RHYTHMIC BEHAVIOR IN LARGE SCALE SYSTEMS: A MODEL RELATED TO MEAN-FIELD GAMES

PAOLO DAI PRA, ELENA SARTORI, AND MARCO TOLOTTI

Abstract. Systems comprised by many interacting components may exhibit, in the thermodynamic limit, time-periodic behavior in some macroscopic observable. When the microscopic dynamics are described by irreducible, continuous-time Markov processes, this is a purely thermodynamic phenomenon, since no periodicity is allowed at the microscopic level. The study of this phenomenon is mainly motivated by its applications to neurosciences, more specifically to systems of interacting neurons, but is occurs in many other contexts, including multi-agent systems in Economics. In this work we suggest a mechanism producing rhythms that is related to strategic behavior of microscopic units, and requires a game-theroretic approach.

1. INTRODUCTION

Recent years have seen a formidable effort in the attempt of explaining rigorously the emergence of collective periodicity in noisy systems of interacting units. Given the impossibility of accounting for the huge related literature, we only mention the inspiring work [9], and few available rigorous results [10, 3, 2, 4]. In these works a key role in the emergence of periodicity is played by *delay* in the information transmission ([10, 4]) and *dissipation* [3, 2].

In the present work we propose a totally different mechanism, in which microscopic units are *agents* whose aim is to maximize an utility function. The state of each agent evolves as a controlled Markov process; since each agent looks for the control maximizing his own utility, it is natural to consider Nash equilibria for the resulting dynamic game. The recent theory of *mean field games* has put forward a class of dynamic games for which the limit behavior, as the number of agents increases to infinity, can be described in analytic terms [8, 6]. In this limit, the solution of the dynamic game is given by a system of two coupled equations: one is the Hamilton-Jacobi-Bellman equation for the value function, the second is the master equation for the optimal evolution of the representative agent. The main aim of this work is to provide a stylized model for the behavior of a network of conformist agents, for which the limit system admits a periodic and non-constant solution. This rhythmic behavior emerges even in absence of external periodic signals, and it is endogenously produced by the strategic behavior of agents. Periodic behavior in mean-field games has been often predicted but, to our knowledge, proved in only one example, the rather celebrated *Mexican wave* model (see [6]). It must be remarked that the Mexican wave model possesses a continuous symmetry, which allows the appearance of traveling wave solution. The model we propose below has a discrete (actually binary) space structure, so there is no continuous symmetry.

2. THE MICROSCOPIC MODEL

Consider a network of N interacting agents, each possessing a binary state $\sigma_i(t) \in \{-1, 1\}$ at time $t \in \mathbb{R}$. We denote by $\mathcal{F}_t$ the σ-field generated by $\{\sigma_i(s) : s \leq t,\, i = 1, \ldots, N\}$. Every agent can control his state by means of the control $u_i = (u_i(t))_{t\geq 0}$, a *progressively measurable* process whose effect on the dynamics is given by

$$\mathbb{P}(\sigma_i(t+h) = -\sigma_i(t)|\mathcal{F}_t) = u_i(t)h + o(h). \tag{1}$$

In other words, $u_i(t)$ is the probability rate of *flipping* the state σ_i. Let

$$m_N(t) := \frac{1}{N}\sum_{i=1}^{N}\sigma_i(t)$$

be the average state of the network at time t. The instantaneous reward of agent i at time t is given by

$$R_i(t) := \sigma_i(t)m_N(t) - \frac{1}{2\mu(\sigma_i(t), m_N(t))}u_i^2(t).$$

The two summands in the reward R_i are easy to interpret. The term $\sigma_i(t)m_N(t)$ favors imitation: agents are conformist, they try to adapt to the majority. The term $-\frac{\mu}{2}u_i^2(t)$ is an *energy* cost: a rapid change of the state would require high values for u_i, which are costly. The factor $\mu(\sigma_i(t), m_N(t))$, that we assume to be nonnegative, modulates the relevance of this cost term: large values of μ allow high mobility to the agents, who can rapidly adapt to a change in the majority. Conversely, small values of μ reduce the adaptive response of agents. We allow μ to depend on the state of agent i and on the average state of the network.
Each agent i aims at maximizing the discounted utility

$$U_i := \mathbb{E}\left[\int_0^{+\infty} e^{-\lambda t}R_i(t)dt\right],$$

where $\lambda > 0$ is a *discount* factor.

