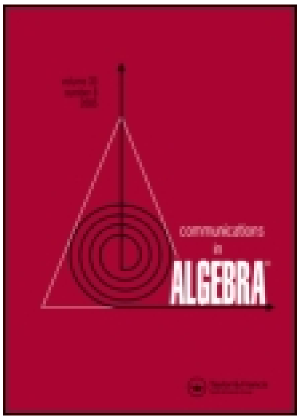

Communications in Algebra
Publication details, including instructions for authors and subscription information:
http://www.tandfonline.com/loi/lagb20

On the Quantum Function Algebra at Roots of One

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Available online: 18 Aug 2006

COMMUNICATIONS IN ALGEBRA®
Vol. 32, No. 6, pp. 2377–2383, 2004

On the Quantum Function Algebra at Roots of One[#]

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ABSTRACT

In this note we show that certain properties of the quantum function algebra at roots of unity hold over any algebraically closed field of characteristic zero.

Key Words: Quantum function algebra; Roots of one; Representations.

2000 Mathematics Subject Classification: 20F55; 20E15.

1. INTRODUCTION

In recent years, many investigations have been devoted to both generic quantum function algebras and quantum function algebras at roots of unity. Both have been studied over fields of characteristic zero but, while in the generic case usually one works over arbitrary algebraically closed fields, in the root of unity case results are mainly obtained over the complex field. One can find results for both cases, as well as a lot of references, in Brown and Goodearl (2002). In this note we deal with the root of unity case: we are mainly interested in the structure and representation

[#]Communicated by T. Lenagan.
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DOI: 10.1081/AGB-120037227
0092-7872 (Print); 1532-4125 (Online)

theory of the quantum function algebra $F_\varepsilon[G]$, as studied in De Concini and Lyubashenko (1994), and De Concini and Procesi (1997) and more generally of the multiparameter quantum function algebra $F_\varepsilon^\varphi[G]$, as studied in Costantini and Varagnolo (1994, 1996a,b) and Costantini (1998). In this paper we show that all results proved for $F_\varepsilon^\varphi[G]$ over $\mathbb{C}$ are valid over any algebraically closed field of characteristic zero. This result is essentially a consequence of Elimination Theory, which in turn can be thought as a variant of Hilbert's Nullstellensatz: we introduce the appropriate formalism and give a direct proof of the statement.

For notation we refer to Costantini (1998). For our purposes it is convenient to use the language of algebraic k-groups, following, for instance, Jantzen (1987). Let $A = (a_{ij})$ be a finite indecomposable Cartan matrix of rank n. To A there is associated a root system Φ. We put $k = \mathbb{Q}$, and we denote by G the connected simple simply connected k-group corresponding to Φ. We fix a maximal torus T of G, and a Borel subgroup B containing T. We denote by $\mathfrak{h}$ the Lie algebra of T: then Φ is the set of roots relative to $\mathfrak{h}$, and B determines the set of positive roots Φ^+. We denote by P the weight lattice and by W the Weyl group. We shall use the capital letters K, L, M to denote algebraically closed fields of characteristic zero. If A, B are algebras over a field F, $A \underset{F}{\cong} B$ means that A, B are isomorphic F-algebras.

