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K-GROUPS AND WREATH PRODUCTS

To Guido Zappa on the occasion of his 90th birthday

ABSTRACT. — We give criteria for a wreath product to have complemented subgroup-lattice.

KEY WORDS: Complemented group; Wreath product; Subgroup lattice.

A group G is called a K-group if its subgroup lattice $\ell(G)$ is a complemented lattice. For basic information concerning K-groups we refer the reader to [9, §3 n. 1]. While finite simple groups are K-groups, [2], and while the structure of solvable K-groups is well understood [11], a characterization of all finite K-groups is still missing. The purpose of the present paper is to give a contribution in this direction, by establishing several criteria for a wreath product G of L by H, $G = L \wr H$, to be a K-group.

The paper is divided in 4 sections. Section 1 contains preliminaries of general nature and shows how to reduce the classification of K-groups to the case where the solvable radical $S(G)$ of G is trivial. In section 2, we give relevant structural information on the H-invariant subgroups of the interval $[B/\Delta B]$, where B is the base group of G and ΔB is the diagonal subgroup of B: Propositions 2.3 and 2.4 are central for our applications. In section 3 and 4 are presented several criteria which guarantee that a wreath product is a K-group; particular relevance in this regard have Theorems 3.2, 3.5, 3.8 and 4.2.

The notation is mainly standard; in case of special symbols, we shall define them when first needed in the course of exposition. We emphasize that throughout the paper H stands for a transitive permutation group of degree $n \geq 2$ on a set Ω whose elements are either the digits $1, 2, \ldots, n$ or the right cosets $H_i h$ with H_i the stabilizer of i and $h \in H$. All groups are meant to be finite.

1. We recall that in a K-group G, an interval $[E/D]$ is a complemented lattice as soon as D is a Dedekind subgroup ([9], 2.1) and E is a dual-Dedekind subgroup ([9], 2.4) of G; moreover if $A \leq B \leq G$ with A normal in G and B subnormal in G, then the Frattini subgroup $\Phi(B/A)$ is trivial. In particular the generalized Fitting subgroup $F^*(G)$ is a direct product of simple groups. Useful in this context is the well known statement:

(1.1) *Let A be a nilpotent subnormal subgroup of a group G. Then $A^G\Phi(G)/\Phi(G)$ is a direct product of minimal normal subgroups of $G/\Phi(G)$ and has a complement in $G/\Phi(G)$* [4, 12].

Proposition 1.1. *Let S be a solvable subnormal subgroup of G. Then G is a K-group if and only if G/S^G is a K-group and $\Phi(G/F_i(S^G)) = 1$ for all terms of the ascending Fitting series of S^G.*

Proof. The necessity is clear. Conversely, let G be a minimal counterexample. Then $S \neq 1$ and, since $\Phi(G) = 1$, by (1.1) we have $G = F(S^G) : C$, where $F(S^G)$ is a direct product of minimal normal subgroups, while $C \cong G/F(S^G)$ is a K-group. But then, by [9] 3.1.9, G itself is a K-group, a contradiction. □

Corollary 1.2. *The group G is a K-group if and only if:*
i) *$G/S(G)$ is a K-group, and*
ii) *$\Phi(G/F_i(S(G))) = 1$ for all i's.* □

As one may note, Corollary 1.2 reduces the study of K-groups essentially to the semisimple case.

Corollary 1.3. *Let G be a K-group and $N \trianglelefteq G$. Then N is a K-group if and only if $N/S(N)$ is a K-group.*

Proof. Assume $N/S(N)$ a K-group and N a minimal counterexample. Then $S(N) \neq 1$ and, since $G/F(S(N))$ is a K-group, such is $N/F(S(N))$. Moreover $\Phi(N) \leq \Phi(G) = 1$ so that, by (1.1), $N = F(S(N)) : C$, with $F(S(N))$ a direct product of minimal normal subgroups of N, while C is a K-group. But then N is a K-group by [9] 3.1.9, a contradiction. □

By Corollary 1.3, if R is subnormal in $S(G)$ and G is a K-group, then R is a K-group.

Given a non-trivial group L, let L^Ω be the group of all functions of Ω in L, group which can be identified with the direct product B of n copies of L, $B = L_1 \times \cdots \times L_n$. The position $f^h(\omega) = f(\omega^{h^{-1}})$ defines a right action of h on L^Ω. The semidirect product G of B by H defined by $(f, h)(f_1, h_1) = (g, hh_1)$, with $g = ff_1^h$, is called the *wreath product* of L by H and is denoted by $G = L \wr H = B : H$. The group $B = L^\Omega$ is called the *base group* and the subgroup ΔB of constant functions is called the *diagonal subgroup*. The group H permutes the elements of $\{L_1, \dots, L_n\}$ via conjugation according to the rule $L_i^h = L_{i^h}$. We recall that $H_G = 1$, $C_G(B) = Z(B)$, $C_G(H) = Z(H) \times \Delta B$, $N_G(H) = H \times \Delta B$, while H_i is the normalizer as well as the centralizer of L_i in H.

(1.2) *Given $G = L \wr H$, assume $S(L) \neq L$. Then $\Phi(G) \leq S(G) = S(B) \cong (S(L))^n$.*

Proof. We have $G/S(B) \cong (L/S(L)) \wr H$ and $S(G) \cap B = S(B)$; hence $S(G)/S(B) \leq$ $\leq C_{G/S(B)}(B/S(B)) \leq B/S(B)$, since $S(L) \neq L$. □

Proposition 1.4. *Given $G = L \wr H$ we have*
a) *if G is a K-group then $(L/S(L)) \wr H$, H and $S(L)$ are K-groups. L itself is a K-group if (and only if) $L/S(L)$ is a K-group.*

b) *G is a K-group if and only if* $(L/S(L)) \wr H$ *is a K-group and* $\Phi((L/F_i(S(L)) \wr H) = 1$ *for all i's.*

PROOF. *a)* Since $G/S(B) \cong (L/S(L)) \wr H$ and $H \cong G/B$, they are K-groups and, by Corollary 1.3, such is $S(B)$ and so is $S(L)$. Also, again by Corollary 1.3, B is a K-group if and only if $B/S(B)$ is a K-group, so that L is a K-group if and only if $L/S(L)$ is a K-group.

b) By Proposition 1.1, G is a K-group if and only if $G/S(B)$ is a K-group and $\Phi(G/F_i(S(B))) = 1$ for all i's, and we are done since $G/F_i(S(B)) \cong (L/F_i(S(L))) \wr H$. □

Given a group G and a subgroup X, the interval $[G/X]$ is called *monocoatomic* with *coatom* M if M is the unique maximal subgroup of G containing X. For later references we recall the following criterion established in [2].