A control $u^* = (u_1^*, u_2^*, \ldots, u_N^*)$ is called a *Nash equilibrium* if for every $i = 1, \ldots, N$, assuming that all agents $j \neq i$ use the control u_j^*, we have $U_i(u_i^*) \geq U_i(u_i)$ for every other control u_i: in equilibrium no agent has interest in changing his strategy. Note that this dynamic game is invariant for permutation of agents, so it fall within the domain of *mean-field games* ([7, 8]).

3. THE MACROSCOPIC MODEL

The limit as $N \to +\infty$ of the dynamic game described above it is easy to obtain at a heuristic level. One expects that the average state $m_N(t)$ obeys a *Law of Large Numbers*, so it converges to a deterministic limit $m(t)$. The *representative agent* aims at maximizing

$$J(u) := \mathbb{E}\left[\int_0^{+\infty} e^{-\lambda t}\left(\sigma(t)m(t) - \frac{1}{2\mu(\sigma(t), m(t))}u^2(t)\right)dt\right]. \tag{2}$$

An equilibrium control u^* must satisfy the following consistency relation: if we denote by $\sigma^*(t)$ the process produced by the control u^*, then

$$m(t) = \mathbb{E}[\sigma^*(t)].$$

This problem is solved in two steps: first one writes the Dynamic Programming Equation corresponding to the maximization problem for $J(u)$ *given* $m(t)$; then one imposes that $m(t)$ is consistent with the master equation for the optimal process $\sigma^*(t)$. Denoting by $V(\sigma, t)$ the value function of the control problem of maximizing $J(u)$, the Dynamic Programming Equation reads, defining $\nabla V(\sigma, t) := V(-s, t) - V(\sigma, t)$,

$$-\lambda V(\sigma, t) + \frac{\mu(\sigma, m)}{2}\left[[\nabla V(\sigma, t)]^+\right]^2 + \frac{\partial V}{\partial t}(\sigma, t) + \sigma m(t) = 0, \tag{3}$$

and yields the optimal (feedback) control

$$u^* = \mu(\sigma, m)\,[\nabla V(\sigma, t)]^+.$$

Inserting u^* in (1) one derives a differential equation for $m(t)$. It is convenient to write $\mu(\sigma, m)$ in the form $\mu(\sigma, m) = \sigma a(m) + b(m)$, and set $z(t) := \nabla V(1, t)$. By (3) and (1) we obtain the following system of coupled equations:

$$\begin{cases} \dot{z}(t) = \frac{b(m(t))}{2} z(t)|z(t)| + \frac{a(m(t))}{2} z^2(t) + \lambda z(t) + 2m(t) \\ \dot{m}(t) = -(m(t)b(m(t)) + a(m(t)))|z(t)| - (m(t)a(m(t)) + b(m(t)))z(t) \end{cases} \tag{4}$$

We remark that the rigorous derivation of (4) as limit of the microscopic dynamic games does not follow from standard results, and is the subject of present work. Rigorous convergence results have been obtained recently for diffusion models (see [5, 1]).

Some remarks are needed concerning equation (4). It is relevant to note that equation (4) should not be meant as an initial-value problem: only the initials $m(0)$, i.e. the initial information on agents' proportion, is assigned. On the other hand the value function in this problem is necessarily bounded, so only bounded solutions of (4) matter. Conversely, every bounded solution of (4) determines an equilibrium u^* for the control problem (2).

4. SPECIAL MODELS

In this section we consider two special models, for which we determine the bounded solutions of (4).

4.1. The standard model: $\mu(\sigma, m) = \mu = const$.

In this model the mobility is constant. Equation (4) takes the form:

$$\begin{cases} \dot{z}(t) = \frac{\mu}{2} z(t)|z(t)| + \lambda z(t) + 2m(t) \\ \dot{m}(t) = -\mu m(t)|z(t)| - \mu z(t) \end{cases} \tag{5}$$

We are interested in finding bounded solutions to (5). Note that $(z^*, m^*) = (0, 0)$ is always an equilibrium.

Theorem 4.1. (a) Low mobility regime*: $\mu \leq \frac{\lambda^2}{8}$. For every $m(0) \in [-1, 1]$ equation* (4) *admits a unique bounded solution. For $m(0) \neq 0$ complete* consensus *occurs:* $\lim_{t \to +\infty} m(t) = sign(m(0)) \in \{-1, 1\}$.

