

Exercises - Set 3

Exercise 1 Let B be a Brownian Motion and $S_t = \sup\{B_s : 0 \leq s \leq t\}$ its running maximum.

- (a) For $a \geq 0$ and $b \in \mathbb{R}$ compute

$$P(S_t \geq a, B_t \geq b).$$

(Hint: use the proof of the reflection principle).

- (b) Show that the pair of real random variables (S_t, B_t) admits a joint density $f(x, y)$, i.e for all $A \in \mathcal{B}(\mathbb{R}^2)$

$$P((S_t, B_t) \in A) = \int_A f(x, y) dx dy,$$

and compute $f(x, y)$.

Soluzione.

- (a) For $b \geq a$

$$P(S_t \geq a, B_t \geq b) = P(B_t \geq b) = P(N(0, 1) \geq b/\sqrt{t}),$$

while for $a < b$ and the reflection principle

$$\begin{aligned} P(S_t \geq a, B_t \geq b) &= P(S_t \geq a) - P(S_t \geq a, B_t < b) \\ &= P(S_t \geq a) - P(S_t \geq a, B_t > 2a - b) = 2P(B_t \geq a) - P(B_t > 2a - b). \end{aligned}$$

- (b) By the above computation, $F(a, b) := P(S_t \geq a, B_t \geq b)$ is of class $\mathcal{C}^2$ except on the line $a = b$, so

$$P(S_t \geq a, B_t \geq b) = \int_{[a, +\infty) \times [b, +\infty)} \frac{\partial^2 F}{\partial x \partial y}(x, y) dx dy.$$

Since the rectangles $[a, +\infty) \times [b, +\infty)$ for a basis for $\mathcal{B}(\mathbb{R}^2)$, it follows that

$$f(x, y) = \frac{\partial^2 F}{\partial x \partial y}(x, y).$$

By point (a), we have

$$f(x, y) = \begin{cases} 0 & \text{for } x \leq y \text{ or } x \leq 0 \\ -\frac{\partial^2}{\partial x \partial y} P(B_t > 2x - y) = \frac{2}{t\sqrt{2\pi t}}(2x - y) \exp\left[-\frac{1}{2t}(2x - y)^2\right] & \text{otherwise.} \end{cases}$$

Exercise 2 Let τ be a $(\mathcal{F}_t)_{t \geq 0}$ stopping time. Let σ be a random time such that $\sigma(\omega) \geq \tau(\omega)$ for all ω , and σ is $\mathcal{F}_\tau$ -measurable. Show that σ is a stopping time.

Soluzione. Since σ is $\mathcal{F}_\tau$ -measurable, then for every t , $\{\sigma \leq t\} \in \mathcal{F}_\tau$. Thus

$$\{\sigma \leq t\} = \{\sigma \leq t\} \cap \{\tau \leq t\} \in \mathcal{F}_t.$$

Exercise 3 Recall the reflection principle: for $a > 0$

$$P(\tau_a \leq t) = P(|B_1| \geq a/\sqrt{t}).$$

- (a) Show that for every $a > 0$, $P(\tau_a < +\infty) = 1$.

- (b) * Deduce that, with probability one, (without using the Law of Iterated Logarithm)

$$\limsup_{t \rightarrow +\infty} B_t = +\infty, \quad \liminf_{t \rightarrow +\infty} B_t = -\infty$$

Soluzione.

(a) Just observe that

$$P(\tau_a = +\infty) = \lim_m P(\tau_a > n) = \lim_n P(|B_1| \leq a/\sqrt{n}) = 0.$$

