# Exercises - Set 2

**Exercise 1** Let $X = (X_t)_{t\geq 0}$ be a real valued stochastic process such that, for all $\omega \in \Omega$, the map $t \mapsto X_t(\omega)$ is right continuous and for every $t > 0$ the left limit

$$\lim_{s\uparrow t} X_s(\omega)$$

exists. Consider the set, for $T > 0$

$$\{\omega \in \Omega : t \mapsto X_t(\omega) \text{ is continuous on } [0,T]\}.$$

Show that it belongs to

$$\mathcal{F}_T^X := \sigma(X_t : t \in [0,T]).$$

*Hint*: Let fix $T > 0$ and define $D := (\mathbb{Q} \cap [0,T]) \cup \{T\}$. If $f : [0,T] \to \mathbb{R}$ we denote by $f_{|D}$ its restriction to $D$. Prove and use the following
*Lemma*. Let $f : [0,T] \to \mathbb{R}$ be right continuous. Then $f$ is uniformly continuous if and only if $f_{|D}$ is uniformly continuous.

**Solution.** We prove the Lemma later. Since a function $f : [0,T] \to \mathbb{R}$ is continuous if and only if it is uniformly continuous, we have:

$$\begin{aligned}
&\{X \text{ is continuous in } [0,T]\} = \{X \text{ is unif. continuous in } [0,T]\} = \{X_{|D} \text{ is unif. continuous in } [0,T]\} \\
&= \{\forall k \geq 1\, \exists n \geq 1 \text{ such that for every } s,t \in D \text{ with } |s-t| \leq \frac{1}{n} \text{ we have } |X_t - X_s| \leq \frac{1}{k}\} \\
&= \bigcap_{k\geq 1} \bigcup_{n\geq 1} \bigcap_{s,t\in D: |s-t|\leq \frac{1}{n}} \{|X_t - X_s| \leq \frac{1}{k}\}.
\end{aligned}$$

Noting that $\{|X_t - X_s| \leq \frac{1}{k}\} \in \sigma(X_s, X_t) \subseteq \mathcal{F}_T^X$ we conclude that $\{X \text{ is continuous in } [0,T]\} \in \mathcal{F}_T^X$.
*Proof of the Lemma*. Since uniform continuity is preserved by restriction, it is enough to show that if $f_{|D}$ is uniformly continuous then so it is $f$. Take $\epsilon > 0$. We must show that there exists $\delta > 0$ such that $|t-s| \leq \delta$ implies $|f(t) - f(s)| \leq \epsilon$. By assumption, such $\delta$ exists for $s,t \in D$. Take now $0 \leq s < t < T$ with $t - s \leq \delta$. Since $D$ is dense, there are sequences $s_n, t_n \in D$, $s_n > s$, $t_n > t$, s.t. $s_n \downarrow s$ and $t_n \downarrow t$. By right continuity

$$|f(t) - f(s)| = \lim_{n\to+\infty} \lim_{m\to+\infty} |f(t_m) - f(s_n)| \leq \epsilon$$

since, for every fixed $n$, $|t_m - s_n| < \delta$ for $m$ sufficiently large. The case in which $t = T$ is dealt with similarly, taking only the sequence $s_n$.

NOTE: existence of left limit is not needed!

**Exercise 2** (a) Let $B = (B^{(1)}, B^{(2)})$ be a two-dimensional Brownian Motion. Define

$$\beta_t^{(1)} = \frac{1}{\sqrt{2}}(B_t^{(1)} + B_t^{(2)} \quad \beta_t^{(2)} = \frac{1}{\sqrt{2}}(B_t^{(1)} - B_t^{(2)}.$$

Show that $(\beta^{(1)}, \beta^{(2)})$ is a two-dimensional Brownian Motion.

(2) Generalize the argument above. Let $B$ be a $d$-dimensional Brownian Motion, and let $A$ be a $d \times d$ matrix. Define

$$\beta_t := AB_t.$$

Show that $\beta$ is a two-dimensional Brownian Motion if and only if $A$ is orthogonal, i.e. $AA^* = I$.