(b) High mobility regime*: $\mu > \frac{\lambda^2}{8}$. For $|m(0)| \neq 0$ sufficiently small there is more than one bounded solution to* (4)*. All such solutions reach consensus* ($\lim_{t \to +\infty} m(t) \in \{\pm 1\}$)*, but exhibit a transient oscillatory regime, in which the orbits of the solutions spiral around* $(0, 0)$ *before reaching consensus.*

Thus, in the high mobility regime, the equilibrium control may be not unique: there are equilibrium controls leading to transient oscillating behavior.

Proof of Theorem 4.1. We first observe that (4), besides the origin O, admits two other equilibria P and Q, symmetric with respect to the origin: $\pm \left(\frac{\sqrt{\lambda^2 + 4\mu} - \lambda}{\mu}, -1 \right)$. Linear analysis shows that P and Q are saddle points for all values of the parameters; the origin O is linearly unstable:

- for $\mu \leq \frac{\lambda^2}{8}$ it is repellent, i.e. the eigenvalues of the linearized system are both negative reals;
- for $\mu > \frac{\lambda^2}{8}$ is an unstable spiral, i.e. the eigenvalues of the linearized system have both negative real part but nonzero imaginary part.

I order to perform a global analysis, we first consider the nullcline $\mathcal{N}$ given by the equation $\frac{\mu}{2} z|z| + \lambda z + 2m = 0$. Off the nullcline, solutions to (5) have trajectories that are locally graphs of a function $m = m(z)$. By implicit differentiation, assuming $(z, m) \in [0, +\infty) \times [-1, 1]$, one checks that $m''(z) > 0$ if and only if $\varphi^-(z) < m < \varphi^+(z)$, with

$$\varphi^{\pm}(z) = -\frac{z}{4}\left[\lambda \mp \sqrt{\lambda^2 - 8\mu + 6\lambda\mu z + 4\mu^2 z^2}\right].$$

For $(z, m) \in (-\infty, 0) \times [-1, 1]$, similar convexity conditions are obtained by reflection w.r.t. the origin.

Consider the fixed point Q and its stable manifold $\mathcal{M}_s$, i.e. the trajectory of a solution of (5) converging to Q.

Low mobility regime: $\mu \leq \frac{\lambda^2}{8}$. In this case the graphs of φ^+ and φ^- meet at the origin (see Fig. 1). Moreover, tha graph of φ^- meets the nullcline $\mathcal{N}$ at the equilibrium point Q. A linear analysis at Q and the study of the direction of the vector field of (5) at the points of the graph of φ^- show that $\mathcal{M}_s$ is at the left of the graph of φ^-. In particular $\mathcal{M}_s$ is concave, so it cannot intersect the nullcline $\mathcal{N}$, that can be intersected only vertically by a solution of (5). It follows that $\mathcal{M}_s$ is within the area between $\mathcal{N}$ and the graph of φ^-. Since the origin is stable for the time-reversal of (5), necessarily $\mathcal{M}_s$ joins the origin with Q. Moreover, in the area between $\mathcal{N}$ and the graph of φ^-, it easily checked that $\frac{dm}{dz} = \frac{\dot{m}}{\dot{z}} < 0$, so it is the graph of a strictly decreasing function. Thus, for every $m_0 \in (-1, 0)$, there is a unique point of $\mathcal{M}_s$ with $m = m_0$, which is the starting point of a solution of (5) converging to Q; in particular $m(t) \to -1$ as $t \to +\infty$. It is actually the only bounded solution starting from a point of the form (m_0, z). This can be seen as follows. The point (m_0, z), with $m_0 < 0$, cannot belong to the stable manifold of P, which is ite image of $\mathcal{M}_s$ under reflection w.r.t the origin. Thus the solution starting from (m_0, z) cannot converge to any fixed point. Moreover, since the divergence of the vector field driving (5) is constantly equal to $\lambda > 0$, then periodic orbits are not allowed. Thus, by the Theorem of Poicaré-Bendixon, the solution starting from (m_0, z) must be unbounded.

High mobility regime: $\mu > \frac{\lambda^2}{8}$. In this case the graphs of φ^+ and φ^- do not reach the origin (see Fig. 2). As in the low mobility regime, the stable manifold $\mathcal{M}_s$, as departing from Q, forms a concave curve between $\mathcal{N}$ and the graph of φ^-. If we show that $\mathcal{M}_s$ gets arbitrarily close to the origin then the previous linear analysis implies that it must spiral around the origin, in particular it is not that graph of an injective function.

