{ odd (note that } S \leq PSL(4, q^2))$. We have Aut $S = PGU(4, q) : \langle \varphi \rangle$, where φ is a field automorphism of order 2m. The elements $x_{\alpha_1}(1)x_{\alpha_3}(1)$ and $x_{\alpha_2}(k)$ for $k \in GF(q^2)^{\sharp}$ are not conjugate in Aut S (since they are not conjugate in PGL(4, q), and φ is a field automorphism), hence $\langle x_{\alpha_1}(1)x_{\alpha_3}(1) \rangle$ and $\langle x_{\alpha_2}(1) \rangle$, which have order p, are not conjugate in Aut S.

(v) S = PSL(4, q), $q \equiv 5 \mod 8$, $q = p^m$, p odd. We have Aut $S = PGL(4, q) : \langle \varphi \rangle : \langle \delta \rangle$, where φ is a field automorphism of order m, δ is the graph automorphism. The elements $x_{\alpha_1}(1)x_{\alpha_3}(1)$ and $x_{\alpha_2}(k)$ for $k \in GF(q)^{\sharp}$ are not conjugate in Aut S (since they are not conjugate is PGL(4, q) and $\langle \varphi, \delta \rangle$ fixes the set $\{x_{\alpha_2}(k) | k \in GF(q)^{\sharp}\}$), hence $\langle x_{\alpha_1}(1)x_{\alpha_3}(1) \rangle$ and $\langle x_{\alpha_2}(1) \rangle$, which have order p, are not conjugate in Aut S.

(vi) S = PSL(3, q), $q = p^m$, p odd. We have Aut $S = PGL(3, q) : \langle \varphi \rangle : \langle \delta \rangle$, where φ is a field automorphism of order m, δ is the graph automorphism. The elements $x_{\alpha_1}(1)x_{\alpha_2}(1)$ and $x_{\alpha_1}(k)$ for $k \in GF(q)^{\sharp}$ are not conjugate in Aut S (since $x_{\alpha_1}(1)x_{\alpha_2}(1)$ is regular, so that it lies in a unique Sylow p-subgroup of S, while $x_{\alpha_1}(k)$ is not regular), hence $\langle x_{\alpha_1}(1)x_{\alpha_2}(1) \rangle$ and $\langle x_{\alpha_1}(1) \rangle$, which have order p, are not conjugate in Aut S.

(vii) S = PSU(3, q), $q = p^m$, p odd (note that $S \le PSL(3, q^2)$). We have Aut $S = PGU(3, q) : \langle \varphi \rangle$, where φ is a field automorphism of order 2m. In S there are regular and nonregular unipotent elements of order p. Then, as for PSL(3, q), we conclude that there are subgroups X, Y of order p of S which are not conjugate in Aut S.

(viii) $S = PSL(2, q), q = p^m, p$ odd. We have Aut $S = PGL(2, q) : \langle \varphi \rangle$, where φ is a field automorphism of order *m*. By Huppert [8], Satz 8.5, the groups PSL(2, q) are groups with partition and the Sylow *r*-subgroups are cyclic for $r \neq 2$, *p*. Assume *X* and *Y* are cyclic subgroups of *S* of the same order *k*. Then *k* divides only one among $p, \frac{q-1}{2}, \frac{q+1}{2}$ (which are pairwise coprime), so that *X* and *Y* are conjugate in *S* unless k = p.

So assume k = p. Then S has only one class of subgroups of order p if and only if the subgroups of order p in a Sylow p-subgroup P of S are conjugate FINITE csc GROUPS

in N(P) (since two distinct Sylow *p*-subgroups intersect trivially). We may assume *P* are the unitriangular upper matrices, and *H* are the diagonal matrices in *S*. Then N(P) = HP, and the unipotent elements in *P* fall in 2 classes under the action of *H*. Therefore, there is a unique class of subgroups of order *p* in *P* under *H* if and only if $GF(q)^{\sharp} = (GF(q)^{\sharp})^2 GF(p)^{\sharp}$, i.e., if and only if *m* is odd.

Therefore, the finite simple group PSL(2, q), with $q = p^m$, p odd is a csc-group if and only if m is odd. We note that the group $PSL(2, 3) \simeq \mathbb{A}_4$ is a csc-group. Hence we may state that the groups $PSL(2, p^m)$ with odd p are csc-groups if and only if m is odd.

Now assume $S < G \le \text{Aut } S$. We determine in which cases G is a csc-group. We have $\text{Out } S \simeq C_2 \times C_m$. By Lucchini et al. [14], Aut S splits over S if and only if $(\frac{p^m-1}{2}, 2, m) = 1$, i.e., if and only if m is odd. So let us first assume m is odd. Then we know that $PSL(2, p^{m'})$ is a csc-group for each divisor m' of m; we use the following lemma.

Lemma 3.5. Let S = PSL(2, q), $q = p^m$, p odd, m odd, q > 3, $S < G \le Aut S$. Then G is a csc-group if and only if $G = S : \langle \psi \rangle$, where ψ is the field automorphism of order k (hence $k \mid m$), where $(\mid S \mid, k) = 1$.

Proof. Assume G is a csc-group. Since Out S is cyclic, and the only subgroup of order 2 in Out S corresponds to PGL(2, q) which always splits over S, we must have $G \cap PGL(2, q) = 1$, so that $G = S : \langle \psi \rangle$, where $\psi = \varphi^{m/k}$. Moreover, we must have (|S|, k) = 1. On the other hand, if $G = S : \langle \psi \rangle$, with $\psi = \varphi^{m/k}$, with (|S|, k) = 1, then G is a csc-group, since $C_S(\varphi^{m/h}) \simeq PSL(2, p^h)$ is a csc-group for each divisor h of m.

To conclude the case $S = PSL(2, p^m)$, assume finally that *m* is even, m = 2n. Then we have seen that *S* is not a csc-group, since there are 2 classes of subgroups of order *p*. We show that there are no groups *G* with socle *S* which are csc-groups. In this case, Aut *S* does not split over *S*, and Out $S \simeq C_2 \times C_{2n}$. Let τ be the involution in $\langle \varphi \rangle$. Suppose for a contradiction that such a *G* exists. We note that $G \cap PGL(2, q) = G \cap (S : \langle \tau \rangle) = 1$, since PGL(2, q) splits over *S*, $PGL(2) = S : \langle \sigma \rangle$. There exists $\alpha \in G$ such that

$$\left\langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right\rangle^{\alpha} = \left\langle \begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix} \right\rangle,$$