2. PRELIMINARIES

In this section, we introduce the object of our investigation. Having fixed Φ, let L be any algebraically closed field of characteristic zero and fix a primitive ℓth root of unity ε in L (for the restrictions on ℓ see below). The construction of $F_\varepsilon[G(\mathbb{Q}(\varepsilon))]$ given in De Concini and Lyubashenko (1994) over the cyclotomic field $\mathbb{Q}(\varepsilon)$ (see also Lusztig, 1990), can be used to define the quantum function algebra $F_\varepsilon[G(L)]$ over L (in fact here L could be any overfield of $\mathbb{Q}(\varepsilon)$, or even any $\mathbb{Q}(\varepsilon)$-algebra). At this point, then one may consider the multiparameter version $F_\varepsilon^\varphi[G(L)]$ of $F_\varepsilon[G(L)]$ by just applying the procedure of Costantini and Varagnolo (1996a) to L: here the multiparameter φ is a certain endomorphism of P. We note that the first tool in this construction is the perfect pairing as introduced in Costantini and Varagnolo (1994), which is defined over arbitrary fields. Then one can proceed in the definition of $F_\varepsilon^\varphi[G(L)]$ and show that it contains a central Hopf-subalgebra $F_0(L)$ which is isomorphic to the affine algebra $L[G(L)]$. Moreover, $F_\varepsilon^\varphi[G(L)]$ is a finitely generated projective (in fact free, by Brown et al., 2000) module over $F_0(L)$. We observe that $F_0(L)$ is a UFD by Marlin (1976): from this one can prove that $F_\varepsilon^\varphi[G(L)]$ is a maximal order in its quotient division algebra (see Theorem 7.4 and its proof in De Concini and Lyubashenko, 1994, and Theorem 3.8 in Costantini and Varagnolo, 1996a). Therefore all the theory developed in De Concini and Procesi (1993) can be applied. We remark that we are assuming that ℓ is an odd positive integer, not divisible by 3 if Φ is of type G_2, and coprime to $\det(1-\varphi)$. We also assume that ℓ is a *φ-good* integer, that is ℓ is also coprime with a certain integer $\ell(\varphi)$, which depends only on Φ and φ (for the precise definition, see Costantini, 1998, Sec. 2). In particular, if $\varphi = 0$, one may assume ℓ to be a good integer.

The crucial part in our investigation is the following. The main obstruction in trying to adjust the arguments used over $\mathbb{C}$ is when one switches from polynomial

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functions on $G(\mathbb{C})$ to complex valued smooth functions on $G(\mathbb{C})$. This is done to show, using transcendal methods, that the representation theory of $F_\varepsilon^\varphi[G(\mathbb{C})]$ is constant over the symplectic leaves of $G(\mathbb{C})$ coming from quantization (De Concini and Lyubashenko, 1994, Sec. 9). This is a key step in the proof that the representation theory of $F_\varepsilon^\varphi[G(\mathbb{C})]$ is constant over the strata $X_{w_1,w_2}(\mathbb{C}) = B(\mathbb{C})^- n_{w_1} B(\mathbb{C})^- \cap B(\mathbb{C}) n_{w_2} B(\mathbb{C})$, w_1, $w_2 \in W$.

In fact, the right and left winding automorphisms corresponding to $T(\mathbb{C})$, can be defined over L, and one may check that the results obtained about the center and the representation theory of $F_\varepsilon^\varphi[G(\mathbb{C})]$ (Costantini, 1998, Theorem 2.8, Theorem 3.5 and Corollary 3.6) can be obtained step by step for $F_\varepsilon^\varphi[G(L)]$, once one knows that the representation theory of $F_\varepsilon^\varphi[G(L)]$ is constant over $X_{w_1,w_2}(L) = B(L)^- n_{w_1} B(L)^- \cap B(L) n_{w_2} B(L)$.

Our aim is to show that this last condition holds, and the method we use is comparison with the complex case.

3. THE MAIN RESULT

Recall we are assuming G to be an algebraic k-group. This means that for every k-algebra R we have

$$G(R) = \mathrm{Hom}_k(F_0, R)$$

where F_0 is the affine k-algebra of G. Moreover, the affine algebra $L[G(L)]$ of $G(L)$ is $L \otimes_k F_0$. All root subgroups X_α, for $\alpha \in \Phi$, are defined over k, and so are B, T and the opposite Borel subgroup B^-. Finally, for each $w \in W$, we may choose a representative n_w of w in $G(k)$. From the Bruhat decomposition of $G(L)$ we get

$$G(L) = \bigcup_{(w_1,w_2)\in W\times W} X_{w_1,w_2}(L),$$

where

$$X_{w_1,w_2}(L) = B(L) n_{w_1} B(L) \cap B^-(L) n_{w_2} B^-(L)$$

for every w_1, $w_2 \in W$. It is clear that

$$X_{w_1,w_2}(L) \subseteq X_{w_1,w_2}(M) \quad \text{if } L \leq M.$$

Each $X_{w_1,w_2}(L)$ is an affine L-variety of dimension $n + \ell(w_1) + \ell(w_2)$, where ℓ is the length function on W.