Thus we are left to show that $\mathcal{M}_s$ gets arbitrarily close to the origin. This amounts to show that the solution $(\hat{z}(t), \hat{m}(t))$ of the time-reversed system starting from a point in $\mathcal{M}_s$ close to Q, converges to the origin as $t \to +\infty$. Due to the spiraling around the origin, $(\hat{z}(t), \hat{m}(t))$ cannot converge to the origin following the graph of a monotone function. Thus it must intersect first the positive z-axis and then the positive m axis at some $m^* > 0$. Suppose $m^* < 1$. Note that $\mathcal{M}_s$ intersects the m-axis horizontally, so, again by convexity, after having touched $(0, m^*)$ it continues downward. Since $\mathcal{M}_s$, in the half-plane $z < 0$ cannot touch the stable manifold of P, it follows it is trapped in a bounded region. Due to the absence of periodic orbits, necessarily $(\hat{z}(t), \hat{m}(t)) \to (0, 0)$ as $t \to +\infty$.

Finally, we need to show that $m^* < 1$. By continuity from the low mobility regime, this is certainly true for $\mu - \frac{\lambda^2}{8}$ sufficiently small. If our claim is false, then there must be a value of μ for which $m^* = m^*(\mu) = 1$. In this situation, $\mathcal{M}_s$ continuous horizontally up to P. It follows that the union of $\mathcal{M}_s$ with the stable manifold of P form a closed curve, tangent to the vector field driving (5); this is impossible by the Divergence Theorem.

□

4.2. **Introducing crowding effects.** Here we set $\mu(\sigma, m) := \mu(1 + \epsilon\sigma m)$, for some $\mu > 0$ and $\epsilon \in [0, 1]$: changing state is more costly for an agent belonging to the minority. Equation (4)

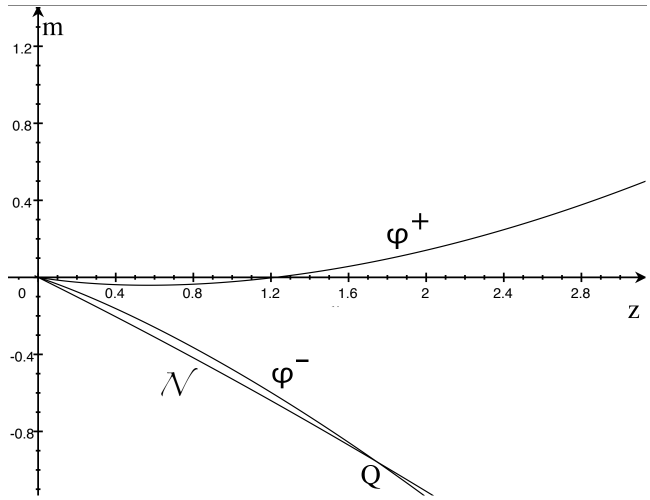

FIGURE 1. Low mobility regime

becomes:

$$\begin{cases} \dot{z}(t) = \frac{\mu}{2} z(t)|z(t)| + \frac{\mu \epsilon m}{2} z^2(t) + \lambda z(t) + 2m(t) \\ \dot{m}(t) = -(1+\epsilon)\mu m(t)|z(t)| - \mu(1+\epsilon m^2(t))z(t) \end{cases} \tag{6}$$

For the following result we do not have a full proof yet. The numerical evidence appears solid, however, and leads to a rather clear picture of the bifurcations.

Theorem 4.2. *There exists $\hat{\mu}$, with $\frac{\lambda^2}{8} < \hat{\mu} < +\infty$ such that the following statements hold.*

(a) Low mobility regime*: $\mu \le \frac{\lambda^2}{8}$. For every $m(0) \in [-1,1]$ equation (4) admits a unique bounded solution. For $m(0) \neq 0$ complete* consensus *occurs:* $\lim_{t\to+\infty} m(t) = sign(m(0)) \in \{-1,1\}$.

(b) Moderate mobility regime*: $\frac{\lambda^2}{8} < \mu \le \hat{\mu}$. For $|m(0)| \neq 0$ sufficiently small there is more than one bounded solution to (4). All such solutions reach consensus ($\lim_{t\to+\infty} m(t) \in \{\pm 1\}$), but, for $|m(0)|$ small enough, they exhibit a transient oscillatory regime, in which the orbits of the solutions spiral around $(0,0)$ before reaching consensus.*

(c) High mobility regime*: $\mu > \hat{\mu}$. For every $m(0) \in [-1,1]$ equation (4) admits two bounded solutions leading to consensus:* $\lim_{t\to+\infty} m(t) \in \{-1,1\}$*. Moreover (6) admits a unique non-constant periodic orbit: thus, for $|m(0)|$ sufficiently small, there are two periodic solutions which differ for a time shift.*

We remark that in the high mobility regime there is an equilibrium control leading to *permanent* oscillatory behavior.