**Solution.** We only solve (2). First observe that $\beta$ is Gaussian, since $(\beta_{t_1}, \ldots, \beta_{t_n})$ is a linear transformation of $B_{t_1}, \dots, B_{t_n}$. Moreover $\beta$ is continuous since $B$ is continuous, and $E(\beta_t) = 0$. Thus, $\beta$ is a Brownian motion if and only if the covariance matrix of $(\beta_s, \beta_t)$ equals $\min(t,s)\mathbb{I}$. But

$$Cov(\beta_s^{(i)}, \beta_t^{(j)}) = E[\beta_s^{(i)}\beta_t^{(j)}] = \sum_{h,k=1}^{d} A_{ih}A_{jk}E[B_t^{(h)}B_s^{(k)}] = \min(s,t)\sum_h A_{ih}A_{jh}.$$

This amounts to say that the covariance matrix of $(\beta_s, \beta_t)$ is $\min(s,t)AA^*$, from which the concluison follows.

**Exercise 3** We recall that a function $f : I \to \mathbb{R}$, where $I$ is an interval of $\mathbb{R}$, is called *Hölder continuous* with exponent $\alpha > 0$ in $x \in I$, if there exists $C > 0$ such that for every $y \in I$

$$|f(y) - f(x)| \leq C|y - x|^\alpha.$$

Let now $B$ be a Brownian motion, and assume $\alpha > \frac{1}{2}$.

(a) Show that, for every $n > 0$,

$$\lim_{s \downarrow 0} P(|B_s| \leq ns^\alpha) = 0.$$

(b) Let

$$H := \{\omega \in \Omega : B(\omega) \text{ is Hölder continuous in 0 with exponent } \alpha\}.$$

Show that $P(H) = 0$.

**Solution.**

(a)

$$P(|B_s| \leq ns^\alpha) = P\left(|N(0,1)| \leq ns^{\alpha - \frac{1}{2}}\right) \to 0$$

as $s \downarrow 0$.

(b) Let

$$H_n := \{\omega \in \Omega : |B_t(\omega)| \leq nt^\alpha \text{ for all } t \geq 0\}.$$

Then

$$H = \bigcup_n H_n,$$

so it is enough to show that $P(H_n) = 0$ for all $n$. But, for all $s > 0$

$$H_n \subseteq \{|B_s| \leq ns^\alpha\},$$

and the conclusion follows from point (a).

**Exercise 4** Let $B$ be a Brownian motion for a filtration $(\mathcal{F}_t)_{t \geq 0}$. Define $W_0 = 0$ and, for $t > 0$,

$$W_t = B_t - \int_0^t \frac{B_s}{s} ds.$$

(a) Show that $W$ is a.s. well defined, in the sense that the above integral is a.s. finite. (*Hint*: use the Law of the Iterated Logarithm)

(b) Show that $W$ is a Brownian motion.

(c) Assume the filtration is complete. Show that $W$ is $(\mathcal{F}_t)$-adapted, but it is *not* a Brownian Motion for the filtration $(\mathcal{F}_t)$.

**Solution.**

(a) By the law of the iterated logarithm and the fact that if $X$ is a Brownian Motion also $(tX_{1/t})_{t\geq 0}$ is a Brownian motion

$$\limsup_{t\to+\infty} \frac{|B_t|}{\sqrt{t}\sqrt{2\log\log t}} = \limsup_{t\downarrow 0} \frac{|B_t|}{\sqrt{t}\sqrt{2\log\log(1/t)}} = 1$$

almost surely. Thus there is an event $A$ with $P(A) = 1$ such that for all $\omega \in A$ there exists $\bar{t}(\omega)$ such that for every $t \leq \bar{t}(\omega)$

$$|B_t(\omega)| \leq 2\sqrt{t}\sqrt{2\log\log(1/t)} \leq C(\omega)t^{1/3}$$

for some constant $C(\omega)$. But then, for every $t \leq \bar{t}(\omega)$,

$$\frac{|B_t(\omega)|}{t} \leq \frac{C(\omega)}{t^{2/3}},$$

so that $\frac{|B_t(\omega)|}{t}$ is integrable in any interval of the form $[0,T]$.