Lemma 3.1. *Let K be any (algebraically closed) subfield of L with finite transcendency degree over k. Then there exist a subfield K' of $\mathbb{C}$ and a field isomorphism $\psi : K \to K'$, which is the identity on $\overline{\mathbb{Q}}$ and such that, under the group isomorphism $G(\psi) : G(K) \to G(K')$ induced by ψ, $X_{w_1,w_2}(K)$ is mapped onto $X_{w_1,w_2}(K')$.*

Proof. Let $(t_1, \ldots, t_m)$ be a transcendency basis of K over $\mathbb{Q}$. Then K is the algebraic closure of $\mathbb{Q}(t_1, \ldots, t_m)$ in L. Let us choose elements $z_1, \ldots, z_m$ in $\mathbb{C}$ algebraically

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independent over $\mathbb{Q}$, and consider the field isomorphism $\psi : \overline{\mathbb{Q}}(t_1, \ldots, t_m) \to \overline{\mathbb{Q}}(z_1, \ldots, z_m)$, which is the identity on $\overline{\mathbb{Q}}$, and maps $t_i \mapsto z_i$ for $i = 1, \ldots, m$. Since $K/\overline{\mathbb{Q}}(t_1, \ldots, t_m)$ is an algebraic extension, ψ extends to a field homomorphism $\psi : K \to \mathbb{C}$. We take $K' = \operatorname{Im}\psi$, and denote by ψ the isomorphism of K onto K' induced by ψ. Let $f = G(\psi) : G(K) \to G(K')$ be the group isomorphism induced by ψ, which is the unique group homomorphism such that $x_\alpha(u) \mapsto x_\alpha(\psi(u))$ for every $\alpha \in \Phi$, $u \in K$. To conclude, we have to show that $f(X_{w_1,w_2}(K)) = X_{w_1,w_2}(K')$. By the definition of f, we have $f(B(K)) = B(K')$, $f(B^-(K)) = B^-(K')$ and, since $n_w \in G(k)$, $f(n_w) = n_w$. Hence

$$\begin{aligned} f(X_{w_1,w_2}(K)) &= f(B(K)n_{w_1}B(K)) \cap f(B^-(K)n_{w_2}B^-(K)) \\ &= B(K')n_{w_1}B(K') \cap B^-(K')n_{w_2}B^-(K') = X_{w_1,w_2}(K') \end{aligned}$$

and we are done. □

We now come back to our object of investigation $F_\varepsilon^\varphi[G(L)]$ which, for convenience, we shall denote by $A(L)$. From the construction of $A(L)$ it follows that $A(L) = L \otimes_{\overline{\mathbb{Q}}} A(\overline{\mathbb{Q}})$. If $\alpha : L \to M$ is a $\overline{\mathbb{Q}}$-algebra map, then we get $A(\alpha) : A(L) \to A(M)$, and its restriction $F_0(\alpha) := A(\alpha)_{|F_0(L)} : F_0(L) \to F_0(M)$. In particular, if α is a field isomomorphism (inducing the identity on $\overline{\mathbb{Q}}$), then $A(\alpha)$ is an $F_0(\alpha)$-semilinear isomorphism from the $F_0(L)$-module $A(L)$ onto the $F_0(M)$-module $A(M)$.