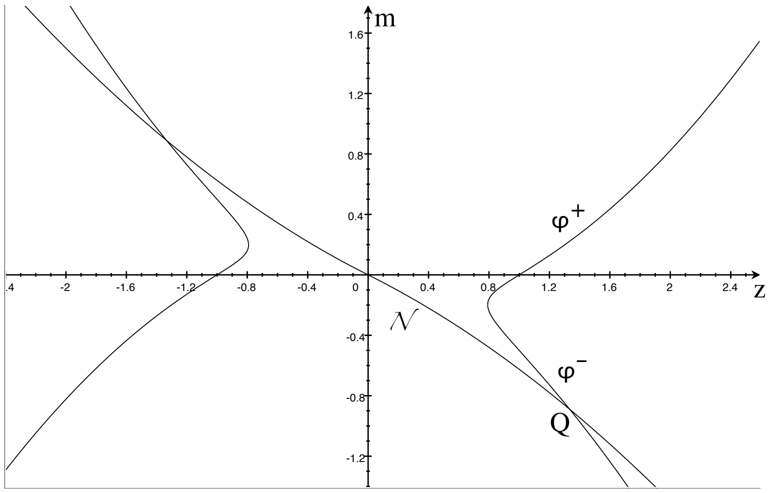

FIGURE 2. High mobility regime

In order to illustrate the bifurcations stated in Theorem 4.2, we first observe that system (6) has three equilibria: the origin O, whose linear properties are identical to those of the standard model treated in the previous subsection, and the points P and Q with coordinates $\pm\left(-\frac{2}{\lambda}, 1\right)$. Both P and Q are easily seen to be saddle points, for all values of the parameters. In Figures 3 and 4 we plot the stable manifolds of P and Q: exactly as for the two phases of the standard model, these manifolds are graphs of a monotone function in the low mobility regime (Fig. 3), while they spiral around the origin in the moderate mobility regime (Fig. 4). What fails here is that the divergence of the driving vector field is not of constant sign, so that limit cycles cannot be ruled out. Numerics suggest that the m coordinate of the first intersection of the stable manifold of Q with the m-axis is increasing in μ, and it equals 1 at some $\mu = \hat{\mu}$. Then, the manifold continues horizontally to reach P (Fig. 5). Thus, by symmetry, the two stable manifolds join to form a *separatrix*. By increasing μ further, a periodic orbit bifurcates from the separatrix through a *homoclinic bifurcation* (Fig. 6).

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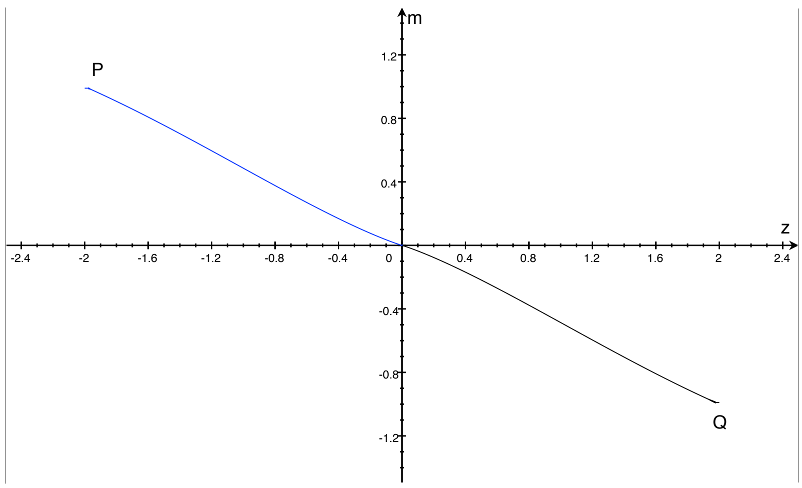

Figure 3. Stable manifolds in the low mobility regime

E-mail address: daipra@math.unipd.it

Dipartimento di Matematica Pura ed Applicata, Università degli Studi di Padova, via Trieste 63, I-35121 Padova, Italy

E-mail address: esartori@math.unipd.it

Dipartimento di Matematica Pura ed Applicata, Università degli Studi di Padova, via Trieste 63, I-35121 Padova, Italy

E-mail address: tolotti@unive.it

Department of Management, Ca' Foscari University of Venice, Cannaregio 873 - 30121 Venezia, Italy.