(b) If we define

$$W_t^{(N)} = B_{\lfloor Nt\rfloor/N]} - \frac{1}{N}\sum_{k=1}^{\lfloor Nt\rfloor} \frac{B_{k/N}}{k/N},$$

we have that for every $t$, $W_t^{(N)} \to W_t$ almost surely. Moreover, given times $0 \leq t_1 < t_2 < t_n$ the vector $(W_{t_1}^{(N)},\ldots,W_{t_n}^{(N)})$ is a linear transform of $(B_{1/N},\ldots,B_{\lfloor Nt_n\rfloor/N]})$ and so it is Gaussian. Thus also $(W_{t_1},\ldots,W_{t_n})$ is Gaussian, being a.s. limit of Gaussian vectors. Moreover the a.s. continuity of $W$ follows from its definition, as the fact that $E(W_t) = 0$. We only need to check that $E(W_sW_t) = \min(s,t)$. For $s \leq t$

$$\begin{aligned}
E(W_sW_t) &= E(B_sB_t) - \int_0^t \frac{E(B_sB_u)}{u}du - \int_0^s \frac{E(B_tB_u)}{u}du + \int_0^s du\int_0^t dv\frac{E(B_uB_v)}{uv}\\
&= s - \int_0^s ds - \int_s^t \frac{s}{u}du - \int_0^s ds + \int_0^s du\left[\int_s^t \frac{u}{uv}dv\right] + \int_0^s du\int_0^s dv\frac{E(B_uB_v)}{uv}\\
&= -s + \int_0^s du\int_0^s dv\frac{E(B_uB_v)}{uv} = -s + 2\int_0^s du\int_0^u dv\frac{E(B_uB_v)}{uv}\\
&= -s + 2\int_0^s du\int_0^u dv\frac{1}{u}dv = s.
\end{aligned}$$

(c) The approximated process $W^{(N)}$ is clearly adapted. So $W_t$ is the a.s. limit of $\mathcal{F}_t$-measurable random variables. To conclude that it is $\mathcal{F}_t$-measurable we actually need to assume that the filtration is complete. If $W$ were a $(\mathcal{F}_t)$ Brownian motion, then $W_t - W_s$ would be independent of $B_s$, that would imply $E[(W_t - W_s)B_s] = 0$. But

$$E[(W_t - W_s)B_s] = E[(B_t - B_s)B_s] - \int_s^t \frac{E(B_sB_u)}{u}du = -(t-s) \neq 0.$$

**Exercise 5** Let $B$ be a Brownian motion, that we assume to have continuous trajectories *for every* $\omega \in \Omega$. Let

$$A_n := \left\{\sup_{t\in[0,1/n]} B_t > 0\right\} \quad A = \bigcap_n A_n.$$

(a) Show that $P(A_n) \geq \frac{1}{2}$.

(b) Show that $P(A) = 1$ (*hint*: use Blumenthal 0-1 Law)

**Solution.**

Note that

$$P(A_n) \geq P(B_{1/h} > 0) = \frac{1}{2}.$$

The $A_n$'s form a decreasing family of events. Thus, since $A_n \in \mathcal{F}^B_{1/n}$,

$$A = \bigcap_n A_n = \bigcap_{n \geq N} A_n \in \mathcal{F}^B_{1/N}$$

for every $N$, so $A \in \mathcal{F}^B_{0+}$. By the Blumenthal 0-1 Law $P(A) \in \{0,1\}$. But being $P(A) = \lim P(A_n) \geq \frac{1}{2}$, necessarily $P(A) = 1$.

**Exercise 6** Let $B$ be a Brownian motion, and define

$$\mathcal{H}_t := \sigma(B_s : s \geq t) \quad \mathcal{H} = \cap_{t \geq 0} \mathcal{H}_t.$$

Show that if $A \in \mathcal{H}$ then $P(A) \in \{0,1\}$.

**Solution.** Set $W_t := tB_{1/t}$. We know that $W$ is a Brownian motion. Note that

$$\mathcal{H}_t = \sigma(W_s : s \leq 1/t) = \mathcal{F}^W_{1/t},$$

which gives

$$\mathcal{H} = \mathcal{F}^W_{0+}.$$

The conclusion follows from the Blumenthal 0-1 Law.