Lab exercises

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Padova, June 8, 2015

1 Trapezoidal formula and Simpson's formula

For the approximation of definite integrals, we saw two important methods: the *trapezoids formula* and the *Simpson's formula*. Both are *interpolatory* formulas.

We have studied their *simple* expressions and the *composite or generalized* ones.

If we call by $I(f) = \int_a^b f(x)dx$, and $I_T(f)$, $I_T^c(f)$, $I_S(f)$ and $I_S^c(f)$ the corresponding simple and composite formulas, we have

- `Simple trapezoidal formula and its error.`

$$I_T(f) = \frac{b-a}{2}\left(f(a) + f(b)\right) ,$$

$$E_T(f) = I(f) - I_T(f) = -\frac{(b-a)^3}{12} f''(\xi) , \quad \xi \in (a,b).$$

- `Trapezoidal formula and its error.` Here we must take an equispaced subdivision of $[a,b]$ such as $\{x_0 = a, \ldots, x_i = a + ih, \ldots, x_n = b\}$, with $h = (b-a)/n$:

$$I_T^c(f) = \frac{h}{2}\left(f(a) + f(b)\right) + h\sum_{i=1}^{n-1} f(x_i) ,$$

$$E_T^c(f) = I(f) - I_T^c(f) = -\frac{(b-a)^3}{12n^2} f''(\xi) , \quad \xi \in (a,b).$$

- `Simple Simpson's formula and its error.` The quadrature points are $x_0 = a$, $x_1 = \frac{a+b}{2}$ and $x_2 = b$, i.e. with $h = (b-a)/2$. The integral is then approximated as

$$I_S(f) = \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] ,$$

  with error

$$E_S(f) = I(f) - I_S(f) = -\frac{1}{90}\frac{(b-a)^5}{32} f^{(4)}(\xi), \quad \xi \in (a,b)\, , \; h = (b-a)/2\, .$$

- `Composite Simpson's formula and its error.` Here we take, as in the case of the trapezoids formula, an equispaced subdivision of $[a,b]$. Hence in the general interval $I_k = [x_{k-1}, x_k]$, we consider the points x_{k-1}, $\bar{x}_k = \frac{x_{k-1}+x_k}{2}$ and x_k. Setting $h = (b-a)/(2n)$, this is equivalent to take the points $2n+1$ points $a = z_0$, z_1, z_2, $\ldots$, $z_{2n} = b$.

we have:

$$I_S^c(f) = \frac{h}{3}(f(z_0) + f(z_{2n})) + \frac{h}{3}\left[4 \sum_{k=1}^{2n-1''} f(z_k) + 2 \sum_{k=2}^{2n-2''} f(z_k)\right] ,$$

where the $''$ indicate that the sum is done either on the odd or the even indeces, respectively

$$E_S^c(f) = I(f) - I_S^c(f) = -\frac{(b-a)^5}{2880n^4} f^{(4)}(\xi), \quad \xi \in (a,b) .$$

Solve the following problems in Matlab

$\boxed{1.}$ Write the M-functions that implements the simple and composite trapezoidal and Simposon formulas.

$\boxed{2.}$ Compute the approximation of

$$\int_{-1}^{1} \frac{x}{2} \mathtt{e}^{-\frac{x}{2}} \cos(x)\, dx \tag{1}$$

by means of the simple Simpson's formula and calculate the absolute error. As exact value, take the output of the Matlab function

`quadl(inline('x/2.*exp(-x/2).*cos(x)'),-1,1)`

which is approximately -0.122.

$\boxed{3.}$ Compute numerically

$$\int_0^{2\pi} x\mathrm{e}^{-x} \cos(2x)\, dx = \frac{3(\mathrm{e}^{-2\pi} - 1) - 10\pi\mathrm{e}^{-2\pi}}{25} \approx -0.12212260462 ,$$

by means of **composite** formulas of trapezoids and Simpson,

(a) Find *a priori* the number of quadrature points so that the absolute errors $E_T^c(f)$ and $E_S^c(f)$ be in modulus less than $\mathtt{tol} = \mathtt{1.e-6}$.

(b) Determine also the absolute error with respect to the exact value of the integral.

Time: **2 hours**.