where $\mu \in GF(q)^{\sharp} \setminus ((GF(q)^{\sharp})^2)$, and α must be of the form $\alpha = i_s \sigma \varphi^i$, for a certain $s \in S$, $\varphi^i \neq 1$ (otherwise, $PGL(2, q) = S\langle \alpha \rangle \leq G$, a contradiction). Hence $S < S\langle \sigma \varphi^i \rangle \leq G$. Let $[S\langle \sigma \varphi^i \rangle : S] = 2^a f$, with odd f, $a \geq 1$. By taking the f-power of $\sigma \varphi^i$, we get $\sigma \varphi^j \in S\langle \sigma \varphi^i \rangle$, with $[S\langle \sigma \varphi^j \rangle : S] = 2^a$. However, if a > 1, the minimal subgroup of $S\langle \sigma \varphi^j \rangle/S$ is $S : \langle \tau \rangle/S$, so that $G \geq S : \langle \tau \rangle$, a contradiction. Hence $[S\langle \sigma \varphi^j \rangle : S] = 2$, i.e., $S\langle \sigma \varphi^j \rangle = S\langle \sigma \tau \rangle$ (which does not split over S). We prove that there exists an element of order 4 in $S\langle \sigma \tau \rangle \setminus S$, so in $G \setminus S$, there is an element of order 4. This is a contradiction, since S always has elements of order 4, m being even.

To show that in $S\langle \sigma \tau \rangle \backslash S$ there is an element of order 4, it is enough to exhibit an element $\delta \in PGL(2, q) \backslash S$ such that $\delta \delta^{\tau}$ has order 2 (if $\beta = i_{\delta} \tau$ then $\beta^2 = i_{\delta \delta^{\tau}}$). Suppose $p^n \equiv 3 \mod 4$, $2^h || (p^n + 1)$, and take $u \in GF(q)^{\sharp}$ of order 2^{h+1} . Let

$$\delta = \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix} \in PGL(2, q) \backslash S.$$

Then

$$\delta\delta^{\tau} = \begin{pmatrix} u^{1+p^n} & 0\\ 0 & 1 \end{pmatrix}.$$

Since $(u^{1+p^n})^2 = 1$, while $u^{1+p^n} \neq 1$, it follows that $\delta \delta^{\tau}$ has order 2.

Suppose $p^n \equiv 1 \mod 4$, $2^h || (p^n - 1)$, and take $u \in GF(q)^{\sharp}$ of order 2^{h+1} . Let

$$\delta = \begin{pmatrix} 0 & u \\ 1 & 0 \end{pmatrix} \in PGL(2, q) \backslash S.$$

Then

$$\delta\delta^{\tau} = \begin{pmatrix} u & 0\\ 0 & u^{p^n} \end{pmatrix} \in PGL(2, q) \backslash S$$

Then $u^2 = u^{2p^n}$, while $u \neq u^{p^n}$, hence $\delta \delta^{\tau}$ has order 2.

We now deal with simple groups of Lie type in characteristic 2:

(i) $S = {}^{2}B_{2}(q)$, $q = 2^{2m+1}$, $m \ge 1$ ($S \le Sp_{4}(q)$). We have Aut $S = S : \langle \varphi \rangle$, where φ is a field automorphism of order 2m + 1. We can deal with this case in the same way as in case PSL(2, q), since S is a group with partition. However, in this case the cyclic subgroups of order 4 are all conjugate in S, since there are 2 classes of elements of order 4, and x is not conjugate to x^{-1} if x has order 4 (the Sylow 2-subgroup has exponent 4). Hence S is a csc-group. We observe that also ${}^{2}B_{2}(2) \simeq$ 5 : 4, which is solvable, is a csc-group.

If $S < G \le \text{Aut } S$, then G is a csc-group if and only if $G = S : \langle \psi \rangle$, with $\psi = \varphi^{(2m+1)/k}$, with (|S|, k) = 1 (since $C_S(\psi) \simeq {}^2B_2(2^{(2m+1)/k})$ is a csc-group even when it is not simple).

(ii) S = PSL(3, q), $q = 2^m$. Let r be a prime divisor of q - 1, and assume $r \neq 3$. If

$$x = \operatorname{diag}(\alpha, \alpha, \alpha^{-2}), \quad y = \operatorname{diag}(\alpha, 1, \alpha^{-1}),$$

then x and y have order r and act in different ways on the projective plane. In particular, $\langle x \rangle$ and $\langle y \rangle$ are not conjugate in Aut S. We are left with PSL(3, 2) and PSL(3, 4).

We have $PSL(3, 2) \simeq PSL(2, 7)$ which is a csc-group, while $Aut(PSL(3, 2)) \simeq PGL(2, 7)$ is not a csc-group.

Finally, PSL(3, 4) is not a csc-group since it has 3 classes of elements of order 3. Moreover, if $S < G \le \text{Aut } S$, then G splits over S, since Aut S splits over S, and G is not a csc-group, since in S there are elements of order 2 and 3.

(iii) $S = PSL(2, q), q = 2^m, m \ge 2$. Here we argue as in the case q odd. However, here the Sylow 2-subgroup is elementary abelian, so that S is a csc-group. We note that $PSL(2, 2) \simeq \mathbb{S}_3$ is a csc-group. Since Aut $S = S : \langle \varphi \rangle$, where φ is a field automorphism, then $S < G \le$ Aut S is a csc-group if and only if $G = S : \langle \psi \rangle$, ψ in $\langle \varphi \rangle$ of order k, with (|S|, k) = 1 (since $C_S(\psi) \simeq PSL(2, 2^{m/k})$ is a csc-group).

(iv) PSU(3, q), $q = 2^m$, $m \ge 2$ (note that $S \le PSL(3, q^2)$). We have Aut $S = PGU(3, q) : \langle \varphi \rangle$, where φ is a field automorphism of order 2m.

Let r be a primitive prime divisor of $2^{2m} - 1$: r exists if m = 2 or $m \ge 4$. Then r divides $2^m + 1$, and $r \ne 3$, since 3 divides $2^2 - 1$. S contains a copy of $C_r \times C_r$: for a suitable basis of the 3-dimensional vector space over \mathbb{F}_{q^2} , the nonsingular Hermitian scalar product can be represented by the identity matrix. Therefore, the elements

$$x = \operatorname{diag}(\alpha, \alpha, \alpha^{-2}), \quad y = \operatorname{diag}(\alpha, 1, \alpha^{-1})$$

for $\alpha \in GF(q^2)^{\sharp}$ of order *r* are in *S*, and act in different ways on the projective plane over $GF(q^2)$. In particular, $\langle x \rangle$ and $\langle y \rangle$ are not conjugate in Aut *S*. We are left with PSU(3, 8). In this case, there are elements of order 3 in *S* with centralizers of different orders.