PROPOSITION 1.5. *Let* $\{[G/X_i]\}_i$ *be a family of monocoatomic intervals with* $\{M_i\}_i$ *the family of its coatoms. Then G is a K-group if each X_i is a K-group and* $(\cap_i M_i)_{P(G)} = 1$, *$P(G)$ being the group of all autoprojectivities of G.* □

2. Given $G = L \wr H$ and a non-empty subset I of Ω, set $\Delta_I = \{(x_1, \dots, x_n) \in B \mid x_i = x_j$ for all $i,j \in I\}$. Thus Δ_I is the subgroup of B of all functions constant on I: we have $\Delta_I = \Delta(\underset{k\in I}{\times} L_k) \times \prod_{k\notin I} L_k \cong L^{|\Omega\setminus I|+1}$ and, for $h \in H$, $\Delta_I^h = \Delta_{I^h} = \Delta(\underset{k\in I^h}{\times} L_k) \times \prod_{k\notin I^h} L_k = \Delta(\underset{k\in I}{\times} L_{k^h}) \times \prod_{k\notin I^h} L_k$. The following intersection formulas hold for non-empty subsets I, J of Ω

$$(*) \qquad \Delta_I \cap \Delta_J = \begin{cases} \Delta_{I\cup J} & \text{if } I \cap J \neq \emptyset \\ \Delta(\underset{k\in I}{\times} L_k) \times \Delta(\underset{k\in J}{\times} L_k) \times \prod_{k\notin I\cup J} L_k & \text{if } I \cap J = \emptyset \end{cases}$$

The map $X \mapsto X \cap B$ defines an isomorphism of $[G/H]$ onto the lattice $[B/1]_H$ of H-invariant subgroups of B; in what follows we are mainly interested in describing the structure of maximal subgroups as well of maximal H-invariant subgroups of B. If one puts $\hat{L}_i = \{f \in L^\Omega \mid f(i) = 1\}$, then $\hat{L}_i \lhd \hat{L}_i \Delta B = B$ and $X \mapsto X \cap \hat{L}_i$ defines an isomorphism of $[B/\Delta B] \to [\hat{L}_i/1]_{\Delta B}$. For $T \leq B$ we set $T^u = \underset{i\in I}{\times} T^{\pi_i}$, with $\pi_i : B \to L_i$ the projection map, and $T_\ell = \underset{i\in I}{\times} T_i$, $T_i = L_i \cap T \lhd T$. T is called a *standard subgroup* of B if $T = T^u (= T_\ell)$ and non-standard otherwise: since $(\Delta B)^u = B$, all elements of $[B/\Delta B]$ different from B are non-standard subgroups. Clearly a standard subgroup T of B is H-invariant if and only if $T_i^h = T_{i^h}$ for all i's and it is a maximal standard H-invariant subgroup of B if moreover $T_i \lessdot L_i$.

According to [10, §1], to get a maximal subgroup F in $[B/\Delta B]$ one can proceed in the following way: consider in B an H-invariant standard subgroup $S = S_1 \times \cdots \times S_n$, $S_i^h = S_{i^h}$, with S_i maximal normal in L_i, and set $F = R \times \prod_{k\notin u} L_k$, $u = \{r, s\}$, $R = (S_r \times S_s)\,\Delta(L_r \times L_s)$. Then F is a maximal subgroup of B containing ΔB. We are now interested in determining the structure of F_H. To this end, we introduce the set $\mathcal{U}$ of all subsets of Ω of cardinality 2. On $\mathcal{U}$ we define a relation $\sim$ by setting

(**) $u \sim v$ if there exist sequences $u_0, \ldots, u_t$ in $\mathcal{U}$ and $h_1, \ldots, h_t$ in H, $t > 0$,

$$\text{such that } u_{i-1}^{h_i} = u_i \text{ and } u_{i-1} \cap u_i \neq \emptyset \text{ for all } i > 0.$$

Clearly $\sim$ is an equivalence relation, and it is an H-congruence [3, Exercise 1.5.4], in the sense that $u \sim v$ if and only if $u^h \sim v^h$: hence H acts on the quotient space $\mathcal{U}/\sim$. We note that $u \sim v$ implies $v = u^h$ for some $h \in H$.

LEMMA 2.1. *Given $u \in \mathcal{U}$, let $\mathcal{V}$ be the congruence class of u and let $\mathcal{V}^H = \{\mathcal{U}_i\}_{i \in I}$ be the orbit of $\mathcal{V}$ in $\mathcal{U}/\sim$. Then*

a) *if $\{r, s\} \in \mathcal{U}_i$ and $\{r, s'\} \in \mathcal{U}_j$, with $i, j \in I$, then $i = j$;*

b) *for $i \in I$, set $\Omega_i := \underset{v \in \mathcal{U}_i}{\times} v \subseteq \Omega$. Then $\{\Omega_i\}_{i \in I}$ is a complete system of (imprimitivity) blocks for H with $\Omega_i = \Omega_j$ if and only if $i = j$, $H_{\mathcal{U}_i} = H_{\Omega_i}$, $|I| = |H : H_{\Omega_i}|$, $|\Omega_i| = |H_{\Omega_i} : H_r|$, $r \in \Omega_i$.*

c) *let $i \in I$ be such that $u \in \mathcal{U}_i$, and let $r \in u$. Then Ω_i is the intersection of all blocks for H containing u or, equivalently, if $u = \{H_r, H_r x\}$, then $\Omega_i = \{H_r h \mid h \in \langle H_r, x\rangle\}$; Ω_i is a minimal block* [3, Example 1.5.1] *if and only if $H_r <\cdot \langle H_r, x\rangle$.*