(b) For $n, m \geq 1$ define

$$\tau_m^{(n)} := \inf\{t \geq n : B_t \geq m\}.$$

We will show that $\tau_m^{(n)} < +\infty$ a.s.. This means that there exists $A_{m,n} \subseteq \Omega$ with $P(A_{m,n} = 1)$ such that $\tau_m^{(n)}(\omega) < +\infty$ for every $\omega \in A_{m,n}$. Thus, setting

$$A := \bigcap_{m,n} A_{m,n},$$

also $P(A) = 1$ and for $\omega \in A$, $B(\omega)$ crosses the level m for arbitrarily large times, so

$$\limsup_{t \rightarrow +\infty} B_t(\omega) \geq m.$$

Since this holds for every m , we have

$$\limsup_{t \rightarrow +\infty} B_t(\omega) = +\infty.$$

The “liminf” case can be obtained by reflection. So we are left to prove that $\tau_m^{(n)} < +\infty$ a.s.. Define $W_t := B_{t+n} - B_n$. W is a BM, and it is independent of B_n . Moreover

$$\tau_m^{(n)} - n := \inf\{t \geq 0 : W_t \geq m + B_n\} =: \tau.$$

So it is enough to show that $P(\tau < +\infty) = 1$. Note that τ is a measurable function of W and B_n , i.e.

$$\mathbf{1}_{\{\tau < +\infty\}} = f(W, B_n).$$

Since W and B_n are independent

$$P(\tau < +\infty) = E[E(f(W, B_n)|B_n)] = E[\varphi(B_n)]$$

with

$$\varphi(b) = E[f(W, b)] = P(\inf\{t \geq 0 : W_t \geq m + b\} < +\infty) = 1$$

for each b , by what seen in (a). Thus the conclusion follows.

Exercise 4 Let B be a Brownian motion. Show that the following processes are martingales for the filtration generated by the Brownian motion:

$$B_t^3 - 3tB_t \quad B_t^4 - 6tB_t^2 + 3t^2$$

Soluzione. Recall that, if $Z \sim N(0, 1)$, $E(Z^3) = 0$ and $E(Z^4) = 3$. Observe that

$$E(B_t^3|\mathcal{F}_s) = E[(B_t - B_s)^3|\mathcal{F}_s] + E[-3B_tB_s^2 + 3B_t^2B_s + B_s^3|\mathcal{F}_s] = B_s^3 + 3(t-s)B_s$$

where we have used the facts that

$$E(B_t|\mathcal{F}_s) = B_s, \quad E(B_t^2|\mathcal{F}_s) = B_s^2 + t - s.$$

It follows immediately that

$$E(B_t^3 - 3tB_t|\mathcal{F}_s) = B_s^3 - 3sB_s.$$

Similarly

$$E(B_t^4|\mathcal{F}_s) = E[(B_t - B_s)^4|\mathcal{F}_s] + E[4B_tB_s^3 - 6B_t^2B_s^2 + 4B_t^3B_s - B_s^4|\mathcal{F}_s] = 3(t-s)^2 + 6(t-s)B_s^2 + B_s^4,$$

where we have also used the above formula for $E(B_t^3|\mathcal{F}_s)$. The fact that

$$E[B_t^4 - 6tB_t^2 + 3t^2|\mathcal{F}_s] = B_s^4 - 6sB_s^2 + 3s^2$$

follows easily.

Exercise 5 Let B be a Brownian motion and $\tau := \inf\{t > 0 : B_t \in \{-a, b\}\}$, with $a, b > 0$. Define, for $\gamma \in \mathbb{R}$,

$$M_t := e^{\gamma B_t - \frac{1}{2}\gamma^2 t}.$$

(a) Show that M is a martingale.

(b) Show that $E[M_\tau] = 1$.

Hint: use the fact that $M_\tau = \lim_n M_{\tau \wedge n}$.

(c) Set $\alpha := E\left(e^{-\frac{1}{2}\gamma^2\tau} \mathbf{1}_{B_\tau = -a}\right)$ and $\beta := E\left(e^{-\frac{1}{2}\gamma^2\tau} \mathbf{1}_{B_\tau = b}\right)$. Show that $\alpha e^{-\gamma a} + \beta e^{\gamma b} = 1$.

(d) Show that

$$E\left(e^{-\frac{1}{2}\gamma^2\tau}\right) = \frac{\sinh(\gamma a) + \sinh(\gamma b)}{\sinh(\gamma(a+b))}.$$

Soluzione.