Alternating groups. We have $\mathbb{A}_5 \simeq PSL(2, 5)$, $\mathbb{A}_6 \simeq PSL(2, 9)$, so that \mathbb{A}_5 is a csc-group, $\mathbb{S}_5 = \operatorname{Aut} \mathbb{A}_5$ is not a csc-group since it splits over \mathbb{A}_5 . If G is such that $\mathbb{A}_6 \leq G \leq \operatorname{Aut} \mathbb{A}_6$, then G is not a csc-group; \mathbb{A}_7 is not a csc-group since it has 2 elements of order 3 with centralizers of different order, \mathbb{S}_7 is not a csc-group, since it splits over \mathbb{A}_7 .

Sporadic groups. By Conway et al. [3], the groups M_{11} , M_{23} , J_1 are csc-groups, while M_{22} , ON, Ly, Th are not csc-groups (since they contain elements of order 4 with centralizers of different orders) and J_3 , McL are not csc-groups (since they contain elements of order 3 with centralizers of different orders). If $S < G \le \text{Aut } S$, then S is not a csc-group since, by Conway et al. [3], G splits over S and [G:S] = 2.

The proof of Lemmas 3.2 and 3.3 is completed.

Lemma 3.6. Let G be a quasisimple csc-group which is not simple. Then $G \simeq SL(2, p^m)$ with $p \neq 2$ and m odd.

Proof. The group $SL(2, p^m)$ is certainly a csc-group if $PSL(2, p^m)$ is. The groups $SL(2, 2^n)$ and $Sz(2^{2n+1})$ with $n \ge 2$ do not admit central extensions, and the same holds for M_{11} , M_{23} , and J_1 (see Conway et al. [3]). Again using Conway et al. [3], one can check that no nontrivial central extension of Sz(8) is a csc-group.

4. MONOLITHIC csc-GROUPS

We introduce the following definition.

Definition 4.1. A csc-group is called csc-*monolithic* (or, simply, *monolithic*) if either $F^*(G)$ is a *p*-group or $F^*(G) = E(G)$.

Lemma 4.2. Let G be a group with a normal elementary abelian subgroup N of order p^{2m} and such that G/N is isomorphic to a subgroup of $\Gamma L(2, p^{2m})$

containing SL(2, p^{2m}). If $N = C_G(N)$ and if the action induced by G on N is the natural one, then G is not a \csc_p -group.

Proof. Let G_0 be the normal subgroup of G containing N such that $G_0/N \simeq$ SL(2, p^{2m}). It is enough to show that there exists an element of order p in $G_0 \setminus N$. We distinguish two cases.

(i) $p \neq 2$.

Let z be an element of order 2 of G_0 , and let $\overline{G} = G/N$. Then $Z(\overline{G}) = \langle \overline{z} \rangle$, and z induces inversion on N. Let \overline{x} be an element of order 2p of \overline{G} , and let x be a preimage of \overline{x} in G. Then $x^{2p} \in N$, and x has order either 2p or $2p^2$. If the order of x is 2p, then $x^2 \in G \setminus N$ is an element of order p.

If x has order $2p^2$, then x^{2p} would be a nontrivial element of N, and as such it should be inverted by x^{p^2} , a contradiction.

(ii) p = 2.

Let x be an element of order 3 of G_0 , and let \bar{x} be the corresponding element of $\overline{G} = G/N$. The minimal polynomial of \bar{x} as an element of SL(2, 2^{2m}) is $T^2 + T + 1$; hence $C_N(x) = \{1\}$, for every $y \in N$, we have $yy^x y^{x^2} = 1$ and, in particular, $\langle y, y^x \rangle$ is a x-invariant subgroup of N.

In G_0/N there exists an element \bar{a} of order 2 inverting \bar{x} . Let a be a preimage of \bar{a} in G which is a 2-element. Let us fix $y \in C_N(a)$ such that $y \neq 1$: then $(y^x)^a = (y^a)^{x^{-1}} = y^{x^2} \in \langle y, y^x \rangle$. Therefore, if $T = \langle y, x, a \rangle$ and $L = \langle y, y^x \rangle$, it follows that L is an elementary abelian normal subgroup of order 4 of T, such that $T/L \simeq S_3$.

As x is an element of order 3 of T acting fixed-point-freely on L, it follows that $T \simeq \mathbb{S}_4$. In particular, in T there exists an element b of order 2 inverting x. Certainly, $b \notin N$.

Lemma 4.3. Let G be a group with a normal elementary abelian subgroup N of order 3^n . Assume that:

(i) $G/N \simeq SL(2, q)$ with q odd;

(ii) The involutions of G induce inversion on N.

Then G is not a csc_3 -group.

Proof. It is enough to exhibit an element of order 3 in $G \setminus N$. Let $\overline{G} = G/N$, $\overline{x} \in \overline{G}$ be an element of order 6 (such an element exists since the order of SL(2, q) is divisible by 3 and $Z(\overline{G})$ has order 2), and let x be a preimage of \overline{x} in G. Then $x^6 \in N$ and x has order 6 or 18. If |x| = 6, then x^2 is an element of order 3 not in N. If |x| = 18, then x^6 would be a nontrivial element of N and as such, it would be inverted by x^9 , a contradiction.

Lemma 4.4. Let G be a csc-group, and suppose $F^*(G)$ is a p-group. Then $G/\Phi(F^*(G))$ is isomorphic to a subgroup of the semilinear affine group $A\Gamma(p^m)$, where $p^m = |F^*(G)/\Phi(F^*(G))|$ or G is a Frobenius group with kernel $F^*(G)$ which is elementary abelian of order p^2 , and one (and only one) of the following holds:

- (i) p = 5 and $G/F^*(G) \simeq SL(2, 3)$;
- (ii) p = 11 and $G/F^*(G) \simeq SL(2, 3)$;
- (iii) p = 11 and $G/F^*(G) \simeq SL(2, 3) \times C_5$;

(iv) p = 11 and $G/F^*(G) \simeq SL(2, 5)$; (v) p = 19 and $G/F^*(G) \simeq SL(2, 5)$; (vi) p = 29 and $G/F^*(G) \simeq SL(2, 5)$; (vii) p = 29 and $G/F^*(G) \simeq SL(2, 5) \times C_7$; (viii) p = 59 and $G/F^*(G) \simeq SL(2, 5)$; (ix) p = 59 and $G/F^*(G) \simeq SL(2, 5) \times C_{29}$.

Moreover, $F^*(G)$ is a Sylow p-subgroup of G whose structure is described in Lemma 2.2.

Proof. We may assume, up to considering $G/\Phi(F^*(G))$, that $F^*(G)$ is elementary abelian, of order p^m say. One then has $C_G(F^*(G)) = F^*(G)$ and $\overline{G} = G/F^*(G)$ acts on $F^*(G)$ as a subgroup of GL(m, p). Since \overline{G} permutes transitively the subgroups of order p of $F^*(G)$, we may obtain a group $\widetilde{G} = \overline{GZ}$ (where Z is the center of GL(m, p)) which permutes transitively the elements of order p of $F^*(G)$, and such that $\overline{G} \leq \widetilde{G}$.

