Degree in Mechanical Engineering - Lab exercises

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1 Solution of linear systems with iterative methods

1. Write the M-functions

   `function [x, iter, err] = Jacobi(A,b,tol,maxit)`

   `function [x, iter, err] = GaussSeidel(A,b,tol,maxit)`

   that implement the Jacobi and Gauss-Seidel methods, respectively. Define the initial solution `x0=zeros(length(b),1)` in the body of the functions.

   The output parameters are: `x` the solution vector, `iter` the number of iterations and `err` the vector of *relative errors* among iterations.

2. Consider the linear system `Ax=b` with `A = toeplitz([4 1 0 0 0 0])` and `b` chosen so that the exact solution of the system is `x=[2,2,2,2,2,2]'`. As initial vector, take `x0=zeros(6,1)`. Make also a plot, using `semilogy`, of the error.

   Do the same for the Gauss-Seidel iteration.

3. Take $n = 10$ and build the tridiagonal matrix

   `A=diag(ones(n,1)*10)+diag(ones(n-1,1)*3,+1)+diag(ones(n-1,1)*3,-1)`

   and the vector `b` so that the solution is `x=[ones(9,1); 0]`.

   (a) Why do the Jacobi and Gauss-Seidel iterations converge for the solution of $Ax = b$.

   (b) Consider the Jacobi matrix J (which has for any natural norm $\|J\| < 1$) and the tolerance $\epsilon = 1.e - 9$, by using this inequality

   $$\frac{\|P\|^k}{1 - \|P\|}\|x^1 - x^0\| < \epsilon \tag{1}$$

   where P is the iteration matrix, compute a priori the minimum number k_{min} of iterations necessary to solve the linear system $Ax = b$ with the Jacobi method, $\epsilon = 1.e - 9$, `x0=zeros(n,1)`, `maxit=50` and $\|\cdot\|_2$.

   (c) Do the same for the Gauss-Seidel iteration.

4. Construct the matrix `A=pentadiag(-1,-1,`α`, -1,-1)` for $n = 10$, $\alpha \in [0.5, 1.5]$, i.e. by using `A=toeplitz([-1 -1`α`, -1,-1, 0,0,0,0,0])`. Decompose it as $A = M + D + N$ with `D=diag(`$[\alpha - 1, \ldots, \alpha - 1]$`)`, `M=pentadiag(-1,-1,1,0, 0,0)` and $N = A - D - M$.

   (a) For which α^* the iterative method $(M + N)x^{(k+1)} = -Dx^{(k)} + q$ results converging faster?

(b) Letting `b=(1:n)'`, compute by the previous iterative method, the solution of `Ax=b` with `x0=[ ones(m,1); zeros(n-m,1)]`, with $m < n$. Use `tol=1.e-6, maxit=50`.

Time: **2 hours**.

1.1 Home works

1. Consider the matrix `A=diag(1:n)`, for some chosen `n`.

   Study the iteration
   $$x^{(k+1)} = (I - \theta A)x^{(k)} + \theta b, \quad k \geq 0\,.$$

   (a) For which $\theta \in [0, 2/3]$ does the method converge? (*Hint: plot* $\rho(\theta) = \rho(I - \theta A)$).

   (b) Let $\theta_0 = \min_{0 \leq \theta \leq 2/3} \rho(\theta)$ and let `b` be such that `x=ones(n,1)`. Solve the linear system with the iteration above for θ_0, `x0=zeros(n,1), tol=1.e-6, maxit=100`.

2. Consider the random values `x=rand(20,1)` and stretch them to $[0, 5]$. Call this vector again `x`. The corresponding values along y are defined as follows

$$y_i = x_i^2, \quad for\ i\ odd \tag{2}$$
$$y_i = x_i^2 + 0.5, \quad for\ i\ even. \tag{3}$$

   Find the polynomial of degree 2 that approximates the couples (x_i, y_i), $i = 1, ..., 20$ in the least-squares sense. Try then with a polynomial of degree 3