Degree in Mechanical Engineering - Lab exercises

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1 Solution of linear systems by gaussian elimination

- **Elimination**

  Given the linear system $Ax = b$, the Matlab code for the **elimination step** of Gauss elimination method is as follows

```
for i=1:n-1,
  for j=i+1:n,
     m=A(j,i)/A(i,i); %multiplier
%------------------------------------
      A(j,i:n)=A(j,i:n)-m*A(i,i:n); %row elimination
%------------------------------------
   b(j)=b(j)-m*b(i);
   end
end
```

  the result is an upper triangular matrix, stored again on matrix `A` and a new right hand side (r.h.s.) vector, again stored on `b`.

- **Back substitution**

  The algorithm for back substitution of a linear system $Ux = b$, with U upper triangular matrix, can be written in Matlab as in Table 1. The lines

```
function x = BS(U,b)
% x = BS(U,b)
n = length(b);
x = b;
x(n) = x(n)/U(n,n);
for i = n-1:-1:1
 for j =i+1:n
  x(i) = x(i)-U(i,j)*x(j);
 end
x(i) = x(i)/U(i,i);
end
```

Table 1: Back substitution

```
for j = i+1:n
  x(i) = x(i)-U(i,j)*x(j);
end
x(i) = x(i)/U(i,i);
```

using matrix operators, can be substituted by the following ones

```
x(i) = (x(i)-U(i,i+1:n)*x(i+1:n))/U(i,i);
```

where, in this case, the operator * is the scalar product of vectors.

1.1 Exercises

- Consider the linear system $Ax = b$ with `A = toeplitz([4 1 0 0 0 0])` and b chosen so that the exact solution is $x = [2, 2, 2, 2, 2, 2]^T$. Solve it by the *gaussian elimination*. In the gaussian elimination, do you need to apply the pivoting?

- Solve by gaussian elimination, the system $Ax = b$ with

$$A = \begin{bmatrix} 8 & 1 & 2 & 0.5 & 2 \\ 1 & 0.5 & 0 & 0 & 0 \\ 2 & 0 & 4 & 0 & 0 \\ 0.5 & 0 & 0 & 7 & 0 \\ 2 & 0 & 0 & 0 & 16 \end{bmatrix}.$$

  and b chosen so that `x=[0 0 1 1 1]'`.

2 Solution of linear systems with iterative methods

Write the M-functions

(i) `function [x, iter, err] = Jacobi(A,b,tol,maxit)`

(ii) `function [x, iter, err] = GaussSeidel(A,b,tol,maxit)`

that implement the Jacobi and Gauss-Seidel iterative methods, respectively.
The output parameters are: `x` the solution vector, `iter` the number of iterations and `err` the vector of *relative errors* among iterations.

In the body of the functions define

- the initial solution `x0=zeros(length(b),1)` .
- the matrices `J` and `GS` of Jacobi and Gauss-Seidel.

2.1 Exercises

1. Consider the linear system `Ax=b` with `A = toeplitz([4 1 0 0 0 0])` and `b` chosen so that the exact solution of the system is `x=[2,2,2,2,2,2]'`.

   As initial vector, take `x0=zeros(6,1)`. Solve the system with Jacobi and plot the relative errors (using `semilogy`).

   Do the same for the Gauss-Seidel iteration.

2. Take now $n = 10$ and consider the tridiagonal matrix

   `A=diag(ones(n,1)*10)+diag(ones(n-1,1)*3,+1)+diag(ones(n-1,1)*3,-1)`

   and the vector `b` so that the solution is `x=[ones(9,1); 0]`.

   (a) Why do the Jacobi and Gauss-Seidel iterations converge for the solution of $Ax = b$.

   (b) Consider the Jacobi matrix J (which has for any natural norm $\|J\| < 1$) and the tolerance $\epsilon = 1.e - 9$. By using the inequality

   $$\frac{\|P\|^k}{1 - \|P\|}\|x^1 - x^0\| < \epsilon \qquad (1)$$

   where P is the iteration matrix, compute a priori the minimum number k_{min} of iterations necessary to solve the linear system $Ax = b$ with the Jacobi method, $\epsilon = 1.e - 9$, `x0=zeros(n,1)`, `maxit=50` and $\|\cdot\|_2$.

   (c) Do the same for the Gauss-Seidel iteration.

3. Construct the matrix `A=pentadiag(`α`,-1, -1)` for $n = 10$, $\alpha \in [0.5, 1.5]$, i.e. by using `A=toeplitz([` α`, -1, -1,0, 0, 0, 0, 0, 0,0])`. Decompose it as $A = M + D + N$ with `D=diag(`$[\alpha - 1, \ldots, \alpha - 1]$`)`, `M=tril(A)-D` and $N = A - D - M$.

   (a) For which α^* the iterative method $(M + N)x^{(k+1)} = -Dx^{(k)} + q$ results converging faster?

   (b) Letting `b=(1:n)'`, compute by the previous iterative method, the solution of `Ax=b` with `x0=[ ones(m,1); zeros(n-m,1)]`, with $m < n$. Use `tol=1.e-6, maxit=50`.

Time: **2 hours**.

2.2 Home works

1. Take the `Hilbert matrix` of order n, in Matlab is `H=hilb(n)`, with n chosen by the user, and the linear system $Hx = b$. The vector `b` is taken so that the solution is $x = (1, ..., 1)^T$

   - Find the diagonal vector `d`, the upper triangular `U` and lower triangular `L` matrices of H. What do you see?
   - Using the vector `d`, do the command `D=diag(d)`: what do you see?

- Solve the system $H\,x = b$ by using Matlab elementary functions.
- Perturb the vector b by the vector $\delta b = (0, \dots, 0, 1.e-4)^T$. Solve the new system $H\hat{x} = b + \delta\, b$ whose solution is $\hat{(x)}$.

  Estimate the `2-norm condition number`, $\kappa_2(H)$ of H by means of the relation

$$\frac{\|x - \hat{x}\|}{\|x\|} \leq \kappa_2(H)\frac{\|\delta b\|}{\|b\|}\,.$$

  Compare κ_2 with `cond(H)`.

Notice: given a vector `v`, $\|v\|$ is computed by the command `norm(v)` which gives by default the 2-norm of `v`.

2. Consider the matrix `A=diag(1:n)`, for some chosen `n`.

   Study the iteration

$$x^{(k+1)} = (I - \theta A)x^{(k)} + \theta b, \quad k \geq 0\,.$$

   (a) For which $\theta \in [0, 2/3]$ does the method converge? (*Hint: plot* $\rho(\theta) = \rho(I - \theta A)$*).*

   (b) Let $\theta_0 = \min_{0 \leq \theta \leq 2/3} \rho(\theta)$ and let `b` be such that `x=ones(n,1)`. Solve the linear system with the iteration above for θ_0, `x0=zeros(n,1)`, `tol=1.e-6, maxit=100`.

3. Consider the random values `x=rand(20,1)` and stretch them to $[0, 5]$. Call this vector again `x`. The corresponding values along y are defined as follows

$$y_i = x_i^2, \quad \text{for } i \text{ odd} \qquad (2)$$

$$y_i = x_i^2 + 0.5, \quad \text{for } i \text{ even.} \qquad (3)$$

   Find the polynomial of degree 2 that approximates the couples $(x_i, y_i),\ i = 1, ..., 20$ in the least-squares sense. Try then with a polynomial of degree 3