We may apply the already mentioned classification theorem by Hering [7], see also Remark XII.7.5 in Huppert and Blackburn [10], to conclude that for \tilde{G} there are the following possibilities:

- (1) There exist $h, k \in \mathbb{N}$ with m = kh and $SL(k, p^h) \leq \widetilde{G} \leq \Gamma L(k, p^h)$. Since \overline{G} is normal in \widetilde{G} , we must have $SL(k, p^h) \leq \overline{G} \leq \Gamma L(k, p^h)$. On the other hand, \overline{G} is a csc-group, so that, by Lemma 3.2, k = 2, and Lemma 4.2 allows to exclude this case.
- (2) There exists $h, k \in \mathbb{N}$ with m = kh, $\widetilde{G} \simeq Sp(k, p^h)$. Then also $\overline{G} \simeq Sp(k, p^h)$, hence, by Lemma 3.2, k = 2, and Lemma 4.2 allows to exclude this case.
- (3) We have p = 2, m = 6h, and $\tilde{G} \simeq G_2(2^h)$. Then also $\overline{G} \simeq G_2(2^h)$, but, by Lemmas 3.2, 3.3, $G_2(2^h)$ is not a csc-group, and this case is excluded.
- (4) \tilde{G} contains a normal extraspecial subgroup of order 2^{m+1} . If m = 2, then $p \in \{3, 5, 7, 11, 23\}$ and, by Proposition 2.5 and the observations in Examples 1–4 we are in one of the cases (i)–(iii). If m > 2, then m = 4 and p = 3: this case cannot occur due to the observations in Example 5.
- (5) We have G̃^(∞) ≃ SL(2, 5), where G̃^(∞) denotes the last term of the derived series of G̃. Then also G̃^(∞) ≃ SL(2, 5) and p^m ∈ {3⁴, 11², 19², 29², 59²}. We have to exclude the case p^m = 3⁴ by Lemma 4.3, while the other possibilities give rise to one of the cases (iv)–(ix) (see Examples 6–9).
- (6) We have $\widetilde{G} \simeq \mathbb{A}_6$ and $p^m = 2^4$; this case cannot occur since \mathbb{A}_6 is not a csc-group.
- (7) We have $\widetilde{G} \simeq \mathbb{A}_7$ and $p^m = 2^4$; this case cannot occur since \mathbb{A}_7 is not a csc-group.
- (8) We have $\tilde{G} \simeq SL(2, 13)$ and $p^m = 3^6$; this case cannot occur by Lemma 4.3.
- (9) We have G̃ ≃ PSU(3, 3²) and p^m = 2⁶; this case cannot occur since PSU(3, 3²) is not a csc-group.

We prove the last statement. If G is solvable, since by hypothesis $F^*(G) = F(G)$ is a p-group, by Lemma 2.1, it follows that F(G) is a Sylow p-subgroup of G. If G is not solvable, then G is isomorphic to one of the groups in (iv)–(ix), and from a direct inspection it follows that again $F^*(G) = F(G)$ is a Sylow p-subgroup of G.

If $p \neq 2$, then one can show that $F^*(G)$ is cyclic or elementary abelian using the same arguments used in the proof of Lemma 2.2. If p = 2, then G/F(G) has odd order, so that G is solvable by Feit–Thompson; by Lemma 2.2, one concludes that F(G) has the structure stated in that lemma.

We have, therefore, proved the following proposition.

Proposition 4.5. Let G be a monolithic csc-group. Then either $F^*(G) = F(G)$ and G has the structure described in Lemma 4.4, or $F^*(G)$ is a simple or quasisimple group as described in Lemmas 3.2, 3.6 and $G/F^*(G)$ is cyclic of order coprime to the order of $F^*(G)$.

5. THE GENERAL CASE

In this section, we prove the theorem stated in the Introduction. We begin by showing that in the csc-group, at most one composition factor is non-abelian.

Lemma 5.1. Let S_1, S_2, \ldots, S_n be non-abelian simple groups, $n \ge 2$. Then the direct product $S = S_1 \times S_2 \times \cdots \times S_n$ is not isomorphic to a normal subgroup of a csc-group G.

Proof. By Feit–Thompson, there exists an involution $x_i \in S_i$ for each i = 1, ..., n. The subgroups $\langle (x_1, x_2, ..., x_{n-1}, 1) \rangle$ and $\langle (x_1, x_2, ..., x_{n-1}, x_n) \rangle$ have the same order, but centralizers of different order in S, hence they are not conjugate in G.

Lemma 5.2. Let G be a csc-group. Then at most one composition factor of G is non-abelian.

Proof. If G is solvable, then the result is clear. Let us suppose that G is nonsolvable. By Lemma 1.2, without loss of generality, we may assume $O_{\infty}(G) = \{1\}$ and that $F^*(G) = E(G)$ is a direct product of simple groups. By Lemma 5.1, $F^*(G)$ is simple and, since $C_G(F^*(G)) \leq F^*(G)$, we must have $C_G(F^*(G)) = \{1\}$. Then $G/F^*(G)$ is isomorphic to a subgroup of Out $F^*(G)$ which, by the classification of finite simple groups, is solvable.

Lemma 5.3. Let G be a csc-group with $E(G) \neq \{1\}$. Then $F^*(G) = O_{2'}(F(G)) \times E(G)$, $(|O_{2'}(F(G))|, |E(G)|) = 1$ and $O_2(F(G))$ has order 1 or 2.

Proof. It is well known that $F^*(G)$ is a central product of F(G) and E(G) (see 31.12 in Aschbacher [1]).

Let us first consider the case $Z(E(G)) = \{1\}$; then E(G) is simple, and it is the unique non-abelian composition factor of G. In this case, we clearly have $F^*(G) =$ $F(G) \times E(G)$. If $p \in \pi(F(G)) \cap \pi(E(G))$, then, taken $x \in F(G)$ and $y \in E(G)$ both of order p, we would have $C_G(x)$ nonsolvable since it contains E(G), while $C_G(y)$ is solvable, E(G) being the unique non-abelian composition factor of G. Therefore, $\langle x \rangle$ and $\langle y \rangle$ are not conjugate in G, a contradiction.

If $Z(E(G)) \neq \{1\}$ then, by Lemma 3.6, |Z(E(G))| = 2, and we conclude by considering G/Z(E(G)).