PROOF. *a)* Let $h \in H$ be such that $\mathcal{U}_i^h = \mathcal{U}_j$. Then $\{r, s\}^h \sim \{r, s'\}$, so that $\{r, s'\} = \{r, s\}^{hh'}$ for a certain $h' \in H$; but then $\{r, s\} \sim \{r, s'\}$, *i.e.* $\mathcal{U}_j = \mathcal{U}_i$, and $j = i$.

b) Assume $r \in \Omega_i \cap \Omega_j$. Then $\{r, s\} \in \mathcal{U}_i$, $\{r, s'\} \in \mathcal{U}_j$ for certain s, s', so by a), $i = j$. Let $h \in H_{\Omega_i}$ and let $\mathcal{U}_i^h = \mathcal{U}_j$. Then $\Omega_i = \Omega_i^h = \underset{v \in \mathcal{U}_j}{\cup} v = \Omega_j$, so $i = j$, and $\mathcal{U}_i^h = \mathcal{U}_i$.

c) Let $\overline{\Omega}$ be the intersection of all blocks containing u, and suppose $\overline{\Omega} \subset \Omega_i$. Then there exists a $v \in \mathcal{U}_i$ such that $v\overline{\Omega}$. Since $u \sim v$, there are sequences (u_i), (h_i) as in (* *). Hence there exists j such that $u_{j-1} \in \overline{\Omega}$, but $u_j \notin \overline{\Omega}$, but this is a contradiction, since $u_j = u_{j-1}^{h_j}$, so that $\overline{\Omega} \cap \overline{\Omega}^{h_j} \supseteq u_{j-1} \cap u_j \neq \emptyset$. Therefore $\Omega_i = \overline{\Omega}$. If $r \in u$ and we identify u with $\{H_r, H_r x\}$ (for a certain $x \in H \setminus H_r$), then the minimal block containing u has setwise stabilizer the subgroup $\langle H_r, x\rangle \in [H/H_r]$ [3, Theorem 1.5A], that is $\Omega_i = \{H_r h \mid h \in \langle H_r, x\rangle\}$. The conclusion follows. □

For simplicity and without loss of generality, further on we shall assume that $u = \{1, r\}$ and call $\mathcal{U}_1$ the congruence class of u.

LEMMA 2.2. *Set $\Omega_1 = \underset{v \in \mathcal{U}_1}{\cup} v$, $F = \Delta_u$. Then*

a) $F_{H_{\mathcal{U}_1}} = F_{H_{\Omega_1}} = \Delta_{\Omega_1}$;

b) $F_{(H_{\Omega_1})^h} = F^h_{H_{\Omega_1}} = \Delta_{\Omega_1^h}$.

PROOF. *a)* We have $H_{\mathcal{U}_1} = H_{\Omega_1}$ by Lemma 2.1 *b)*. For $h \in H_{\mathcal{U}_1}$, we get $F^h = \Delta_{u^h}$ with $u \sim u^h$; so if (u_i), (h_i) are sequences as in (* *), using (*) we get

$$F \wedge F^h \geq \Delta_{u_0 = u} \wedge \cdots \wedge \Delta_{u_n = u^h} \geq \Delta_{\Omega_1}.$$

Since Δ_{Ω_1} is H_{Ω_1}-invariant, we get $F_{H_{\Omega_1}} \geq \Delta_{\Omega_1}$. But since $\Omega_1 = \bigcup_{h \in H_{u_1}} u^h$, also the other inclusion holds.

b) $F_{(H_{\Omega_1})h} = F_{(H_{u_1})h} \bigwedge_{y \in H_{u_1} h} F^y = \left(\bigwedge_{x \in H_{u_1}} F^x \right)^h = \Delta_{\Omega_1^h}$. □

PROPOSITION 2.3. *Set* $H = \dot{\bigcup}_{1 \leq i \leq m} H_{\Omega_1} h_i$, *with* $h_1 = 1$, $F = \Delta_u$. *Then*

a) $F_H = \times_{1 \leq i \leq m} \Delta\left(\times_{k \in \Omega_1^{h_i}} L_k \right) \cong L^m$, $\Delta B \leq F_H$, $F_H \wedge L_i = 1$;

b) *there exists an order-inverting embedding* φ *of* $[H/H_1]$ *into* $[B/\Delta B]_H$ *such that* $X^\varphi \wedge L_i = 1$ *for all i's, if* $X \neq H_1$.

c) *if L is simple non-abelian, then* $(F_H H)_G = 1$.

PROOF. *a)* $F_H = \bigwedge_{1 \leq i \leq m} F_{H_{\Omega_1} h_i} = \bigwedge_i F_{H_{\Omega_1}}^{h_i} = \bigwedge_i \Delta_{\Omega_1}^{h_i} = \times_{1 \leq i \leq m} \Delta\left(\times_{k \in \Omega_1^{h_i}} L_k \right) \cong L^m$ using Lemma 2.2 *b)*.

b) For $X \in [H/H_1]$ consider the imprimitivity system $\{\overline{\Omega}_j\}_{1 \leq j \leq m}$ determined by X. Then the map $\varphi : X \mapsto \times_{1 \leq j \leq m} \Delta\left(\times_{k \in \overline{\Omega}_j} L_k \right)$ has the required properties.

c) Set $N = (F_H H)_G$. Then $N \wedge B = 1$, hence $N \leq C_G(B) = 1$. □

COROLLARY 2.4. *Let T be an element of* $[B/\Delta B]_H$ *such that* $\overline{L}_i = L_i / L_i \wedge T$ *is simple non-abelian. Then T is a maximal element in* $[B/\Delta B]_H$ *if and only if* $T/T_\ell = \times_{1 \leq i \leq m} \Delta\left(\times_{k \in \Omega_i} \overline{L}_k \right)$, *where* $m = |H : H_{\Omega_1}|$ *and* $\{\Omega_i\}$ *is a complete system of minimal blocks for H, afforded by an* $R \in [H/H_1]$ *such that* $H_1 \lessdot R$.