For every $g \in G(L)$ we consider the ideal m_g of $F_0(L)$ corresponding to g, and we denote by $A(L)_g$ the finite dimensional L-algebra $A(L)/m_gA(L)$. Our aim is to show that $A(L)_g \underset{L}{\cong} A(L)_h$ if g, $h \in X_{w_1,w_2}(L)$.

Assume E/F is a field extension, A a finite dimensional F-algebra, $A_E := E \otimes_F A$ the E-algebra obtained by scalar extension. We shall make use of the following

Lemma 3.2. *Let E/L be a field extension and A, B be finite dimensional L-algebras. Then*

(1) $A \underset{L}{\cong} B \Longrightarrow A_E \underset{E}{\cong} B_E$.
(2) $\operatorname{rad} A_E = E \otimes_L \operatorname{rad} A$.
(3) $A_E/\operatorname{rad} A_E = (A/\operatorname{rad} A)_E$.
(4) $A_E \underset{E}{\cong} B_E \Longrightarrow A \underset{L}{\cong} B$.

Proof. The first three assertions are standard (see Curtis and Reiner, 1962, Corollary (29.22) or Lam, 1991, Ex. 2, p. 123), since L, being algebraically closed, is a splitting field for both A and B. The last one is a consequence of Hilbert's Nullstellensatz. More precisely, assume $A_E \underset{E}{\cong} B_E$, and let $\alpha : A_E \to B_E$ be an E-isomorphism of algebras, with inverse β. Choose L-bases $\mathscr{A} = (a_1, \ldots, a_r)$, $\mathscr{B} = (b_1, \ldots, b_r)$ of A, B, respectively, and let X, Y be the matrices of α, β with respect to the bases $1 \otimes_L \mathscr{A}$, $1 \otimes_L \mathscr{B}$, respectively. This means that it is possible to solve over E a system of equations in the entries of X and Y with coefficients in L, equations coming from preservation of algebra structure constants and from $XY = 1_r$. From the fact that L is algebraically closed, it follows that the same system of equations has a solution over L: this implies that A, B are isomorphic L-algebras. □

We can finally prove

Proposition 3.3. *For any algebraically closed field L of characteristic zero, the representation theory of $F_\varepsilon^\varphi[G(L)]$ is constant over the stratum $X_{w_1,w_2}(L)$, w_1, w_2 in W.*

Proof. Consider two elements $g,\ h \in X_{w_1,w_2}(L)$. Let F be any finitely generated subfield of L such that $g,\ h \in G(F)$. Let K be the algebraic closure of F in L: then $g,\ h \in X_{w_1,w_2}(K)$. Consider the isomorphism $\psi : K \to K'$ from Lemma 3.1, and the group isomorphism $f := G(\psi) : G(K) \to G(K')$. We have to show that $A(L)_g \underset{L}{\cong} A(L)_h$. By Lemma 3.2 (1), it is enough to show that $A(K)_g \underset{K}{\cong} A(K)_h$ and, via ψ and f, this is equivalent to $A(K')_{f(g)} \underset{K'}{\cong} A(K')_{f(h)}$. By Lemma 3.1, both $f(g)$ and $f(h)$ lie in $X_{w_1,w_2}(K')$, which is contained in $X_{w_1,w_2}(\mathbb{C})$. But over $\mathbb{C}$ we know that $A(\mathbb{C})_{f(g)} \underset{\mathbb{C}}{\cong} A(\mathbb{C})_{f(h)}$. Hence, by Lemma 3.2 (4), we conclude that $A(K')_{f(g)} \underset{K'}{\cong} A(K')_{f(h)}$, and we are done. □

We summarize certain properties that now can be proved for $F_\varepsilon^\varphi[G(L)]$. First, one has to define the various objects over L, following, e.g., the lines of Costantini (1998). Here we just recall that for every $(w_1, w_2) \in W \times W$, the L-algebra $F_\varepsilon^\varphi[G(L)]_{w_1,w_2}$ is a certain localization of a suitable quotient of $F_\varepsilon^\varphi[G(L)]$.