Lemma 5.4. Let G be a csc-group, and let P be a noncyclic Sylow subgroup of F(G). Then P is a (normal) Sylow subgroup of G, and P has the structure described in Lemma 2.2. Moreover, $G/PC_G(P)$ is isomorphic to a subgroup of $\Gamma(p^m)$, where $p^m = |P/\Phi(P)|$ or P is abelian, with $|P| \in \{5^2, 11^2, 19^2, 29^2, 59^2\}$ and $G/C_G(P)$ has the structure of one of the Frobenius complements described in Lemma 4.4(i)–(ix).

Proof. We argue by induction on the order of *G*. If *G* is monolithic, we conclude by Lemma 4.4 and Proposition 4.5. So let *N* be a minimal normal p'-subgroup of *G*, and let $P_1 \in \text{Syl}_p(G)$. If $\overline{G} = G/N$, then by induction, we have $\overline{P}_1 \trianglelefteq \overline{G}$ and \overline{P}_1 has the structure described in Lemma 2.2. Since $\overline{P}_1 \simeq P_1$, also P_1 has the structure described in Lemma 2.2. If $P = P_1$, we are done. We show that if we suppose that $P \neq P_1$, then we get a contradiction. We distinguish two cases:

(i) P_1 is elementary abelian. Let $x \in P_1 \setminus P$, and let y be a nontrivial element of P; clearly, the subgroups $\langle x \rangle$ and $\langle y \rangle$ have the same order and are not conjugate in G, a contradiction.

(ii) p = 2, and P_1 has the structure described in Lemma 2.2(3). If $P_1 \setminus P$ contains an involution, or if P contains an element of order 4, we may conclude by an argument similar to the one used in the previous case. So let $P = \Omega_1(P_1) = Z(P_1)$. Then P centralizes N, and the elementary abelian 2-group P_1/P acts faithfully on N; since $|P_1/P| > 2$, we cannot have N = E(G); hence $F^*(G) = F(G)$, and N is a minimal normal q-subgroup of G, N is not cyclic, where $q \in \pi(G) \setminus \{2\}$. If $Q \in Syl_q(F(G))$ then, since Q is not cyclic, from the previous case we get that Q is an elementary abelian Sylow q-subgroup of G. By induction, $\overline{G}/C_{\overline{G}}(\overline{Q})$ has the required structure, in particular the Sylow 2-subgroups of $\overline{G}/C_{\overline{G}}(\overline{Q})$ can not be elementary abelian, a contradiction.

Hence $P \in Syl_p(G)$; to prove the last statement, we may assume, up to considering $G/\Phi(P)$, that P is elementary abelian. Then $G/C_G(P)$ is a csc-group, and a p'-group permuting transitively the cyclic subgroups of P. Again we conclude by using the above mentioned classification theorem by Hering [7], and the proof of Lemma 4.4.

We remark that due to Lemmas 3.2, 3.3, 3.6, 4.4, 5.3, 5.4, the theorem stated in the Introduction is proved.

To obtain a more detailed classification of csc-groups, we shall use the following definition and the forthcoming notation.

Definition 5.5. A csc-group G is called *minimal* in one of the following cases:

(1) $F^*(G) = F(G)$ and the following conditions hold:

- (i) F(G) contains a unique noncyclic Sylow *p*-subgroup *P*;
- (ii) Every proper normal subgroup of G containing P in not a csc-group;
- (iii) $\pi(O_{p'}(G)) \subseteq \pi(G/F(G));$

(2) G = E(G);

(3) G is cyclic or metacyclic.

Let p be a prime, and let

$$p-1 = \prod_{q \in \pi(p-1)} q^{\alpha(q)}$$

be the factorization of p - 1. If $m \in \mathbb{N}^*$, we put

$$\rho_p(m) = \pi(p-1) \cap \pi((p^m - 1)/(p-1))$$

and

$$\boldsymbol{\epsilon}_p(m) = \prod_{q \in \rho_p(m)} q^{\alpha(q)}, \qquad \delta_p(m) = \frac{p^m - 1}{p - 1} \cdot \boldsymbol{\epsilon}_p(m).$$

We remark that $\epsilon_2(m) = 1$ and that, in general, $\epsilon_p(m)$ depends on $m \mod p - 1$.

Lemma 5.6. Let G be a subgroup of $A\Gamma(p^m)$ containing the Fitting subgroup P of $A\Gamma(p^m)$, and let us write G = PH with $H \leq \Gamma(p^m)$. Then G is a minimal csc-group if and only if G is a Frobenius group with complement of order $\delta_p(m)$. Moreover, if $Z = Z(\Gamma(p^m))$, then $P\hat{H}$ is a sharply 2-transitive group, where $\hat{H} = HZ$.

Proof. Let $\pi = \pi(H)$, and for a fixed $g \in P^{\sharp}$ let $C = C_H(g)$. Let $\Gamma(p^m) = \Gamma_0(p^m) \langle \alpha \rangle$ with $|\langle \alpha \rangle| = m$ and $H_0 = H \cap \Gamma_0(p^m)$.

Suppose for a contradiction that there exists $q \in \pi \setminus \pi(H_0)$. Then a Sylow q subgroup Q of H is conjugate in $\Gamma(p^m)$ to a Sylow q-subgroup of $\langle \alpha \rangle$. In particular, Q is cyclic and $C_P(Q) \neq \{1\}$; hence we may assume $Q \leq C$. Since H_0 and H/H_0 are cyclic, if T is a Hall q'-subgroup of H, then $T \leq H$ and H = TQ. Therefore, T permutes transitively the subgroups of order p of P, and since T is clearly a csc-group, PT turns out to be a csc-group, a contradiction to minimality of G.

Therefore, $\pi(H_0) = \pi$; if $C \neq \{1\}$ and if $r \in \pi(C)$, then there exists an element c of order r in C, and an element x of order r in H_0 . It follows that $\langle c \rangle$ and $\langle x \rangle$ are subgroups of order r of H, not conjugate in H, and this is a contradiction, since H is a csc-group. Hence $C = \{1\}$, and PH is a Frobenius group.

The minimality condition on G and elementary arithmetic considerations give $|H| = \delta_p(m)$ (in fact $O_{\pi}(Z) \leq H$).

We have $Z = O_{\pi}(Z) \times O_{\pi'}(Z)$ and $\hat{H} = H \times O_{\pi'}(Z)$; since *H* permutes transitively the subgroups of order *p* of *P*, we get that \hat{H} permutes transitively the elements of P^{\sharp} . Moreover, $C_{\hat{H}}(g) = C = \{1\}$, and due to the fact that $|\hat{H}| = p^m - 1$, $P\hat{H}$ is a sharply 2-transitive group.

Remark 5.7. Let G be a subgroup of $A\Gamma(p^m)$ containing the Fitting subgroup of $A\Gamma(p^m)$, and let $\pi = \pi((p^m - 1))$. If G is a csc-group, and if H is a Hall π -subgroup of G, then, from the proof of Lemma 5.6, it follows that |H| divides $p^m - 1$.