PROOF. The sufficiency is clear from our previous discussion. Assume now that T is a maximal element in $[B/\Delta B]_H$. Since $T \neq B$, there exists $u = \{1, r\}$ such that $T/T_\ell \leq F/T_\ell = \Delta_u T_\ell / T_\ell = \Delta(\overline{L}_1 \times \overline{L}_r) \times \prod_{k \notin u} \overline{L}_k \lessdot B/T_\ell$. So $T \leq F_H \in [B/\Delta B]_H$ implies $T = F_H$ and T/T_ℓ has the indicated structure by Proposition 2.3 *a)*. Moreover, since T is maximal H-invariant, according to Lemma 2.1*c)*, $\Omega_1 = \{H_1 h \mid h \in R\}$, with $H_1 \lessdot R$. The conclusion follows. □

COROLLARY 2.5. *Let* $F = \Delta_u$, $u = \{H_1, H_1 x\}$. *Then*

a) $F_H = \Delta B$ *if and only if* $H = \langle H_r, x \rangle$ i.e. $x \notin \cup M_i$, $H_1 \leq M_i \lessdot H$.

b) *if H is primitive and L is a non-abelian simple group then* $[B/\Delta B]_H$ *has length* 1.

PROOF. *a)* By Proposition 2.3 *a)*, $F_H = \Delta B$ if and only if $H_{\Omega_1} = \langle H_r, x \rangle = H$.

b) follows from Corollary 2.4. □

REMARK 2.1. We note that in general if $[B/\Delta B]_H$ has length 1, then H is primitive and L is simple, since then, by Proposition 2.3 *b)*, we have $H_1 \lessdot H$ and, if $S \triangleleft L$, then $S \wr H \in [B/\Delta B]_H$. However, in the other direction, if L is not assumed to be non-abelian, then $[B/\Delta B]_H$ may have length greater than 1. Consider the following example: let $L = C_p$, $H = \langle (12 \dots n) \rangle$. Then $H_1 = \{1\} \lessdot H$ if and only if n is prime. Assume now $n = 3$. Then the cardinality h of $[B/\Delta B]_H$ is

$$h = \begin{cases} 3 & \text{if } p = 3 \\ 2 & \text{if } p \equiv -1 \bmod 3 \\ 4 & \text{if } p \equiv 1 \bmod 3 \end{cases}$$

and $[B/\Delta B]_H$ has length 1 if and only if $p \equiv -1 \bmod 3$.

In general, if n is an odd prime q, then $[B/\Delta B]_H$ has length 1 if and only if q is a primitive divisor of $p^{q-1} - 1$, that is if and only if $q \neq p$ and q does not divide $p^i - 1$ for $1 \leq i < q - 1$ (see [6]). □

To the expression $F_H = \tilde{L}_1 \times \dots \times \tilde{L}_m$ in Proposition 2.3 *a*), where $\tilde{L}_i$ stands for $\Delta(\underset{k \in \Omega_1^{h_i}}{\times} L_k)$ we can associate, via conjugation, the transitive permutation representation $\theta : H \to \mathrm{Sym}(m)$ of degree m. In the next lemma we collect some useful properties of θ.

Lemma 2.6. *We have*:

a) $\ker\theta = (H_{\Omega_1})_H$ *and, if* $\tilde{H} = H/\ker\theta$, *then* $\tilde{G} = F_H H/\ker\theta = F_H : \tilde{H} = \tilde{L} \wr \tilde{H}$, *with* $\tilde{H}$ *a transitive permutation group of degree* $\deg \tilde{H} = |H : H_{\Omega_1}| < \deg H$ *acting on the complete system of blocks* $\{\Omega_1^H\}$ *for* H *on* Ω.

b) *if* $\Phi(\tilde{G}) = 1$ *and* H *is a K-group, then for any normal subgroup* N *of* $F_H H$ *contained in* $\ker\theta$, $\Phi(F_H H/N) = 1$.

Proof. *a*) We have $\tilde{H}_{\tilde{G}} = 1$ and $N_{\tilde{H}}(\tilde{L}_i) = C_{\tilde{H}}(\tilde{L}_i)$, so that $\tilde{G} = \tilde{L} \wr \tilde{H}$.

b) Set $\mathcal{M} = \{X \lessdot F_H H \mid N \leq X\}$, $\mathcal{M}_{H,N} = \{X \lessdot H \mid N \leq X\}$ and $\tilde{\mathcal{M}} = \{F_H X \mid X \in \in \mathcal{M}_{H,N}\}$; then $F_H X \lessdot F_H H$, $F_H X \wedge H = X$ and $\tilde{\mathcal{M}} \subseteq \mathcal{M}$. Thus $\underset{\mathcal{M}}{\wedge} Y \leq \ker\theta \wedge (\underset{\tilde{\mathcal{M}}}{\wedge} Y) \leq \leq H \wedge (\underset{\tilde{\mathcal{M}}}{\wedge} Y) = \underset{\mathcal{M}_{H,N}}{\wedge} Y = N$ since $\Phi(H/N) = 1$. □

Theorem 2.7. *Given a group* L *consider* $G = L \wr H$. *Let* S_i *be a maximal normal subgroup of* L_i *with* $\bar{L}_i = L_i/S_i$ *non-abelian,* $S_i^h = S_{i^h}$ *for each* i *and* $S = S_1 \times \dots \times S_n$. *Then a* T *in* $[B/S\Delta B]$ *is H-invariant if and only if* $T = B$ *or there exists a complete system of blocks* $\{\Omega_i\}_{1 \leq i \leq m}$ *for* H *on* Ω *such that* $T/S = \underset{1 \leq i \leq m}{\times} \Delta(\underset{k \in \Omega_i}{\times} \bar{L}_k)$.

