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Lab exercises

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1 Solution of linear systems by gaussian elimination

1.1 Elimination

Given the linear system $Ax = b$, the Matlab code for the **elimination step** of Gauss elimination method is as follows

```
for i=1:n-1,
  for j=i+1:n,
     m=A(j,i)/A(i,i);
     for k=i:n,
       A(j,k)=A(j,k)-m*A(i,k);
     end
   b(j)=b(j)-m*b(i);
   end
end
```

the result is an upper triangular matrix, stored again on matrix `A` and a new right hand side (r.h.s.) vector, again stored on `b`.

1.2 Back substitution

The algorithm for back substitution of a linear system $Ux = b$, with U upper triangular matrix, can be written in Matlab as in Table 1. The lines

```
for j = i+1:n
  x(i) = x(i)-U(i,j)*x(j);
end
x(i) = x(i)/U(i,i);
```

using matrix operators, can be substituted by the following ones

```
x(i) = (x(i)-U(i,i+1:n)*x(i+1:n))/U(i,i);
```

where, in this case, the operator `*` is the scalar product of vectors.

```
function x = BS(U,b)
% x = BS(U,b)
n = length(b);
x = b;
x(n) = x(n)/U(n,n);
for i = n-1:-1:1
 for j =i+1:n
  x(i) = x(i)-U(i,j)*x(j);
 end
x(i) = x(i)/U(i,i);
end
```

Table 1: Back substitution

We can generalized to the case of more r.h.s. terms $b_1, b_2, \ldots, b_m$, that is

$$Ux_i = b_i \,. \quad i = 1, 2, \ldots, m \,.$$

Hence, the function in Table 2, can then be used to solve the systems

$$UX = B$$

where U is upper triangular, X is the matrix of the vectors x_i and B the matrix of all r.h.s. b_i, $i = 1, \ldots, m$.

Exercises

Solve the following problem in Matlab

1. Take the `Hilbert matrix` of order n, in Matlab is `H=hilb(n)`, with n chosen by the user, and the linear system $Hx = b$. The vector `b` is taken so that the solution is $x = (1, ..., 1)^T$
   - Find the diagonal vector `d`, the upper triangular `U` and lower triangular `L` matrices of H. What do you see?
   - Using the vector `d`, do the command `D=diag(d)`: what do you see?
   - Solve the system $H\,x = b$ by using Matlab elementary functions.

```
function X = BSS(U,B)
%
% X = BSS(U,B)
%
n = length(B);
X = B;
X(n,:) = X(n,:)./U(n,n);
for i = n-1:-1:1
  X(i,:)=(X(i,:)-U(i,i+1:n)*X(i+1:n,:))/U(i,i);
end
```

Table 2: Back substitution for systems

- Perturb the vector b by the vector $\delta b = (0, \ldots, 0, 1.e-4)^T$. Solve the new system $H\hat{x} = b + \delta\, b$ whose solution is $\hat{(x)}$.

  Estimate the `2-norm condition number`, $\kappa_2(H)$ of H by means of the relation

$$\frac{\|x - \hat{x}\|}{\|x\|} \leq \kappa_2(H) \frac{\|\delta b\|}{\|b\|} \,.$$

  Compare κ_2 with `cond(H)`.

Notice: given a vector `v`, $\|v\|$ is computed by the command `norm(v)` which gives by default the 2-norm of `v`.

2. Consider the linear system $Ax = b$ with `A = toeplitz([4 1 0 0 0 0])` and b chosen so that the exact solution is $x = [2, 2, 2, 2, 2, 2]^T$. Solve it by the *gaussian elimination*. In the gaussian elimination, do you need to apply the pivoting?

3. Solve by gaussian elimination, the system $Ax = b$ with

$$A = \begin{bmatrix} 8 & 1 & 2 & 0.5 & 2 \\ 1 & 0.5 & 0 & 0 & 0 \\ 2 & 0 & 4 & 0 & 0 \\ 0.5 & 0 & 0 & 7 & 0 \\ 2 & 0 & 0 & 0 & 16 \end{bmatrix} .$$

   and b chosen so that `x=[0 0 1 1 1]'`.

4. Solve, by means of gaussian elimination and back substitution, the linear systems

$$A_i x_i = b_i, \quad A_i = (A_1)^i, \quad i = 1, 2, 3, 4$$

with

$$A_1 = \begin{pmatrix} 15 & 6 & 8 & 11 \\ 6 & 6 & 5 & 3 \\ 8 & 5 & 7 & 6 \\ 11 & 3 & 6 & 9 \end{pmatrix} \tag{1}$$

and b_i chosen so that the corresponding solution is $x_i = [1, 1, 1, 1]^{\mathrm{T}}$. Moreover, for every system, compute the error in 2-norm and the condition number of the matrix (using the built-in function `cond`). Then, make a log-log plot (using the command `loglog`) with in abscissas the errors and in ordinates the condition numbers.

Time: **2 hours**.