We introduce the following classes of groups:

- (i) $\mathscr{A}(p^m)$: the class of subgroups of $A\Gamma(p^m)$ considered in Lemma 5.6.
- (ii) $\mathscr{S}(2^m)$: the class of groups having a normal subgroup P of order 2^{2m} with $P' = \Omega_1(P) = Z(P) = \Phi(G)$ of order 2^m extended by a cyclic or metacyclic

3984

group of order $2^m - 1$ permuting transitively the involutions of P' and of P/P'. If $G \in \mathcal{G}(2^m)$ and P = F(G), then $G/P' \in \mathcal{A}(2^m)$.

- (iii) $\mathcal{U}(2^m)$: the class of groups having a normal subgroup P of order 2^{3m} with $P' = \Omega_1(P) = Z(P) = \Phi(G)$ of order 2^m extended by a cyclic or metacyclic group of order $2^{2m} 1$ permuting transitively the involutions of P' and of P/P'. If $G \in \mathcal{S}(2^m)$ and P = F(G), then $G/P' \in \mathcal{A}(2^{2m})$. Moreover, $|C_G(P')| = 2^m(2^m + 1)$.
- (iv) $\mathscr{B}(p^2)$: the class of Frobenius groups with elementary abelian kernel of order p^2 and complement isomorphic to SL(2, 3) with $p \in \{5, 11\}$.
- (v) $\mathscr{C}(p^2)$: the class of Frobenius groups with elementary abelian kernel of order p^2 and complement isomorphic to SL(2, 5) with $p \in \{11, 19, 29, 59\}$.

It is straightforward to verify that all groups in the above classes are minimal csc-groups, with trivial center. We shall refer to these as to (minimal) csc-groups of type $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{S}, \mathcal{U}$, and we shall say that a csc-group is of type \mathcal{E} if it is simple or quasisimple (as described in Lemmas 3.2, 3.6).

Remark 5.8. The order of a *p*-complement of a group *G* in one of the classes $\mathcal{A}(p^m)$, $\mathcal{C}(2^m)$, $\mathcal{U}(p^m)$, $\mathcal{B}(p^2)$, or $\mathcal{C}(59^2)$ is $\delta_p(m)$, and it is therefore the least possible. On the other hand, if $G \in \mathcal{C}(p^2)$ with $p \in \{11, 19, 29\}$, then a *p*-complement of *G* has order $5 \cdot \delta_{11}(2), 3 \cdot \delta_{19}(2)$, and $2 \cdot \delta_{29}(2)$, respectively.

Remark 5.9. From the groups of type \mathcal{A} , \mathcal{B} , \mathcal{C} , \mathcal{S} , \mathcal{U} , only those of type \mathcal{U} are not Frobenius groups, and only those of type \mathcal{C} are not solvable.

Remark 5.10. Each class $\mathcal{A}(p^m)$, $\mathcal{B}(p^2)$, $\mathcal{C}(p^2)$, $\mathcal{C}(2^m)$, $\mathcal{U}(2^m)$ clearly contains, up to isomorphisms, a finite number of groups. Moreover, the following classes contain just one group:

a) $\mathcal{A}(p^m)$ if $(m, p^m - 1) = 1$; b) $\mathcal{S}(2^m)$ if $(m, 2^m - 1) = 1$; c) $\mathcal{U}(2^m)$ if $(2^{2m}, m) = 1$; d) $\mathcal{B}(p^2)$; e) $\mathcal{C}(p^2)$.

Lemma 5.11. Let G be a minimal csc-group. If $G \neq E(G)$, and if P is the unique noncyclic Sylow p-subgroup of F(G), then, if $\overline{G} = G/O_{p'}(F(G))$, one of the following holds:

- F(G) is an elementary abelian p-group of order p^m, and G belongs to one of the classes A(p^m), B(p²), or C(p²);
- (2) F(G) is a non-abelian 2-group; in this case, if $|F(\overline{G})'| = 2^m$, then \overline{G} belongs to one of the classes $\mathcal{S}(2^m)$ or $\mathcal{U}(2^m)$.

Proof. The proof follows immediately from the definition of minimal csc-group and Lemmas 5.4, 5.6. \Box

Remark 5.12. It is not difficult to show that under the hypothesis and with the notation of Lemma 5.11, if $\overline{G} \in \mathcal{B}(p^2)$, then $O_{p'}(G)$ is a (possibly trivial) 3-group, and if $\overline{G} \in \mathcal{C}(p^2)$, then $O_{p'}(G) = \{1\}$.

Lemma 5.13. Let G be a csc-group with $E(G) \neq \{1\}$. Then E(G) is a Hall subgroup of G and a minimal csc-group. Moreover, there exists in G a complement R of E(G) which is a solvable csc-group (and such that (|E(G)|, |R|) = 1).

Proof. We prove that E(G) is a Hall subgroup of G by induction on the order of G/E(G).

By Lemma 5.3, $F^*(G) = O_{2'}(F(G)) \times E(G)$. If $O_{2'}(F(G)) \neq \{1\}$, we conclude by the induction hypothesis applied to $G/O_{2'}(F(G))$. On the other hand, if $O_{2'}(F(G)) = \{1\}$, then $F^*(G) = E(G)$ is simple (we may in fact without loss of generality assume $Z(E(G)) = \{1\}$ by considering G/Z(E(G))), and we conclude by Lemma 3.3.

The existence of *R* follows by the Schur–Zassenhaus theorem. Morever, *R* is solvable by Lemma 5.2, and a csc-group since $R \simeq G/E(G)$.

Lemma 5.14. Let G be a csc-group, and let P be a noncyclic Sylow p-subgroup of F(G). Then there exists a Hall subgroup H of G containing P, such that H is a minimal csc-group. Any subgroup of G with the same properties is conjugate to H in G.

Proof. We make induction on the order of G. If $E(G) \neq \{1\}$ then, by Lemma 5.13, in G there exists a complement R of E(G). Applying to R the inductive hypothesis, and observing that all complements of E(G) are conjugate in G, we conclude.

We may, therefore, assume $F^*(G) = F(G)$ and write $F(G) = P \times T$. If $T = \{1\}$, then *G* is a monolithic csc-group, and we are done by Lemma 4.4. So let $T \neq \{1\}$, and put $\overline{G} = G/T$. Then $\overline{P} \in \text{Syl}_p(F(\overline{G}))$ (in fact $P \in \text{Syl}_p(G)$ by Lemma 5.4). By the inductive hypothesis applied to \overline{G} , we get that \overline{P} is contained in a Hall subgroup \overline{H} of \overline{G} , such that \overline{H} is a csc-group, and every pair of such subgroups are conjugate in \overline{G} . Let *K* be the preimage of \overline{H} in *G*, and let $\pi = \pi(\overline{H})$. Since $O_{\pi'}(T) \leq K$ and $(|O_{\pi'}(T)|, |K/O_{\pi'}(T)|) = 1$, we may apply the Schur–Zassenhaus theorem to conclude that *K* contains a Hall π -subgroup *H* and that every Hall π -subgroup of *K* is conjugate to *H*. Since $P \leq G$ and $p \in \pi$, clearly $P \leq H$; but \overline{H} is a Hall π -subgroup of \overline{G} , so that *H* is a Hall π -subgroup of *G*. Moreover, by what have been said above, any Hall π -subgroup of *G* is conjugate to *H*.