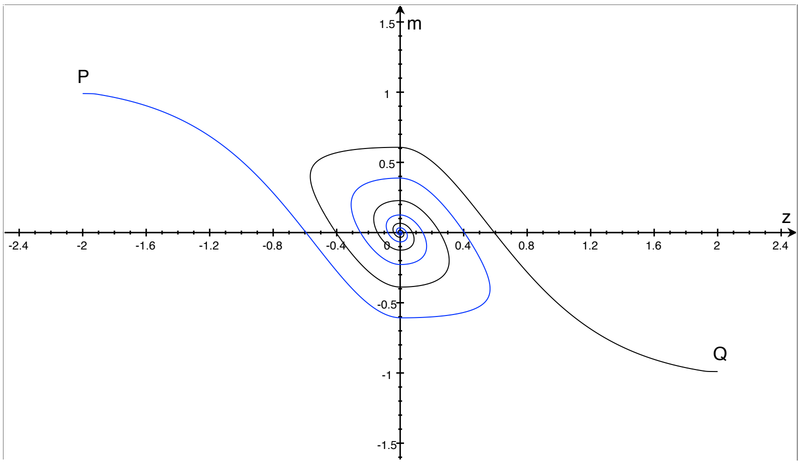

FIGURE 4. Stable manifolds in the moderate mobility regime

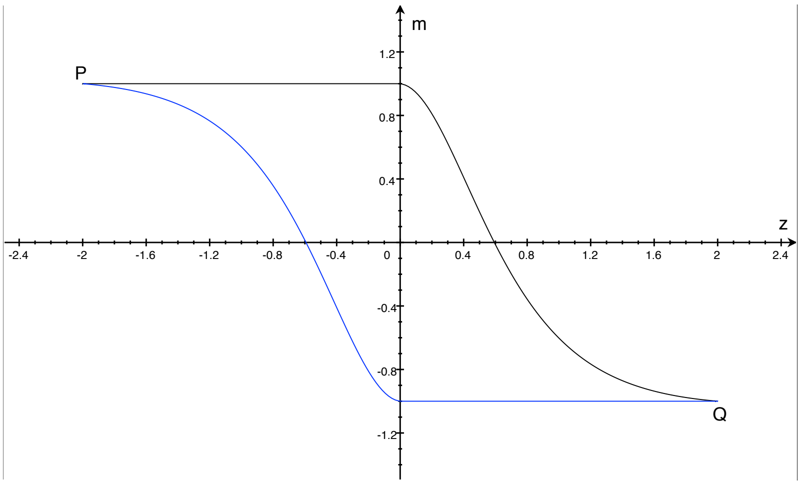

FIGURE 5. Separatrix at $\mu = \hat{\mu}$

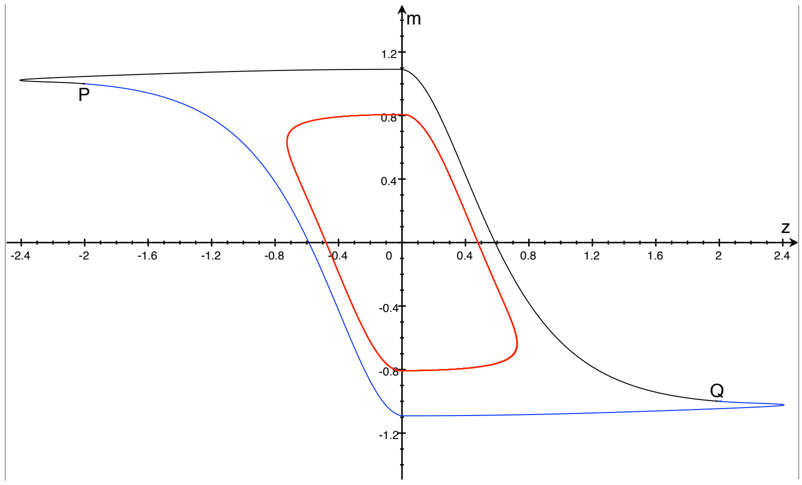

FIGURE 6. Stable manifolds and periodic orbit in the high mobility regime