Proof. Assume T to be H-invariant in $[B/S\Delta B]$, and $T \neq B$. We have $S_i \leq T_i$, hence $S = T_\ell < T < T^u = B$ and by Corollary 2.4 there exists $F = S\Delta_u$, with $u = \{H_1, H_1 x\}$ and $H_1 \lessdot \langle H_r, x \rangle$ such that F_H is a maximal element of $[B/\Delta B]_H$, $T/S \leq F_H/S = \underset{1 \leq i \leq m}{\times} \tilde{L}_i$, where $\tilde{L}_i = \Delta(\underset{k \in \Omega_i}{\times} L_k) \cong L$. By Lemma 2.6, $(F_H/S) : \tilde{H} = \tilde{L} \wr \tilde{H} = \tilde{G}$, $m = \deg \tilde{H} = = |H : H_{\Omega_1}| < \deg H$. Let G be a counterexample with H of minimal degree. If $m = 1$, then $T/S = \tilde{L}_1$, a contradiction. Hence $m \geq 2$. In $\tilde{G} = \tilde{L} \wr \tilde{H}$ we have $\Delta\tilde{B} \leq T < T^u = \tilde{B}$ *i.e.* T is a proper element of $[\tilde{B}/\Delta\tilde{B}]_H$. By minimality, there exists a complete system of blocks $\{\Omega_j\}_{1 \leq j \leq s}$ for $\tilde{H}$, $s \leq m$ such that $T/S = \underset{1 \leq j \leq s}{\times} \Delta(\underset{k \in \Omega_j}{\times} \tilde{L}_k)$. Let d

be the cardinality of Ω_j. Then each block Ω_j is the union of d convenient blocks of $\{\Omega_i\}$: $\Omega_j = \dot{\bigcup}_{1\leq i\leq d} \Omega_i^{(j)}$ and so Ω_j (thought as a subset of Ω) is a block $\tilde{\Omega}_j$ for H with $|\tilde{\Omega}_j| = d\,|H_{\Omega_1} : H_1|$ and $T/S = \times_{1\leq j\leq s} \Delta(\times_{k\in\tilde{\Omega}_j} L_k) \cong L^s$, a contradiction. The converse is clear by Proposition 2.3 *b*). □

(2.1) *Given* $G = L \wr H$, *we have* $(\Delta B \times H)_G = 1$ *if and only if* $Z(L) = 1$.

Proof. $Z(L) \cong Z(\Delta B) \leq \Delta B \times H$ and since $Z(\Delta B) \lhd G$, $(\Delta B \times H)_G \neq 1$ if $Z(L) \neq 1$. Assume now $Z(L) = 1$ and $(\Delta B \times H)_G = N \neq 1$. If $N \wedge B = 1$ we get $N \leq C_G(B) \leq B$, *i.e.* $N \leq Z(B)$, a contradiction. So $D = N \wedge B \neq 1$ and since $N \leq \Delta B \times H$ we have $D \leq (\Delta B \times H) \wedge B = \Delta B$ and $D \lhd G$. Take a non-trivial element $(d, \dots, d) \in D$ and pick $1 \neq (\ell_1, 1, \dots, 1) \in B$. Then $(\ell_1, 1, \dots, 1)^{-1}(d, \dots, d)(\ell_1, 1, \dots, 1) = (d^{\ell_1}, d, \dots, d) \in D$, *i.e.* $d^{\ell_1} = d$ for all $\ell_1 \in L$ so $d \in Z(L) = 1$, a contradiction. Therefore $N = 1$. □

3. We begin with

Lemma 3.1. *Let L be a simple non-abelian group and assume that H has a maximal core-free subgroup* M_0. *For* $R \leq M_0$, *let* H_R *denote the (faithful) right coset representation of H afforded by R. Then* $G_R = L \wr H_R$ *is a K-group if and only if H is a K-group.*

Proof. We begin with $R = M_0$. By Corollary 2.5 *b*), ΔB is a maximal element in $[B/\Delta B]_H$, hence $\Delta B \times H$ is a maximal K-subgroup of G_R, since $\Delta B \cong L$. By (2.1) $(\Delta B \times H)_G = 1$, hence by Proposition 1.5, G is a K-group. For $R \leq M_0$, assume that G_R is a counterexample and choose R with $|H : R| \neq 1$ minimal. Thus $R \neq M_0$ and we can take $u = \{R, Rx\}$ with $R \lessdot \langle H_r, x\rangle = A \leq M_0$. Then for $F = \Delta_u$, F_{H_R} is, by Corollary 2.4, a maximal element in $[B/\Delta B]_{H_R}$, hence $F_{H_R}H_R \lessdot G$. According to Lemma 2.6, $F_{H_R}H_R \cong \tilde{L} \wr \tilde{H}$, with $\tilde{H} \cong H$ since $\ker\theta = A_H = 1$ and $\deg \tilde{H} = |H : A| < |H : R| = \deg H_R$. By the minimality assumption, $\tilde{L} \wr \tilde{H}$ is a K-group and by Proposition 2.3 *c*), $(F_{H_R}H_R)_G = 1$. But then by Proposition 1.5, G_R is a K-group, a contradiction. □

Theorem 3.2. *Let L be a given group such that* $L/S(L)$ *is a direct product of simple groups and let H be a group with a maximal core-free subgroup* M_0. *Then for every* $R \leq M_0$, G_R *is a K-group if and only if H is a K-group and* $\Phi(G/F_i(S(B))) = 1$ *for all i's.*

Proof. The necessity follows from Proposition 1.4 *a*); actually, in our case, L itself is a K-group by Corollary 1.3, since $L/S(L)$ is a K-group by [2]. Conversely, by Proposition 1.4 b), we may assume $S(L) = 1$, *i.e.* $L = S_1 \times \cdots \times S_t$, S_i's simple non-abelian groups. We have to investigate $G_R = L \wr H_R = B : H_R = (B_1 \times \cdots \times B_t) : H_R$, B_i the base group of $S_i \wr H_R$, with H a K-group. If $t = 1$, the conclusion follows by Lemma 3.1. We assume now $t > 1$ and use induction on t. Thus $\hat{B}_i : H_R$ is a K-group for all i's, where $\hat{B}_i = B_1 \times \cdots \times B_{\hat{i}} \times \cdots \times B_t$. Assume, to begin, $R = M_0$ and set $F_i = \Delta B_i \times \hat{B}_i H_R$.

