Zeros of functions with Matlab: Bisection method and fixed point iteration

Emma Perracchione

Corso di Calcolo Numerico per Ingegneria Meccanica - Matr. PARI (Univ. PD)

A.A. 2017/2018, secondo semestre

20 Marzo 2018

Introduction

- Finding the zeros of a function means finding:

$$\bar{x}: \quad f(\bar{x}) = 0.$$

- In the lecture we will see two approaches:

  1. bisection method,
  2. fixed-point iteration.

- The first one is the simplest approach. It divides iteratively the searching interval into two subsets with respect to the midpoint.
- The second one is based on building $\{x_k\}_{k\geq 1}$ converging to the fixed-point of an auxiliary function $g(x) = x - f(x)$.

Bisection

Property

Given a continuous function $f : [a, b] \longrightarrow \mathbb{R}$ *such that* $f(a)f(b) < 0$ *then* $\exists \ \bar{x} \in (a, b)$ *so that* $f(\bar{x}) = 0$.

- Thus, at the first step of the bisection method, we evaluate the mid point m_1 of $[a, b] = [a_1, b_1]$. If $f(m_1)f(a) < 0$ then

$$[a_2, b_2] = [a_1, m_1],$$

  else

$$[a_2, b_2] = [m_1, b_1].$$

- We proceed in this way until a stopping criterion is satisfied, such as $f(m_k) < \tau$, for a fixed tolerance τ, at a certain step k.

Fixed point

We want to find $\bar{x}$ so that $f(\bar{x}) = 0$ or equivalently the fixed point of $g(x) = x$, for $f(x) = x - g(x) = 0$.

Theorem

Let us consider $x^{(k+1)} = g(x^{(k)})$, *for* $k \geq 0$, *with* $x^{(0)}$ *given. If*

1. $g : [a, b] \rightarrow [a, b]$;
2. $g \in C^1([a, b])$;
3. $\exists K < 1$ *such that* $|g^{'}(x)| < K \; \forall x \in [a, b]$;

then g *has a unique fixed point* $\bar{x}$ *in* $[a, b]$ *and* $\{x^{(k)}\}_{k>1}$ *converges to* $x^{(0)} \in [a, b]$.

Built-in routines: roots, fzero and fsolve.

1. `roots` computes the zeros (also complex ones) of polynomials. Call it as `roots(a)`, with a the vector of the polynomial coefficients from the coefficient of higher order to that of the constant term.
2. `fzero` can be called in the following way: `x=fzero(fun,x0,opt)` with `fun` specified using the symbol `@`. For example,

   ```
   f = @(x,c) sin(x^3/c); x0 =2; sol = fzero(f,x0,[],9),
   ```

   here 9 is the value of the parameter `c` and `[]` is for the options.
3. `fsolve` is more general since it works on systems of non linear equations. A typical call is `x = fsolve(fun,x0)` where fun can be specified using `@`. For example,

   ```
   x = fsolve(@myfun,[2 3 4]),
   ```

   where `myfun` is a Matlab function such as:

   ```
   function F = myfun(x)
   F = sin(x);
   ```

   Note that the initial point is specified as a real vector or real array.

Exercise 1

Exercise

Write the functions `bisection.m` *and* `fixedpoint.m` *for computing the zeros of a functions with the bisection's method and fixed point iteration.*

Hint

Use an exit criterion in case the methods do not converge.
For the bisection method the inputs that we need are `f,a,b,tol,maxiter`*, while* `fixedpoint.m` *requires* `g,x0,tol,maxiter`*.*
Fix the stopping criterion for the bisection as `abs(f(c))<tol`*, where* `c` *is the midpoint of the k-th interval.*
Fix the stopping criterion for the fixed point as `abs(x1-x0)<tol`*, where* `x1,x0` *are two successive iterations.*

Exercise 2

Exercise

Take the function $f(x) = x^2 - \log(x^2 + 2)$ *of which we want to compute its zeros.*

- *By plotting the graph of* f*, individuates the two real roots of* $f(x) = 0$ *and the corresponding separation intervals,* I_1 *e* I_2*.*
- *Find two convergent iterative methods, with iteration functions* $g_i(x), \;\; i = 1, 2$*. For each one of them determine the number of necessary iterations. Consider* $k_{max} = 50$*, as max number of iterations and, for the test on the relative error* $tol = 1.e - 6$*.*
- *Compare the results with the ones obtained with fzero.*

Exercise 3

Exercise

Using a suitable iteration function, compare the bisection method, and the fixed point iteration method for computing the only real root of

$$1 = \frac{g}{2x^2}(\sinh(x) - \sin(x)),$$

$g = 9.81$. *Take* $k_{max} = 50$ *and the tolerance* $tol = 1.e-06$ *for the relative error.*

Hint

Remark that

$$\sinh(x) = \frac{e^x - e^{-x}}{2},$$

and

$$\mathrm{arsinh} x = \ln\left(x + \sqrt{x^2+1}\right).$$