Lab exercises

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1 Recalls about trapezoidal and Simpson's formulas

For the approximation of definite integrals, we saw two important methods: the *trapezoidal formula* and the *Simpson's formula*. Both are *interpolatory* formulas of Newton-Côtes.

We have studied their *simple* expressions and the *composite or generalized* ones.

Let $I(f) = \int_a^b f(x)dx$ be the integral we want to compute.

Letting $I_T(f)$, $I_T^c(f)$ and $I_S(f)$, $I_S^c(f)$ the corresponding *simple* and *composite* formulas, we have

- `Simple trapezoids formula and its error.`

$$I_T(f) = \frac{b-a}{2}\left(f(a) + f(b)\right) \ , \tag{1}$$

$$E_T(f) = I(f) - I_T(f) = -\frac{(b-a)^3}{12} f''(\xi) \ , \ \ \xi \in (a, b).$$

- `Trapezoidal formula and its error.`

  Here we must take an equispaced subdivision of $[a, b]$ such as $\{x_0 = a, \dots, x_i = a + ih, \dots, x_n = b\}$, with $h = (b-a)/n$:

$$I_T^c(f) = \frac{h}{2}\left(f(a) + f(b)\right) + h\sum_{i=1}^{n-1} f(x_i) \ , \tag{2}$$

$$E_T^c(f) = I(f) - I_T^c(f) = -\frac{(b-a)^3}{12n^2} f''(\xi) \ , \ \ \xi \in (a, b).$$

- `Simple Simpson's formula and its error.`

$$I_S(f) = \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] \ , \tag{3}$$

$$E_S(f) = I(f) - I_S(f) = -\frac{1}{24}\frac{(b-a)^5}{32} f^{(4)}(\xi), \ \ \xi \in (a, b) \, , \ h = (b-a)/2 \, .$$

- `Composite Simpson's formula and its error.`

  Here we take, as in the case of the trapezoids formula, an equispaced subdivision of $[a, b]$. Hence in the general interval $I_k = [x_{k-1}, x_k]$, we consider the points x_{k-1}, $\bar{x}_k = \frac{x_{k-1}+x_k}{2}$

and x_k. Setting $h = (b-a)/(2n)$, this is equivalent to take the points $2n+1$ points $a = z_0,\, z_1,\, z_2,\, \ldots,\, z_{2n} = b$.

we have:

$$I_S^c(f) = \frac{h}{3}(f(z_0) + f(z_{2n})) + \frac{h}{3}\left[4\sum_{k=1}^{2n-1''} f(z_k) + 2\sum_{k=2}^{2n-2''} f(z_k)\right] , \tag{4}$$

where the $''$ indicate that the sum is done either on the odd or the even indeces, respectively

$$E_S^c(f) = I(f) - I_S^c(f) = -\frac{(b-a)^5}{2880n^4} f^{(4)}(\xi), \;\; \xi \in (a,b) \, .$$

2 Proposed exercises

$\boxed{1.}$ Write the M-function that implements the trapezoidal formula (2). **Notice** that for $n = 1$ it gives the formula (1).

$\boxed{2.}$ Write the M-function that implements the composite Simposon formula (4). Again **notice** that for $n = 1$ it gives the formula (3).

$\boxed{3.}$ Compute the approximation of

$$\int_{-1}^{1} \frac{x}{2} \mathtt{e}^{-\frac{x}{2}} \cos(x)\, dx \tag{5}$$

by means of the simple Simpson's formula and calculate the absolute error. As exact value, take the output of the Matlab function

```
quadl(inline('x/2.*exp(-x/2).*cos(x)'),-1,1,1.e-6)
```

$\boxed{4.}$ Compute numerically

$$\int_0^{2\pi} x\mathrm{e}^{-x} \cos(2x)\, dx = \frac{3(\mathrm{e}^{-2\pi} - 1) - 10\pi\mathrm{e}^{-2\pi}}{25} \approx -0.12212260462 \, ,$$

by means of **composite** formulas of trapezoids and Simpson,

(a) Find *a priori* the number of quadrature points so that the absolute errors $E_T^c(f)$ and $E_S^c(f)$ be in modulus less than $\mathtt{tol} = \mathtt{1.e-6}$. (*Hint*: find the max absolute value of $f^{(2)}$ and $f^{(4)}$ by using a fine discretization of the interval $[0, 2\pi]$)

(b) Determine also the absolute error with respect to the exact value of the integral.

Time: **2 hours**.