Since, by Corollary 2.5, $\Delta B_i \times H_R <\cdot B_i H_R$, F_i is a maximal K-subgroup of G_R. Now by (2.1) $(F_i)_G = \hat{B}_i$, so $\bigwedge_i (F_i)_G = \bigwedge_i \hat{B}_i = 1$ and therefore by Proposition 1.5, G_R is a K-group. For a contradiction, assume that there exists an $R \leq M_0$ such that G_R is not a K-group; choose R such that $|H : R|$ is minimal. From what just seen, $R \neq M_0$. Let x be an element of M_0 such that $R <\cdot \langle H_r, x\rangle = A \leq M_0$; set $u = \{R, Rx\}$ and for a fixed $1 \leq i \leq t$, consider $M_i = \Delta_u <\cdot B_i$ and define $\tilde{M}_i = M_i \times \hat{B}_i$. Then $\tilde{M}_i <\cdot B$ and $(\tilde{M}_i)_{H_R}$ is a maximal element of $[B/\Delta B_1 \times \cdots \times \Delta B_t]_{H_R}$ (see Propositon 2.3 and Corollary 2.4), thus $(\tilde{M}_i)_{H_R} H_R <\cdot G$. According to Lemma 2.6, $(\tilde{M}_i)_{H_R} H_R / \ker \theta_i \cong \tilde{L} \wr \tilde{H}$, $\tilde{L} \cong L$; here $\ker \theta_i = A_H = 1$ and $\deg \tilde{H} = |H : A| < |H : R|$. So, by the minimality assumption, $X_i = (\tilde{M}_i)_{H_R} H_R$ is a maximal K-subgroup of G_R. Since, by Proposition 2.3 *c*), $((\tilde{M}_i)_{H_R} H_R)_G = \hat{B}_i$ and $\bigwedge_i \hat{B}_i = 1$, by Proposition 1.5, G_R is a K-group, a contradiction. □

We recall that given a permutation group L on a set Γ, the group $G = L \wr H$ becomes a permutation group on Γ^Ω via the *product action* by setting $\rho^{(f,h)}(\omega) = (\rho(\omega^{h^{-1}}))^{f(\omega^{h^{-1}})}$. The group G turns out to be a primitive group as soon as L is primitive not regular (see [3, 2.7A]).

Corollary 3.3. *Let* $\{H_i\}_{1 \leq i \leq t}$, $t \geq 2$, *be a family of simple non-abelian primitive permutation groups on the sets* Ω_i *and let* $L_i = H_i \times H_i$ *be the primitive permutation group on the set* $\Gamma_i = \{\Delta(H_i \times H_i)h \mid h \in L_i\}$. *Then*

a) $H = H_1 \wr (H_2 \wr (H_3 \wr \cdots) \cdots)$ *in its product action is a primitive K-group*;
b) $H = L_1 \wr (L_2 \wr (L_3 \wr \cdots) \cdots)$ *in its product action is a primitive K-group.*

Proof. *a*) for $t = 2$ the primitive group $H_1 \wr H_2$ is a K-group by Lemma 3.1. Using induction, $H_2 \wr (H_3 \wr \cdots)$ is a primitive K-group, hence the primitive group H is also a K-group by Theorem 3.2.

b) L_i is a primitive K-group, since $\Delta(H_i \times H_i) <\cdot L_i$. Thus, by Theorem 3.2, $L_1 \wr L_2$ is a K-group and, as in case *a*), with an induction argument, one reaches the conclusion. □

We like to point out that Corollary 3.3 in combination with Theorem 3.2 allows to construct further examples of K-groups.

Proposition 3.4. *Let* L *be a group such that* $L/S(L)$ *is simple and* H *a direct product of simple groups. Then* $G = L \wr H$ *is a K-group if and only if* $\Phi(L/F_i(S(L)) \wr H) = 1$ *for all i's.*

Proof. The necessity follows from Proposition 1.4 *b*). For the converse, we note that H is a K-group and by Proposition 1.4 *b*), we may assume $S(L) = 1$. Pick $u = \{1, r\}$, $F = \Delta_u$, so that $F <\cdot B$ and $F_H H <\cdot G$. For a contradiction, assume that G is not a K-group, and take a counterexample with minimal $\deg H$. Applying Lemma 2.6, we get $F_H H = \tilde{L} \wr \tilde{H} \times \ker \theta$, since $H = \ker \theta \times R$, and $\deg \tilde{H} < \deg H$. By minimality reasons,

$\tilde{L} \wr \tilde{H}$ is a K-group, hence $F_H H$ itself and by Proposition 2.3 *c)* $(M_H H)_G = 1$. So by Proposition 1.5, G is a K-group, a contradiction. □

THEOREM 3.5. *Let L be a group such that $L/S(L)$ is simple and H be a group whose proper normal subgroups are solvable. Then $G = L \wr H$ is a K-group if and only if H is a K-group and $\Phi(G/F_i(S(B))) = 1$ for all i's.*