(a) Recalling that if $X \sim N(0, \sigma^2)$ then

$$E[e^{\gamma X}] = e^{\frac{1}{2}\gamma^2\sigma^2},$$

we have

$$E[e^{\gamma B_t}|\mathcal{F}_s] = e^{\gamma B_s} E[e^{\gamma(B_t - B_s)}|\mathcal{F}_s] = e^{\gamma B_s} E[e^{\gamma(B_t - B_s)}] = e^{\gamma B_s} e^{\frac{1}{2}\gamma^2(t-s)},$$

from which the conclusion easily follows.

(b) By the optional stopping theorem $E(M_{\tau \wedge n}) = 1$ for all n . The conclusion follows observing that $M_{\tau \wedge n} \rightarrow M_\tau$ a.s., $|M_{\tau \wedge n}| \leq e^{|\gamma|(a+b)}$ and dominated convergence.

(c) Just observe that

$$E(M_\tau) = \alpha e^{-\gamma a} + \beta e^{\gamma b}.$$

(d) Note that changing γ in $-\gamma$, α and β remain unchanged, so we also have

$$\alpha e^{\gamma a} + \beta e^{-\gamma b} = 1.$$

Together with the equation obtained in (c), we have a linear system for α and β , that has a unique solution. Solving it and using the fact that

$$\alpha + \beta = E\left(e^{-\frac{1}{2}\gamma^2\tau}\right)$$

the conclusion follows.

Exercise 6 Consider the gambler's ruin problem for an *unfair* game. It can be modeled as follows. Let B be a Brownian motion, $\mu > 0$ and

$$X_t := B_t - \mu t.$$

Moreover, let $\tau := \inf\{t > 0 : X_t \in \{-a, b\}\}$.

(a) Show that X is a supermartingale.

(b) Show that

$$E[e^{2\mu X_\tau}] = 1.$$

(Hint: use the martingale M in Exercise 5 with $\gamma = 2\mu$.)

(c) Let $p := P(X_\tau = -a)$. Show that

$$p = \frac{1 - e^{-2a\mu}}{e^{2b\mu} - e^{-2a\mu}}.$$

(d) Prove that $E[B_\tau] = 0$. Using this compute $E(\tau)$.

Hint. By the Optional sampling Theorem, for every n , $E(B_{\tau \wedge n}) = 0$. Note that

$$|B_{\tau \wedge n}| \leq |X_{\tau \wedge n}| + \mu\tau \wedge n \leq a + b + \mu\tau.$$

Thus the conclusion follows if one can show that $\tau \in L^1$. This can be shown e.g. by showing that $\int_0^{+\infty} P(\tau > t)dt < +\infty$. To estimate $P(\tau > t)$:

$$P(\tau > t) \leq P(|X_t| \leq a + b) \leq P(B_t \geq \mu t - a - b)$$

and this can be estimated by standard bound on Gaussians.

Soluzione.

(a)

$$E(X_t | \mathcal{F}_s) = B_s - \mu t \leq X_s.$$

(b) and (c) By exercise 5,

$$M_t = e^{2\mu B_t - 2\mu^2 t} = e^{2\mu X_t}$$

is a martingale. Since $E(M_{\tau \wedge n}) = 1$, $M_{\tau \wedge n} \rightarrow M_\tau$ a.s. and $|M_{\tau \wedge n}| \leq e^{2\mu(a+b)}$, we have $E(M_\tau) = 1$ so

$$1 = E(M_\tau) = e^{-2\mu a}p + e^{-\mu b}[1 - p]$$

from which one derives p .

(d) $E[B_\tau] = 0$ follows from the hint. Since $X_\tau = B_\tau - \mu\tau$ we have

$$E(\tau) = -\frac{1}{\mu}E(X_\tau),$$

and

$$E(X_\tau) = -ap + b(1 - p),$$

from which $E(\tau)$ can be computed.