If $q \in \pi \setminus \{p\}$, and if $Q \in Syl_q(F(H))$ then, since q divides the order of H/T, Q must be cyclic. Therefore, P is the unique noncyclic Sylow subgroup of F(H), and by construction H is a minimal csc-group.

Definition 5.15. If G is a csc-group, we denote by $\pi_{csc}(G)$ (or simply by π_{csc}) the set of primes $p \in \pi(G)$ such that the Sylow p-subgroup of F(G) is not cyclic. We denote by $H_p(G)$ (or simply by H_p) one of the Hall subgroups of G described in Lemma 5.14.

Lemma 5.16. Let G be a csc-group, and let $p \in \pi_{csc}(G)$. If $P \in Syl_p(F(G))$, then $C_G(P)H_p$ is normal in G.

Proof. We argue by induction on the order of G. We may assume, up to considering $G/\Phi(P)$, that P is elementary abelian, of order p^m say. Let $\overline{G} = G/C_G(P)$, and let \overline{H}_p be

the image of H_p in \overline{G} . It is enough to show that $\overline{H}_p \leq \overline{G}$. We distinguish three cases:

- (i) $H_p/O_{p'}(H_p) \in \mathcal{A}(p^m)$. In this case, \overline{G} is isomorphic to a subgroup of $\Gamma(p^m)$, and since \overline{H}_p contains a Hall $\pi(\frac{p^m-1}{p-1})$ -subgroup of \overline{G} , we easily obtain $\overline{H}_p \leq \overline{G}$.
- (ii) $H_p/O_{p'}(H_p) \in \mathcal{B}(p^2)$. In this case, $\overline{H}_p \simeq SL(2,3)$ is a direct factor of \overline{G} .
- (iii) $H_p/O_{p'}(H_p) \in \mathcal{C}(p^2)$. Also in this case, $H_p \simeq SL(2, 5)$ is a direct factor of G. \Box

Lemma 5.17. Let G be a csc-group, and let $p, q \in \pi_{csc}$ with $p \neq q$. Then $(|\mathsf{H}_p|, |\mathsf{H}_q|) = 1$, and $\mathsf{H}_p\mathsf{H}_q = \mathsf{H}_q\mathsf{H}_p$ is a Hall subgroup of G.

Proof. We argue by induction on the order of *G*. Let $P \in Syl_p(F(G))$ and $Q \in Syl_q(F(G))$, by Lemma 5.4, $P \in Syl_p(G)$ and $Q \in Syl_q(G)$. Let *R* be a Hall $\{p, q\}'$ -subgroup of $F^*(G)$ (see Lemma 5.3); we have $[P, R] = \{1\}$ and $[Q, R] = \{1\}$, and if $R \neq \{1\}$, we conclude by considering G/R.

Therefore, we may assume $F^*(G) = P \times Q$ and that P and Q are elementary abelian (otherwise, we conclude by considering $G/\Phi(P)$ or $G/\Phi(Q)$). Then we have $C_{H_p}(P) = P$ and $C_{H_q}(Q) = Q$, so that H_p and H_q are monolithic csc-groups. We observe that if $|P| = p^m$ and $|Q| = q^n$, then the number of cyclic subgroups of order pq of $P \times Q$ is $v = \frac{p^m-1}{p-1} \cdot \frac{q^n-1}{q-1}$.

Let us first consider the case when $H_p \in \mathcal{A}(p^m)$ and $H_q \in \mathcal{A}(q^n)$. By Lemma 5.4, $G/C_G(P)$ is isomorphic to a subgroup of $\Gamma(p^m)$ and $H_pC_G(P) \simeq H_p/P$ is a Hall subgroup of $G/C_G(P)$ which is normal by Lemma 5.16. Therefore, $|G/C_G(P)| = r \cdot |H_p/P|$ and, by Lemma 4.4 and Remark 5.7, r divides (p-1)m (and clearly $(|H_p|, r) = 1$); we may write $r = r_1r_2$ with $\pi(r_1) \subseteq \pi(|H_p|)$ and $(r_1, r_2) = 1$. Similarly, we get $|G/C_G(Q)| = s \cdot |H_q/Q|$ and $s = s_1s_2$ with $\pi(s_1) \subseteq \pi(|H_q|)$ and $(s_1, s_2) = 1$. Hence $|G/F(G)| = r_2 \cdot s_2 \cdot |H_p| \cdot |H_q|$. Moreover, it is easy to show that $G/H_pC_G(P)$ is cyclic, so that there is a (normal) subgroup N_1 in G of index r_2 containing H_p , H_q , and $C_G(P)$; since G permutes transitively the cyclic subgroups of order pq of $P \times Q$ and $(r_2, v) = 1$, also N_1 has this property. Similarly, there exists a (normal) subgroup N_2 in G of index s_2 containing H_p , H_q , and $C_G(Q)$, and satisfying the same property. If we put $N = N_1 \cap N_2$, then $F(G) \leq N$ and $|G/F(G)| = |H_p/P| \cdot |H_q/Q|$, so that $|G| = |H_p| \cdot |H_q|$. The fact that $|H_p/P| = \epsilon_p(m) \cdot \frac{p^m-1}{p-1}$ and $|H_q/Q| = \epsilon_q(n) \cdot \frac{q^{n-1}}{q-1}$ shows that $\pi(|H_p/P|) \cap \pi(|H_q/Q|) = \emptyset$, and therefore, $\pi(|H_p|) \cap \pi(|H_q|) = \emptyset$.

If one of H_p and H_q belongs to one of the classes \mathscr{B} or \mathscr{C} , then it is clear that the other must lie in the class \mathscr{A} and, arguing as before (and taking into account Remark 5.8), we conclude that $\pi(|H_p|) \cap \pi(|H_q|) = \emptyset$.

Due to the fact that $|N| = |H_p| \cdot |H_q|$, it follows that $N = H_p H_q$, and then $H_p H_q$ is a (Hall) subgroup of G.

Let G be a csc-group. If $\pi_{csc}(G) = \{p_1, p_2, \dots, p_t\}$, then we denote by H(G) or simply by H the subgroup $H_{p_1}H_{p_2}\cdots H_{p_t}$ of G. Clearly, H(G) is a Hall subgroup of G, and it is easy to check that it is a csc-group.