PROOF. The necessity follows from Corollary 1.2 and (1.2). For the converse, by Proposition 1.4 *b)*, we may assume L simple non-abelian. For a contradiction assume G is a counterexample with $\deg H$ minimal and pick $u = \{1, r\}$ such that $\Delta B \leq F = \Delta_u \lessdot B$ and $F_H H \lessdot G$. Since by Lemma 2.6 $F_H H/\ker\theta \cong \tilde{L} \wr \tilde{H}$, $\tilde{L} \cong L$ and $\deg \tilde{H} < \deg H$, by minimality reasons $F_H H/\ker\theta$ is a K-group. Since $S(\tilde{L} \wr \tilde{H}) = S(\tilde{B})$ by (1.2) and $S(\tilde{B}) = 1$, we get $\Phi(\tilde{L} \wr \tilde{H}) = 1$, $(F_H H)_G = S(F_H H) = \ker\theta$, hence $\Phi(F_H H) \leq \ker\theta$. But now by Lemma 2.6 b) $\Phi(F_H H/F_i(\ker\theta)) = 1$ for all i's and so by Proposition 1.1, $F_H H$ is a maximal K-subgroup of G. Since $(F_H H)_G = 1$ by Proposition 2.3 *c)*, by Proposition 1.5 G is a K-group, a contradiction (note that in the case $H = \ker\theta$, *i.e.* $F_H = \Delta B$, then $\Delta B \times H \lessdot B : H$ and $\Delta B \times H \cong L \times H$ is a K-group). □

We recall that a transitive permutation group H is $\frac{3}{2}$-transitive if the stabilizer H_1 has orbits of the same length on $\Omega \setminus \{1\}$. By a theorem of Wielandt [7, Theorem 3.1 *a*] a $\frac{3}{2}$-transitive group is either primitive, or a Frobenius group.

COROLLARY 3.6. *Let L be a group such that $L/S(L)$ is simple and H a $\frac{3}{2}$-transitive permutation group. Then $G = L \wr H$ is a K-group if and only if H is a K-group and $\Phi(G/F_i(S(B))) = 1$ for all i's.*

PROOF. This follows from Theorems 3.2 and 3.5, since a Frobenius K-group is necessarily solvable (if the Frobenius complement has even order, then it has exactly one element of order 2 [3, Theorem 3.4A]). □

Let us denote with $\mathcal{X}$ the class of (simple) groups of Lie type such that $G(q) \in \mathcal{X}$, $q = p^f$, $f > 1$, if for each divisor d of f and each prime divisor r of d, the interval $[G(p^d)/G(p^{d/r})]$ is monocoatomic and $G(p^{d/r})$ is simple (non-abelian). In [1] one can find a list of groups which are members of the class $\mathcal{X}$. In the following we shall denote by φ the field automorphism of $G(q)$ induced by $x \mapsto x^p$.

LEMMA 3.7. *Given $G_0 = G_0(q) \in \mathcal{X}$ and $\psi \in \langle\varphi\rangle$, set $G = G_0 : \langle\psi\rangle$. If r is a prime divisor of f and $G_0(q^{1/r}) \leq M \lessdot G_0(q)$, then the interval $[G \wr H/G_0(q^{1/r})^n H\langle\psi\rangle^n]$ is monocoatomic with coatom $M^n H\langle\psi\rangle^n$.*

PROOF. $G \wr H = G_0(q)^n \langle\psi\rangle^n : H$. Since $M^\psi = M$, $[G_0(q)\langle\psi\rangle/G_0(q^{1/r})\langle\psi\rangle]$ is monocoatomic with coatom $M\langle\psi\rangle$; but then $[G_0(q)^n \langle\psi\rangle^n / G_0(q^{1/r})^n \psi^n]_H$ is monocoatomic with coatom $M^n\langle\psi\rangle^n$, hence $[G_0(q)^n\langle\psi\rangle^n : H/G_0(q^{1/r})^n\langle\psi\rangle^n : H]$ is monocoatomic with coatom $M^n\langle\psi\rangle^n : H$. □

THEOREM 3.8. *Given $G_0 = G_0(q) \in \mathcal{X}$, $q = p^f$, let ψ be an element of $\langle\varphi\rangle$, where φ is the field automorphism of G_0 induced by $x \mapsto x^p$ and set $G = G_0 : \langle\psi\rangle$, $\tilde{G} = G \wr H$ where H is either primitive, or a group with all its proper normal subgroups solvable. Then $\tilde{G}$ is a K-group if and only if $(G/G_0)^n$ is a direct product of minimal H-invariant subgroups and H is a K-group.*

PROOF. If $\tilde{G}$ is a K-group, then $\tilde{G}/G_0^n \cong \langle\psi\rangle \wr H$ is a K-group; in particular $\Phi(\langle\psi\rangle \wr H) = 1$ and so the necessity follows from (1.1). Conversely, since $(G/G_0)^n$ is a direct product of minimal H-invariant subgroups, $|\psi|$ is square-free. Let $\tilde{G}$ be a counterexample with minimal q. Let r be a prime divisor of $|\psi|$. By Lemma 3.6, $[\tilde{G}/G_0(q^{1/r})^n\langle\psi\rangle^n H]$ is monocoatomic with coatom $M^n\langle\psi\rangle^n H$. Now $(M^n\langle\psi\rangle^n H)_{\tilde{G}} \wedge \wedge\, G_0(q)^n = 1$ and since $C_{\tilde{G}}(G^n) = Z(G^n) = 1$, we get $(M^n\langle\psi\rangle^n H)_{\tilde{G}} = 1$. We claim that $G_0(q^{1/r})\langle\psi\rangle^n H$ is a K-group. Set $\langle\psi\rangle = \langle\psi\rangle_r \times \langle\psi\rangle_{r'}$, with ψ_r of order r. By minimality and Theorems 3.2, 3.5, $(G_0(q^{1/r}) : \langle\psi_{r'}\rangle) \wr H$ is a K-group. Moreover $(G_0(q^{1/r}) : \langle\psi_{r'}\rangle) \wr H$ acts on $\langle\psi_r\rangle^n$ as H, hence $\langle\psi_r\rangle^n$ is a completely reducible $(G_0(q^{1/r}) : \langle\psi_{r'}\rangle) \wr H$-module. But then, by [9, Lemma 3.1.9], $G_0(q^{1/r})\langle\psi\rangle \wr H$ is a K-group. According to Proposition 1.5, $\tilde{G}$ is a K-group, a contradiction. □

Note that if in the primitive group H one replaces H_1 with an $R \leq H_1$, then $\tilde{G}_R = G \wr H_R$ is still a K-group as soon as $(G/G_0)^{|H:R|}$ is a completely reducible H-module and H is a K-group: in fact still Theorem 3.2 applies in the proof.