Theorem 3.4. *Let ℓ be a φ-good integer, (w_1, w_2) be in $W \times W$. Then*

(a) *$F_\varepsilon^\varphi[G(L)]_{w_1,w_2}$ is an Azumaya algebra of degree $\ell^{\frac{1}{2}(\ell(w_1)+\ell(w_2)+\mathrm{rk}(\delta^\varphi_{w_1,w_2}))}$.*

(b) *The center $Z^\varphi_{w_1,w_2}(L)$ of $F_\varepsilon^\varphi[G(L)]_{w_1,w_2}$ is the $L[X_{w_1,w_2}(L)]$-algebra generated by $d^\varphi_{\lambda,w_1,w_2}$ for λ in $P^\varphi_{w_1,w_2}$; moreover, if we put $Z^\varphi_{1,w_1,w_2}(L) = L\left[d^\varphi_{\lambda,w_1,w_2} \,|\, \lambda \in P^\varphi_{w_1,w_2}\right]$, then $L\left[X_{w_1,w_2}(L)\right] \cap Z^\varphi_{1,w_1,w_2}(L) = L\left[d^\varphi_{\ell\lambda,w_1,w_2} \,|\, \lambda \in P^\varphi_{w_1,w_2}\right]$ and*

$$Z^\varphi_{w_1,w_2}(L) = L[X_{w_1,w_2}(L)] \otimes_{L\left[d^\varphi_{\ell\lambda,w_1,w_2} | \lambda \in P^\varphi_{w_1,w_2}\right]} Z^\varphi_{1,w_1,w_2}(L).$$

(c) *The spectrum of $F_\varepsilon^\varphi[G(L)]_{w_1,w_2}$ is a Galois covering of $X_{w_1,w_2}(L)$ with Galois group $\{x \in T^\varphi_{w_1,w_2}(L) \,|\, x^\ell = 1\}$.*

In particular

Corollary 3.5. *Let ℓ be a φ-good integer, $g \in X_{w_1,w_2}(L)$. Then*

(a) *Every simple $F_\varepsilon^\varphi[G(L)]_g$-module has L-dimension $\ell^{\frac{1}{2}(\ell(w_1)+\ell(w_2)+\mathrm{rk}(\delta^\varphi_{w_1,w_2}))}$.*

(b) *The number of simple $F_\varepsilon^\varphi[G(L)]$-modules laying over g is $\ell^{n-\mathrm{rk}(\delta^\varphi_{w_1,w_2})}$.*

Finally, one can construct almost all simple $F_\varepsilon^\varphi[G(L)]$-modules following the construction of Costantini (1998, Sec. 3).

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Remark 3.1. Since for our purposes it is enough to deal with simple $A(L)$-modules, one could alternatively show, using Lemma 3.2 (2), (3), that if $g \in X_{w_1,w_2}(L)$, and if h is any element of $X_{w_1,w_2}(\mathbb{C})$, then

$$A(L)_g/\operatorname{rad} A(L)_g \cong \bigoplus_{i \in I} M_{n_i}(L),$$

where the index set I and the n_i's are determined by $A(\mathbb{C})_h/\operatorname{rad} A(\mathbb{C})_h \cong \bigoplus_{i \in I} M_{n_i}(\mathbb{C})$.

Remark 3.2. The arguments we used are based on a feature of the quantum function algebra at roots of one, namely there is an algebraic stratification of the spectrum of a certain central subalgebra such that over each stratum the fibres are isomorphic and each stratum is defined over $\overline{\mathbb{Q}}$: statements on the lines of those we proved for $F_\varepsilon^\varphi[G]$ could be proved for other classes of algebras with the same feature.

ACKNOWLEDGMENT

I am grateful to C. Procesi for helpful suggestions.

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Received December 2002
Accepted March 2003