Lemma 5.18. Let G be a csc-group, and let H = H(G). Then there exists a (possibly trivial) Hall subgroup H_0 of G such that:

(i) $(|\mathsf{H}|, |\mathsf{H}_0|) = 1$ and $(|E(G)|, |\mathsf{H}_0|) = 1;$ (ii) $G = \mathsf{H}_0\mathsf{H}E(G);$

COSTANTINI AND JABARA

(iii) The Sylow subgroups of H₀ are cyclic; in particular, H₀ is a cyclic or metacyclic csc-group.

Proof. To prove the statement, it is enough to show that if we put $\rho = \pi(|\mathsf{H}|) \cup \pi(|E(G)|)$, then there exists a Hall ρ' -subgroup in G. We argue by induction on the order of G.

If $E(G) \neq \{1\}$, then, by Lemma 5.13, E(G) is a Hall subgroup of G, and we may write G = E(G)R with $E(G) \cap R = \{1\}$: then we conclude by considering R. We may, therefore, assume $E(G) = \{1\}$ (and $F^*(G) = F(G)$). Let $\sigma = \pi(|F(G)|) \setminus \pi(|\mathsf{H}|)$, and let T be a Hall σ' -subgroup of F(G). If $T = \{1\}$, then the Sylow subgroups of F(G) are cyclic, hence, since G is a csc-group, every Sylow subgroup of G is cyclic, and we are done.

Otherwise, by the inductive hypothesis applied to $\overline{G} = G/T$, there exists a Hall subgroup \overline{H}_0 in \overline{G} with the required properties.

If $\overline{G} = \overline{H}_0$, then G is solvable and certainly in G there is a Hall ρ' -subgroup. Otherwise, let H be the preimage of \overline{H}_0 in G. Then H is solvable, and therefore, H has a Hall ρ' -subgroup H_0 ; we conclude by observing that $\rho' \subseteq \pi(|H|)$, so that H_0 is a Hall ρ' -subgroup of G.

To prove (iii), we observe that a group with cyclic Sylow subgroups is cyclic or metacyclic (see 10.1.10 in Robinson [15]) and that such a group is csc-group. \Box

From the previous lemmas, we obtain the following characterization of csc-groups.

Proposition 5.19. Let G be a csc-group. Then G is the product of its Hall minimal csc-subgroups. Moreover, among these factors, at most one is nonsolvable, and at most one is cyclic or metacyclic. \Box

Remark 5.20. In fact, at most one among the factors lies in one of the classes $\mathfrak{B}, \mathfrak{C}, \mathfrak{S}$, and \mathfrak{U} , since the groups in $\mathfrak{B}, \mathfrak{C}, \mathfrak{S}$, and \mathfrak{U} are of even order, and the components have pairwise coprime order by Lemma 5.18. It also follows from Proposition 5.19 that if G is a csc-group, then the Sylow subgroups of $G/F^*(G)$ are cyclic or quaternions.

We conclude with a series of examples.

Example 10. Let A be an elementary abelian group of order 4 and $B \in \mathcal{A}(5^3)$ (B is a Frobenius group with elementary abelian kernel of order 5^3 and complement of order 31). Let $\langle x \rangle$ be a cyclic group of order 3, and let x act on A in such a way that $A\langle x \rangle \in \mathcal{A}(2^2)$, and on B so that $B\langle x \rangle$ is (isomorphic) to a subgroup of $A\Gamma(5^3)$. We have $H_2(G) = A\langle x \rangle$ and $H_5(G) = B$; note that $H_2(G)$ is not normal in G but $C_G(A)H_2(G) \trianglelefteq G$.

Example 11. Let A be a Frobenius group with elementary abelian kernel of order 5^2 and complement isomorphic to quaternions. Let B be a Frobenius group with elementary abelian kernel of order 11^3 and complement cyclic of order $7 \cdot 19$. If $\langle x \rangle$ is of order 3^n , with $n \ge 1$, we can make x act on A so that $A\langle x \rangle / \langle x^3 \rangle \in \mathcal{B}(5^2)$. We make x act on B so that $B\langle x \rangle / \langle x^3 \rangle$ is isomorphic to a subgroup of $A\Gamma(11^3)$,

and we consider the semidirect product $G_1 = (A \times B)\langle x \rangle$. It easily follows that G_1 is a csc-group, $Z(G_1) = \langle x^3 \rangle$, $H_5(G_1) = A\langle x \rangle$, and $H_{11}(G_1) = B$; note that if $n \ge 2$, then $(|F(G)|, |G/F(G)|) = 3 \ne 1$.

If we let x act trivially on B (keeping the same action of x on A as above), then we may construct another group $G_2 = (A \times B)\langle x \rangle$. Obviously, $|G_1| = |G_2|$, but $G_1 \not\simeq G_2$.

Example 12. We give an example of a group which is not a csc-group. Let A be elementary abelian of order 8, B a Frobenius group with elementary abelian kernel of order 11³ and complement of order 19, and let $\langle x \rangle$ be of order 7. We make x act on A so that $A\langle x \rangle \in \mathcal{A}(2^3)$ and on B so that $B\langle x \rangle \in \mathcal{A}(11^3)$. Then $G = (A \times B)\langle x \rangle$ permutes transitively the cyclic subgroups of order 2 of $O_2(G)$ and the cyclic subgroups of order 11 of $O_{11}(G)$. However, G is not a csc-group, since it does not permute transitively the $7^2 \cdot 19$ cyclic subgroups of order 22 of F(G).

Example 13. Let $A \in \mathcal{C}(29^2)$, and let *B* be a Frobenius group with elementary abelian kernel of order 11³ and complement of order 19. If $\langle x \rangle$ is of order 7, we can make *x* act on *A* so that $A\langle x \rangle$ is a Frobenius group, and on *B* so that $B\langle x \rangle \in \mathcal{A}(11^3)$. Then $G = (A \times B)\langle x \rangle$ is a nonsolvable csc-group. We have $F^*(G) = F(G)$, $H_{29}(G) = A$, and $H_{11}(G) = B\langle x \rangle$.

Example 14. Let $A \simeq Sz(8)$, and let $B \in \mathcal{A}(11^3)$. Let $\langle x \rangle$ be of order 3, and let x act on A as a (field) automorphism and on B so that $B\langle x \rangle$ is (isomorphic to) a subgroup of $A\Gamma(11^3)$. Then $G = (A \times B)\langle x \rangle$ is a nonsolvable csc-group. We have E(G) = A, $F^*(G) = E(G) \times O_{11}(G)$; moreover, $H_{11}(G) = B$ and $H_0(G) = \langle x \rangle$.

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