COROLLARY 3.9. *Let $G_0 = G_0(q) \in \mathcal{X}$, $q = p^f$ and $\psi \in \langle\varphi\rangle$. If $G = G_0 : \langle\psi\rangle$, then $\tilde{G} = G \wr A_n$ (resp. $G \wr S_n$) is a K-group if and only if $|\psi|$ is square-free and $(n, |\psi|) = 1$.*

By Theorem 3.8, $\tilde{G}$ is a K-group if and only if $(G/G_0)^n$ as an A_n-group (resp. S_n-group) is a direct product of minimal normal subgroups, and this is the case if and only if $|\psi|$ is square-free and $(|\psi|, n) = 1$ [8, 5.3.4]. □

4. In this last section we deal with the case $L = S(L)$. As usual $G = L \wr H$.

LEMMA 4.1. *Let L be a solvable group and U a minimal normal subgroup of L. Denote by $\tilde{U}$ the base group of $U \wr H$. Then $\tilde{U}$ is a minimal normal subgroup of G if and only if $U \not\leq Z(L)$.*

PROOF. If $U \leq Z(L)$, then $\Delta(U^n) \lhd G$, and $\Delta(U^n) < \tilde{U}$. Assume now $U \wedge Z(L) = 1$. If $H = \dot{\bigcup}_i H_1 h_i$, $h_1 = 1$, then $\tilde{U} = U \times U^{h_2} \times \cdots \times U^{h_n}$. Let N be a minimal normal subgroup of G with $N \leq \tilde{U}$. Take a non-trivial element $x \in N$, hence $x = x_1 x_2 \cdots x_n$,

$x_i \in U^{h_i}$ for all i's, and without loss of generality we may assume $x_1 \neq 1$. Pick a $g \in L \setminus C_L(U)$; then $x_1^g \neq x_1$, hence $1 \neq x_1^g x_1^{-1} = x^g x^{-1} \in N \wedge U$. Thus $\langle x_1^g x_1^{-1} \rangle^L = U \leq N$, hence $U^H = \tilde{U} \leq N$. □

THEOREM 4.2. *Given a solvable group L, then $G = L \wr H$ is a K-group if and only if L and $(L/L') \wr H$ are K-groups.*

PROOF. By Proposition 1.4 *a*), $S(L) = L$ is a K-group, and so is $G/(L')^n \cong (L/L') \wr H$. For the converse, let G be a minimal counterexample. Then L is not nilpotent, since $\Phi(L) = 1$ implies $L' = 1$. Let U be a minimal normal subgroup of L contained in $L' \neq 1$. If $\tilde{U}$ is the base group of $U \wr H$, $G/\tilde{U} \cong (L/U) \wr H$ is a K-group for minimality reasons. If $U \leq Z(L)$, $G = \tilde{U} \times L/U \wr H$ is a K-group, a contradiction. So $U \wedge Z(L) = 1$; by Lemma 4.1, $\tilde{U}$ is a minimal normal subgroup of G and we get $G = \tilde{U} : L/U \wr H$, but then by [9, 3.1.9], G is a K-group, a contradiction. □

COROLLARY 4.3. *For a solvable group L, $L \wr A_n$* (resp. *$L \wr S_n$) is a K-group if and only if L is a K-group and $(|L/L'|, n) = 1$.*

PROOF. The conditions $(|L/L'|, n) = 1$ and L a K-group implies that $(L/L')^n$ is a completely reducible A_n-module (resp. S_n-module) [8, 5.3.4]. □

COROLLARY 4.4. *Any twisted wreath product G of the alternating group A_m by A_n in which A_m is twisted by the point-stabilizer A_{n-1} is a K-group, except when $m = 3$ or $m = 4$ and $3 \mid n$.*

PROOF. By [5, 3.4], $G \cong A_m \wr A_n$ and the conclusion follows from Theorem 3.2, 3.5 and Corollary 4.3. □

REFERENCES

[1] N. BURGOYNE - R. GRIESS - R. LYONS, *Maximal subgroups and automorphisms of Chevalley groups*. Pac. J. Math., 71, 1977, 365-403.

[2] M. COSTANTINI - G. ZACHER, *The finite simple groups have complemented subgroup lattices*. Pac. J. Math., 213, 2004, 245-251.

[3] J.D. DIXON - B. MORTIMER, *Permutation Groups*. Springer-Verlag, Berlin-Heidelberg-New York 1996.

[4] W. GASCHÜTZ, *Über die Φ-Untergruppe endlicher Gruppen*. Math. Z., 58, 1953, 160-170.

[5] F. GROSS - L.C. KOVACS, *On normal subgroups which are direct products*. J. Algebra, 90, 1984, 133-168.

[6] B. HUPPERT, *Endliche Gruppen, I*. Springer-Verlag, Berlin-Heidelberg-New York 1967.

[7] B. HUPPERT - N. BLACKBURN, *Finite Groups, II*. Springer-Verlag, Berlin-Heidelberg-New York 1982.

[8] P.B. KLEIDMAN - M.W. LIEBECK, *The subgroup structure of the finite classical groups*. London Math. Soc. Lecture Notes, 129, Cambridge University Press, Cambridge 1990.

[9] R. SCHMIDT, *Subgroup lattices of groups*. de Gruyter, Berlin 1994.
[10] J. THÉVENAZ, *Maximal subgroups of direct products*. J. Algebra, 198, 1997, 352-361.
[11] G. ZACHER, *Caratterizzazione dei gruppi risolubili d'ordine finito complementati*. Rend. Sem. Mat. Univ. Padova, 22, 1953, 113-122.
[12] G. ZACHER, *Costruzione dei gruppi finiti a sottogruppo di Frattini identico*. Rend. Sem. Mat. Univ. Padova, 21, 1952, 383-394.

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